Some basics of $su(1,1)$

(Introdução a $su(1,1)$)

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A basic introduction to the $su(1,1)$ algebra is presented, in which we discuss the relation with canonical transformations, the realization in terms of quantized radiation field modes and coherent states. Instead of going into details of these topics, we rather emphasize the existing connections between them. We discuss two parametrizations of the coherent states manifold $SU(1,1)/U(1)$: as the Poincaré disk in the complex plane and as the pseudosphere (a sphere in a Minkowskian space), and show that it is a natural phase space for quantum systems with $SU(1,1)$ symmetry.

Keywords: $su(1,1)$ algebra, $SU(1,1)$ group, canonical transformations, coherent states, second quantization, pseudosphere.

Uma introdução simples à álgebra $su(1,1)$ é apresentada, na qual discutimos a relação com transformações canônicas, a realização em termos de modos quantizados do campo de radiação e estados coerentes. Ao invés de entrar em detalhes a respeito desses tópicos, preferimos enfatizar as conexões existentes entre eles. Discutimos duas parametrizações da variedade dos estados coerentes $SU(1,1)/U(1)$: como o disco de Poincaré no plano complexo e como a pseudoesfera (uma esfera em um espaço de Minkowski) e mostramos que se trata de um espaço de fase natural para sistemas quânticos com simetria $SU(1,1)$.

Palavras-chave: álgebra $su(1,1)$, grupo $SU(1,1)$, transformações canônicas, estados coerentes, segunda quantização, pseudoesfera.

1. Introduction

What I wish to present here is a very basic and accessible introduction to the $su(1,1)$ algebra and its applications. I present for example how to obtain the energy spectrum of hydrogen without solving the Schrödinger equation. I also present the relation with the symplectic algebra and canonical transformations. But the main focus is on coherent states and the geometry of the quotient space $SU(1,1)/U(1)$. Taking into account the realization of this algebra by creation and annihilation operators, I hope these geometrical considerations may have some importance to the field of quantum optics.

The $su(1,1) \sim sp(2,R) \sim so(2,1)$ algebra is defined by the commutation relations

$$[K_1, K_2] = -iK_0, \quad [K_0, K_1] = iK_2,$$
$$[K_2, K_0] = iK_1, \tag{1}$$

and appears naturally in a wide variety of physical problems. A realization in terms of one-variable differential operators,

$$K_1 = \frac{d^2}{dy^2} + \frac{a}{y^2} + \frac{y^2}{16}, \quad K_2 = \frac{-i}{2}\left(y\frac{d}{dy} + \frac{1}{2}\right),$$
$$K_0 = \frac{d^2}{dy^2} + \frac{a}{y^2} - \frac{y^2}{16}, \tag{2}$$

for example, allows any ODE of the kind

$$\left(\frac{d^2}{dy^2} + \frac{a}{y^2} + by^2 + c\right)f(y) = 0 \tag{3}$$

to be expressed in terms of a $su(1,1)$ element [1]. The radial part of the hydrogen atom and of the $3D$ harmonic oscillator, and also the Morse potential fall into this category, and the analytical solution of these systems is actually due to their high degree of symmetry. In fact, the close relation between the concepts of

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symmetry, invariance, degeneracy and integrability is of great importance to all areas of physics [2].

Just like for $su(2)$, we can choose a different basis

$$K_\pm = (K_1 \pm iK_2), \qquad (4)$$

in which case the commutation relations become

$$[K_0, K_\pm] = \pm K_\pm, \quad [K_+, K_-] = -2K_0. \quad (5)$$

Note the difference in sign with respect to $su(2)$. The Casimir operator, the analog of total angular momentum, is given by

$$C = K_0^2 - K_1^2 - K_2^2 = K_0^2 - \tfrac{1}{2}(K_+K_- + K_-K_+). \qquad (6)$$

This operator commutes with all of the K's.

Since the group $SU(1,1)$ is non-compact, all its unitary irreducible representations are infinite-dimensional. Basis vectors $|k, m\rangle$ in the space where the representation acts are taken as simultaneous eigenvectors of K_0 and C:

$$C|k, m\rangle = k(k-1)|k, m\rangle, \qquad (7)$$
$$K_0|k, m\rangle = (k + m)|k, m\rangle, \qquad (8)$$

where the real number $k > 0$ is called the Bargmann index and m can be any nonnegative integer (we consider only the positive discrete series). All states can be obtained from the lowest state $|k, 0\rangle$ by the action of the "raising" operator K_+ according to

$$|k, m\rangle = \sqrt{\frac{\Gamma(2k)}{m!\Gamma(2k+m)}}(K_+)^m|k, 0\rangle. \qquad (9)$$

2. Energy levels of the hydrogen atom

The hydrogen atom, as well as the Kepler problem, has a high degree of symmetry, related to the particular form of the potential. This symmetry is reflected in the conservation of the Laplace-Runge-Lenz vector, and leads to a large symmetry group, $SO(4, 2)$. Here we restrict ourselves to the radial part of this problem, as an example of the applicability of group theory to quantum mechanics and of $su(1, 1)$ in particular. For more complete treatments see [1, 2]. The radial part of the Schrödinger equation for the hydrogen atom is

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{2Z}{r} - \frac{l(l+1)}{r^2} + 2E\right)R(r) = 0. \quad (10)$$

If we make $r = y^2$ and $R(r) = y^{-3/2}Y(y)$ we have

$$\left(\frac{d^2}{dy^2} - \frac{4l(l+1) - 3/4}{y^2} + 8Ey^2 - 8Z\right)Y(y) = 0, \quad (11)$$

and, as already noted in the introduction, this can be written in terms of the $su(1, 1)$ generators (2). A little algebra gives

$$\left[(\tfrac{1}{2} - 64E)K_0 + (\tfrac{1}{2} + 64E)K_1 - 8Z\right]Y(y) = 0, \quad (12)$$

and the Casimir reduces to $C = l(l+1)$, which gives $k = l + 1$.

Using the transformation equations

$$e^{-i\theta K_2}K_0 e^{i\theta K_2} = K_0\cosh\theta + K_1\sinh\theta \qquad (13)$$
$$e^{-i\theta K_2}K_1 e^{i\theta K_2} = K_0\sinh\theta + K_1\cosh\theta \qquad (14)$$

we can choose

$$\tanh\theta = \frac{64E + 1/2}{64E - 1/2} \qquad (15)$$

in order to obtain

$$K_0\tilde{Y}(y) = \frac{Z}{\sqrt{-2E}}\tilde{Y}(y), \qquad (16)$$

where $\tilde{Y}(y) = e^{-i\theta K_2}Y(y)$. Since we know the spectrum of K_0 from (8) we can conclude that the energy levels are given by

$$E_n = -\frac{Z}{2n^2}, \quad n = m + l + 1. \qquad (17)$$

3. Relation with $Sp(2, R)$

A system with n degrees of freedom, be it classical or quantum, always has $Sp(2n, R)$ as a symmetry group. Classical mechanics takes place in a real manifold, and the kinematics are given by Poisson brackets $(i, j = 1..N)$

$$\{q_i, p_j\} = \delta_{ij}. \qquad (18)$$

Quantum mechanics takes place in a complex Hilbert space, and the kinematics determined by the canonical commutation relations $(i, j = 1..N)$

$$[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}. \qquad (19)$$

These relations can also be written in the form (now $i, j = 1..2N$)

$$\{\xi_i, \xi_j\} = J_{ij}, \qquad (20)$$
$$[\hat{\xi}_i, \hat{\xi}_j] = i\hbar J_{ij}, \qquad (21)$$

where $\xi = (q_1, ..., q_N, p_1, ..., p_N)^T$, $\hat{\xi}_i$ is the hermitian operator corresponding to ξ_i and J is the $2N \times 2N$ matrix given by

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \qquad (22)$$

The symplectic group $Sp(2N, R)$ (in its defining representation) is composed by all real linear transformations that preserve the structure of relations (20). It is easy to see that therefore

$$Sp(2N, R) = \{S | SJS^T = J\}. \qquad (23)$$

For a far more extended and detailed discussion, see [3]

For a classical system with only one degree of freedom, such canonical transformations are generated by the vector fields [4]

$$\{ -q\frac{\partial}{\partial p} + p\frac{\partial}{\partial q} = 2iK_0, \quad -q\frac{\partial}{\partial p} - p\frac{\partial}{\partial q} = 2iK_1, $$
$$-q\frac{\partial}{\partial q} + p\frac{\partial}{\partial p} = 2iK_2, \}. \qquad (24)$$

It is easy to see that these operators have the same commutation relations as the $su(1,1)$ algebra (1).

Note that the symplectic groups $Sp(2n, R)$ are non-compact, and therefore any finite dimensional representation must be nonunitary. In the quantum case, that means that the matrices S implementing the transformations

$$\hat{\xi}'_j = S_{ij}\hat{\xi}_i, \qquad (25)$$

such that $[\hat{\xi}'_i, \hat{\xi}'_j] = i\hbar J_{ij}$, are nonunitary (a 2×2 nonunitary representation of $su(1,1)$ exists for example in terms of Pauli matrices, $K_1 = \frac{i}{2}\sigma_2, K_2 = -\frac{i}{2}\sigma_1, K_0 = \frac{1}{2}\sigma_3$). However, since all $\hat{\xi}_i$ and all $\hat{\xi}'_j$ are hermitian and irreducible, by the Stone-von-Neumann theorem [3, 5] there exists an operator $U(S)$ that acts unitarily on the infinite dimensional Hilbert space of pure quantum states (Fock space). If we now see $\hat{\xi}_i$ and $\hat{\xi}'_i$ as (infinite dimensional) matrices, then $U(S)$ is such that $\hat{\xi}'_i = U(S)\hat{\xi}_i U(S)^{-1}$. Finding this unitary operator in practice is in general a nontrivial task.

4. Optics

4.1. One-mode realization

We know the radiation field can be described by bosonic operators a and $a^\dagger$. If we form the quadratic combinations

$$K_+ = \frac{1}{2}(a^\dagger)^2, \quad K_- = \frac{1}{2}a^2,$$
$$K_0 = \frac{1}{4}(aa^\dagger + a^\dagger a) \qquad (26)$$

we obtain a realization of the $su(1,1)$ algebra. In this case the Casimir operator reduces identically to

$$C = k(k-1) = -\frac{3}{16}, \qquad (27)$$

which corresponds to $k = 1/4$ or $k = 3/4$. It is not difficult to see that the states

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle \qquad (28)$$

with even n form a basis for the unitary representation with $k = 1/4$, while the states with odd n form a basis for the case $k = 3/4$.

The unitary operator

$$S(\xi) = \exp\left(\frac{1}{2}\xi^* a^2 - \frac{1}{2}\xi(a^\dagger)^2\right) = $$
$$\exp(\xi^* K_- - \xi K_+) \qquad (29)$$

is called the squeeze operator in quantum optics, and is associated with degenerate parametric amplification [6]. There is also the displacement operator

$$D(\alpha) = \exp\left(\alpha a^\dagger - \alpha^* a\right), \qquad (30)$$

which acts on the vacuum state $|0\rangle$ to generate the coherent state

$$|\alpha\rangle = D(\alpha)|0\rangle = e^{-|\alpha|^2/2} \sum_{n=o}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle. \qquad (31)$$

Action of $S(\xi)$ on a coherent state gives a squeezed coherent state, $|\alpha, \xi\rangle = S(\xi)|\alpha\rangle$.

4.2. Two-mode realization

It is also possible to introduce a two-mode realization of the algebra $su(1,1)$. This is done by defining the generators

$$K_+ = a^\dagger b^\dagger, \quad K_- = ab,$$
$$K_0 = \frac{1}{2}(a^\dagger a + b^\dagger b + 1). \qquad (32)$$

In this case the Casimir operator is given by $C = \frac{1}{4}(a^\dagger a - b^\dagger b)^2 - \frac{1}{4}$. If we introduce the usual two-mode basis $|n, m\rangle$ then the states $|n + n_0, n\rangle$ with fixed n_0 form a basis for the representation of $su(1,1)$ in which $k = (|n_0| + 1)/2$. A charged particle in a magnetic field can also be described by this formalism [7].

The unitary operator

$$S_2(\xi) = \exp\left(\xi^* ab - \xi a^\dagger b^\dagger\right) =$$
$$\exp(\xi^* K_- - \xi K_+) \qquad (33)$$

is called the two-mode squeeze operator [6], or down-converter. When we consider the other quadratic combinations ($\{a^\dagger b, (a^\dagger)^2, (b^\dagger)^2, a^\dagger a - b^\dagger b\}$ and their hermitian adjoint) we have the algebra $sp(4, R)$, of which $sp(2, R) \sim su(1,1)$ is a subalgebra. More detailed discussions about group theory and optics can be found for example in [3, 4, 8].

5. Coherent states

Normalized coherent states can be defined for a general unitary irreducible representation of $su(1,1)$ as [9]

$$|z, k\rangle = (1 - |z|^2)^k$$

$$\sum_{m=0}^{\infty} \sqrt{\frac{\Gamma(2k+m)}{m!\Gamma(2k)}} z^m |k, m\rangle, \qquad (34)$$

where z is a complex number inside the unit disk, $D = \{z, |z| < 1\}$. Similar to the usual coherent states, they can be obtained from the lowest state by the action of a displacement operator:

$$|z, k\rangle = \exp(\zeta K_+ - \zeta^* K_-)|k, 0\rangle,$$
$$z = \frac{\zeta}{|\zeta|} \tanh|\zeta|. \qquad (35)$$

From (33) we see that $su(1,1)$ coherent states are actually the result of a two-mode squeezing upon a Fock state of the kind $|n_0, 0\rangle$. On the other hand, from the one-mode realization (29) they can be regarded as squeezed vacuum states.

These states are not orthogonal,

$$\langle z_1, k | z_2, k \rangle = \frac{(1 - |z_1|^2)^k (1 - |z_2|^2)^k}{(1 - z_1^* z_2)^{2k}} \qquad (36)$$

and they form an overcomplete set with resolution of unity given by

$$\int_D \frac{2k-1}{\pi} \frac{dz \wedge dz^*}{(1-|z|^2)^2} |z, k\rangle\langle z, k| =$$
$$\sum_{m=0}^{\infty} |k, m\rangle\langle k, m| = 1 \quad \left(k > \frac{1}{2}\right). \qquad (37)$$

From the integration measure we see that the coherent states are parametrized by points in the Poincaré disk (or Bolyai-Lobachevsky plane), which we discuss in the next section. The expectation value for a product of algebra generators like $K_-^p K_0^q K_+^r$ was presented in [10] and is given by

$$\langle z, k | K_-^p K_0^q K_+^r | z, k \rangle = (1 - |z|^2)^{2k} z^{p-r} \sum_{m=0}^{\infty} \frac{\Gamma(m+p+1)\Gamma(m+p+2k)}{m!\Gamma(m+p+1-r)\Gamma(2k)} (m+p+k)^q |z|^{2m}. \qquad (38)$$

Simple particular cases of this expression are

$$\langle z, k | K_- | z, k \rangle = k \frac{2z}{1 - |z|^2},$$
$$\langle z, k | K_0 | z, k \rangle = k \frac{1 + |z|^2}{1 - |z|^2}. \qquad (39)$$

Moreover, for $k > 1/2$ the operator K_0 has a diagonal representation as

$$K_0 = \frac{2k-1}{4\pi} \int_D \frac{d^2 z}{(1-|z|^2)^2} (k-1) *$$

$$\left(\frac{1 + |z|^2}{1 - |z|^2}\right) |z, k\rangle\langle z, k|. \qquad (40)$$

Just as usual spin coherent states are parametrized by points on the space $SU(2)/U(1) \sim S^2$, the two-dimensional spherical surface, $su(1,1)$ coherent states are parametrized by points on the space $SU(1,1)/U(1)$, which corresponds to the Poincaré disk. This space can also be seen as the two-dimensional upper sheet of a two-sheet hyperboloid, also known as the pseudosphere.

6. The pseudosphere

The sphere S^2 is the set of points equidistant from the origin in a Euclidian space:

$$S^2 = \{(x_1, x_2, x_3) | x_1^2 + x_2^2 + x_3^2 = R^2\}. \quad (41)$$

The pseudosphere H^2 plays a similar role in a Minkovskian space, that is, take the space defined by $\{(y_1, y_2, y_0) | y_1^2 + y_2^2 - y_0^2 = -R^2\}$, which is a two-sheet hyperboloid that crosses the y_0 axis at two points, $\pm R$, called poles. The pseudosphere, which is a Riemannian space, is the upper sheet, $y_0 > 0$. The pseudosphere is related to the Poincaré disk by a stereographic projection in the plane (y_1, y_2), using the point $(0, 0, -R)$ as base point. The relation between the parameters is

$$y_0 = R\cosh\tau, \quad y_1 = R\sinh\tau\cos\phi,$$
$$y_2 = R\sinh\tau\sin\phi, \quad (42)$$

and

$$z = e^{i\phi}\tanh\frac{\tau}{2} = \frac{y_1 + iy_2}{R + y_0}. \quad (43)$$

The distance $ds^2 = dy_1^2 + dy_2^2 - dy_0^2$ and the area $d\mu = \sinh\tau d\tau \wedge d\phi$ become

$$ds^2 = d\tau^2 + \sinh\tau d\phi^2 = \frac{dz \cdot dz^*}{(1 - |z|^2)^2}, \quad (44)$$

$$d\mu = \frac{dz \wedge dz^*}{(1 - |z|^2)^2}. \quad (45)$$

Note that the metric is conformal, so the actual angles coincide with Euclidian angles. Geodesics, which are intersections of the pseudosphere with planes through the origin, become circular arcs (or diameters) orthogonal to the disk boundary (the non-Euclidian character of the Poincaré disk appears in some beautiful drawings of M.C. Escher, the "Circle Limit" series [11]). A very good discussion about the geometry of the pseudosphere can be found in [12], and we follow this presentation.

In the pseudosphere coordinates the average values of the $su(1, 1)$ generators are very simple:

$$\langle z, k|K_1|z, k\rangle = \frac{k}{R}y_1, \quad \langle z, k|K_2|z, k\rangle = \frac{k}{R}y_2,$$
$$\langle z, k|K_0|z, k\rangle = \frac{k}{R}y_0. \quad (46)$$

From now on we set $R = k = 1$.

6.1. Action of the group

The symmetry group of the pseudosphere is the group that preserves the relation $y_1^2 + y_2^2 - y_0^2 = -R^2$, the Lorentz-like group $SO(2, 1)$. The $so(2, 1)$ algebra associated with this group is isomorphic to the $su(1, 1)$ algebra we are studying. All isometries can be represented by 3×3 matrices Λ that are orthogonal with respect to the Minkowski metric $Q = \text{diag}(1, 1, -1)$ (actually we must also impose $\Lambda_{00} > 0$ so that we are restricted to the upper sheet of the hyperboloid), and they can be generated by 3 basic types: A) Euclidian rotations, by an angle ϕ_0, on the (y_1, y_2) plane; B) Boosts of rapidity τ_0 along some direction in the (y_1, y_2) plane; C) Reflections through a plane containing the y_0 axis. As examples, we show a rotation, a boost in the y_2 direction and a reflection through the plane (y_1, y_0):

$$A) \begin{pmatrix} \cos\phi_0 & -\sin\phi_0 & 0 \\ \sin\phi_0 & \cos\phi_0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$B) \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\tau_0 & \sinh\tau_0 \\ 0 & \sinh\tau_0 & \cosh\tau_0 \end{pmatrix},$$

$$C) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (47)$$

Incidentally, the geometrical character of the previously used parameters (τ, ϕ) becomes clear.

Using the complex coordinates of the Poincaré disk we have

$$R_{\phi_0}(z) = e^{i\phi_0}z \quad (48)$$

for rotations,

$$T_{\tau_0, \phi_0}(z) = \frac{(\cosh\tau_0/2)z + e^{i\phi_0}\sinh\tau_0/2}{(e^{-i\phi_0}\sinh\tau_0/2)z + \cosh\tau_0/2} \quad (49)$$

for boosts of rapidity τ_0 in the ϕ_0 direction and $S(z) = z^*$ for reflections through the (y_1, y_0) plane. We see that, except for reflections, all isometries can be written as

$$z' = \frac{\alpha z + \beta}{\beta^* z + \alpha^*}, \quad \text{with } |\alpha|^2 - |\beta|^2 = 1, \quad (50)$$

and if, as usual, we represent these transformations by matrices $\begin{pmatrix} \alpha & \beta^* \\ \beta & \alpha^* \end{pmatrix}$ there is a realization of the trans-

formation group by 2×2 matrices, in which

$$R_{\phi_0} = \begin{pmatrix} e^{i\phi_0/2} & 0 \\ 0 & e^{-i\phi_0/2} \end{pmatrix},$$

$$T_{\tau_0,\phi_0} = \begin{pmatrix} \cosh \tau_0/2 & e^{-i\phi_0} \sinh \tau_0/2 \\ e^{i\phi_0} \sinh \tau_0/2 & \cosh \tau_0/2 \end{pmatrix}. \quad (51)$$

This is the basic representation of the group $SU(1,1)$. For other parametrizations of the pseudo-sphere, see [12].

6.2. Canonical coordinates

We present one last set of coordinates, one that has an important physical property. Let us first note that if we define $\mathcal{K}_i = \langle z, k | K_i | z, k \rangle$, then there exists an operation $\{\cdot, \cdot\}$ such that the commutation relations

$$[K_1, K_2] = -iK_0, \quad [K_0, K_1] = iK_2,$$
$$[K_2, K_0] = iK_1 \quad (52)$$

are exactly mapped to

$$\{\mathcal{K}_1, \mathcal{K}_2\} = \mathcal{K}_0, \quad \{\mathcal{K}_0, \mathcal{K}_1\} = -\mathcal{K}_2,$$
$$\{\mathcal{K}_2, \mathcal{K}_0\} = -\mathcal{K}_1, \quad (53)$$

in agreement with the usual quantization condition $\{\cdot, \cdot\} \to i[\cdot, \cdot]$. This Poisson Bracket is written in terms of the Poincaré disk coordinates as

$$\{f, g\} = \frac{(1 - |z|^2)^2}{2ik} \left(\frac{\partial f}{\partial z} \frac{\partial g}{\partial z^*} - \frac{\partial f}{\partial z^*} \frac{\partial g}{\partial z} \right). \quad (54)$$

It is possible to define new coordinates (q, p) that are canonical in the sense that

$$\{f, g\} = \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial q}. \quad (55)$$

These coordinates are given by

$$\frac{q + ip}{\sqrt{4k}} = \frac{z}{\sqrt{1 - |z|^2}} \quad (56)$$

and the classical functions are written in terms of them as

$$\mathcal{K}_1 = \frac{q}{2}\sqrt{4k + q^2 + p^2},$$
$$\mathcal{K}_2 = \frac{p}{2}\sqrt{4k + q^2 + p^2},$$
$$\mathcal{K}_0 = k + \frac{q^2 + p^2}{2}. \quad (57)$$

We thus see that there is a natural phase space for quantum systems that admit $SU(1,1)$ as a symmetry group. Dynamics of time-dependent systems with

this property was examined for example in [13]. This phase space can also be used to define path integrals for $SU(1,1)$ (see [14, 15] and references therein), and obtain a semiclassical approximation to this class of quantum systems.

7. Summary

We have presented a very basic introduction to the $su(1,1)$ algebra, discussing the connection with canonical transformations, the realization in terms of quantized radiation field modes and coherent states. We have not explored these subjects in their full detail, but instead we emphasized how they can be related. The coherent states, for example, can be regarded as one-mode vacuum squeezed states or as two-mode number squeezed states. The coherent states manifold $SU(1,1)/U(1)$ was treated as the Poincaré disk and as the pseudosphere, and shown to be a natural phase space for quantum systems with $SU(1,1)$ symmetry.

Acknowledgments

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