A little subtlety on an electrostatic problem
(Uma pequena sutileza em um problema eletrostático)

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This paper outlines some mathematical considerations regarding a classical problem in elementary electrostatics: the value of electric field at the surface of a conducting sphere with uniform distribution of charge. It is emphasized a consequence following from an elementary but general proof of the, so-called, Gauss’ law.

Keywords: electric field, conducting charged sphere, Gauss’s law.

This short paper outlines some mathematical considerations on a question asked by a few students of an Italian College-level physics course (Liceo Scientifico). The matter was the classical problem of evaluating the electric field inside and outside a uniformly distributed charge on a surface of a sphere, a well known problem explicitly found in elementary physics textbooks [1, 2]. Even if it is clear that any real physical problem in electrostatic must give continuous field everywhere (and one value function), discontinuities in electric field often arise in simple problem solving involving surface charges.

It is known that the field inside any conductor is zero because of the inverse square law as pointed out in the celebrated P.S.S.C. film The Coulomb Law. In the special case of a spherical conductor of radius a, an alternative motivation, usually emphasized that at the centre O the field is zero because of the spherical symmetry; then applying Gauss’s law through a spherical surface of radius r (being r < a) and because no sources of field are in the interior from \( \Phi_s(E) = 0 \) follows \( E = 0 \) being \( \Phi_s(E) \) the electric field flux through the spherical surface. Equally applying the Gauss law for \( r > a \) the well known result of Eq. (1) follows

\[
\begin{align*}
E &= E(r) = 0, & 0 < r < a \\
E &= E(r) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2}, & r > a.
\end{align*}
\]

Usually it is noticed as the field is discontinuous at \( r = a \) being

\[
\lim_{r \to a^-} E(r) = 0; \quad \lim_{r \to a^+} E(r) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{a^2}.
\]

This is problem solving final position.

Because in Italy the physics and mathematics teacher is the same, a subtle question was arisen by a few students: “because the concept of continuity (or discontinuity) refers (at least properly) to a one-valued real function \( E(r) \) in a point a internal to its domain, what is the electric field value at \( x = a \)?” This is a well posed question at least in the context of a simple “problem solving” involving a pure surface charge, because an electric field must exist in all space including \( r = a \).

Hence Eqs. (2) need a requirement of field existence in \( r = a \).

This question is considered in Ref. [3] from a physical point of view and with a correct final answer, but the limit case shown in Fig. 2.12 of this book is misleading because shows an infinite-value function in \( r = a \) (a vertical step in Fig. 2.12 of this book). A correct answer to this question (in the framework of a classically posed problem solving) may simply follow from Gauss’s law. It is noticed that in some textbooks Gauss’s law is simply quoted [1-3], and/or proved in the special case of a spherical surface surrounding a charge located at the centre [4]; other textbooks [3-6] demonstrate Gauss law using a spherical surface (with a charge in the centre) and then extending to the generic surface using the inverse square law. Namely as in Ref. [6]
\[
\int_{\text{any surface}} E_n \, da = \begin{cases} 
0; & q \text{ outside } S \\
\frac{q}{\varepsilon_0}; & q \text{ inside } S 
\end{cases}
\]  

(3)

where \(E_n\) is the field component on the outward oriented infinitesimal surface \(dS\), \(q\) is a charge and \(S\) any closed surface. None of the textbooks quoted, consider the special case of the electric charge at the surface.

An elementary direct proof of Gauss’ law appears on a very old Italian textbook [7] but is always referring to an internal or external charge to a closed surface in free space. The extension to the special case to a charge on the surface follows from an obvious observation on integration domain.

If the charge is in interior, the proof requires an integration of the infinitesimal flux \(d\Phi(E)\)

\[
d\Phi(E) = \frac{Q}{4\pi\varepsilon_0} \, d\omega,
\]

(4)
on the whole solid angle (\(4\pi\) steradians); being \(Q\) the charge, \(\varepsilon_0\) free space permittivity and \(d\omega\) an infinitesimal solid angle. If the charge is in exterior, from the inverse square law and the opposite signs of elementary surface orientations \(dS\) it follows zero flux. In the end, if the charge is located exactly at a point of the surface (and the surface has locally a tangent plane at each point of \(S\)), the flux evaluation requires in this special case an integration on half of the whole solid angle (\(2\pi\) steradians).

Therefore, at \(r = a\) the electric field does exists, and its value is

\[
E(a) = \frac{1}{8\pi\varepsilon_0} \frac{Q \, r}{a^2}.
\]

(5)

Namely the field can be put in compact form valid everywhere for all distances \(r\)

\[
E(r) = \frac{1 + \text{sgn}(r-a)}{2} \frac{1}{4\pi\varepsilon_0} \frac{Q \, r}{r^2},
\]

(6)
as the graphic of \(E\) vs. \(r\) of the Fig. 1 suggests and a simple inspection of Eq. (6) gives a function expressed in a compact form, which is existing everywhere and one-valued function as expected for an electric field. Eq. (6) obviously gives the full correct solution to the feature of problem solving example posed. The “problem solving” example is surely in disagreement with a real physical situation where a continuous function in the neighborhood of \(r = a\) is expected, but this overcomes the aim of an elementary physics course.

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References