Quantum Jumps Versus Stochastic Schrödinger Equation. Applications in Quantum Optics

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Received June 10, 1995

I. Introduction

Traditionally, to study a quantum optical system, one deals with a reduced density matrix approach leading to a master equation[1] of the density matrix of typically one or more modes of the electromagnetic field, after having eliminated all other degrees of freedom such as atoms, reservoirs etc.

This approach has a strong emphasis on the evolution of the whole statistical system and does not deal with the question of how an individual member of such an ensemble evolves in time.

Alternatively, over the last few years, methods have been developed to simulate numerically and up to a point, also analytically, the "individual quantum trajectories" a word coined by Carmichael[2]. Two main line of thought have appeared. On one hand, one method is based on the continuous evolution of the system, under a non Hermitian Hamiltonian, as to include dissipative effects, randomly interrupted by instantaneous quantum jumps[3,4], which, according to Wiseman and Milburn,[5] could be possibly interpreted as a direct counting of decaying quanta. This could, for instance the event describing the absorption of a photon in walls of a resonator. On the other hand, a second approach consists in studying the evolution of a "quantum state diffusion model", where the "individual trajectories" fluctuate with a Wiener process describing the coupling of the system to the environment. From the measurement viewpoint, this would correspond to a heterodyne measurement of a field amplitude.[6-9]

Both methods, of course, converge in the sense that the ensemble average leads to the usual Master Equation describing the system coupled to various reservoirs and leading, typically, to gain and damping.[10]

The plan of the present work is the following: In sections II and III, we briefly describe both the quantum jump and Stochastic Schrödinger approach respectively. Section IV is devoted to apply some of these ideas to the Raman Laser and also explore the possibility of cooperative effects. Finally section V is devoted to discussion. Before concluding this section, we should point out that if one has a small system (s) coupled to one or more large reservoirs, one writes a linear equation for the density matrix of the small system ρ_s as

$$\dot{\rho}_s = \frac{i}{\hbar}[\rho_s, H_s] + L_{\mathrm{relax}}(\rho_s) \ , \tag{1}$$

where $L_{\mathrm{relax}}(\rho_s)$ is a relaxation superoperator describing various possible dissipative mechanisms, leading to the spontaneous emission, damping of the field in a cavity, etc.

For a very large class of relaxation phenomena, one can write:

$$L_{\mathrm{relax}}(\rho_s) = -\frac{1}{2}\sum_m (C_m^\dagger C_m \rho_s^\dagger + \rho_s C_m^\dagger C_m) + \sum_m C_m \rho C_m^\dagger \ , \tag{2}$$

where C_m is an operator acting on the small relevant system. For example, in the case of spontaneous emission, $C = \sqrt{k}\ \sigma_-$ where σ_- is the Pauli lowering spin operator and k is the energy loss rate to the reservoir.

Another example is $C = \sqrt{\gamma(\bar{n}+1)}a$, describing loss (at a rate γ) to the cavity walls at a temperature characterized by the plioton occupation number $\bar{n}$.

II. Quantum jumps

The method is the following[3]

One calculates the time evolution of the wave function from $|\phi(t)\rangle$ to $|\phi(t+\delta t)\rangle$ in 2 steps:

a) A non-normalized wave function $|\phi^{(1)}(t+St)\rangle$ is obtained from a "continuous evolution" with a non-Hermitian Hamiltonian:

$$H = H_S - \frac{i\hbar}{2}\sum_m C_m^\dagger C_m \ , \tag{3}$$

in such a way that, for small St:

$$|\phi'(t+\delta t) >= \left(1 - \frac{iH\delta t}{\hbar}\right)|\phi(t)> \ . \tag{4}$$

Since H is non-Hermitian, one has to renormalize the wave function:

$$< \phi'(t+\delta t)|\phi'(t+St)>=< \phi^{(1)}(t)|\left[1+\frac{iH^\dagger\delta t}{\hbar}\right]\left[1+\frac{iH^\dagger\delta t}{\hbar}\right]|\phi(t)>= 1-\delta p\ , \tag{5}$$

where

$$\delta p = (\delta t)\frac{i}{\hbar} < \phi(t)|H - H^\dagger|\phi(t)>= \sum \delta p_m \ , \tag{6}$$

with:

$$\delta p_m = \delta t < \phi(t)|C_m^\dagger C_m|\phi(t)> \ . \tag{7}$$

b) Between (t) and $(t+\delta t)$ there is a finite possibility of a quantum jump. To decide wliether a quantum juinp has occured, one cliooses a random number r between 0 and 1.

Since we choose (St) small, one can always have $\delta p \ll 1$.

Now, usually $r > \delta p$ and no quantum jump occurs and

$$|\phi(t+\delta t)>_{\text{normalized}} = \frac{|\phi'(t+\delta t)>}{\sqrt{1-\delta p}} \ . \tag{7}$$

However, at some rare occasions, $r < \delta p$ and a quantum jump occurs, and the wave function can be one of the states:

$$|\phi(t+\delta t)>= \frac{C_m|\phi(t)>}{\sqrt{\delta p_m/\delta t}} \ , \tag{8}$$

with a relative probability $(\delta p_m/\delta p)$ with respect to the various jump processes. It is not difficult to prove, that with the rules given above, if one averages the possible outcomes at time t, starting from some initial state, the result coincides with $\rho_s(t)$, provided the initial state is the saine in both cases.

III. The state diffusion picture

The other approach is to write the time evolution described in equation (1) as a Stochastic Schrodinger Equation, given by:[6-8]

$$\begin{aligned} d|\psi> &= \frac{i}{\hbar}H|\psi> dt + \sum_m\left[< C_m^\dagger >_\psi C_m - \frac{1}{2}C_m^\dagger C_m - \frac{1}{2} < C_m^\dagger >_\psi < C_m >_m\right]|\psi> dt \\ &+ \sum_m (C_m - < C_m >_\psi)|\psi> d\xi_m \ , \end{aligned} \tag{9}$$

where $d\xi_m$ is a complex Wiener process characterized by

$$
\begin{aligned}
&M[Re(d\xi_n)Re(d\xi_m)] = M[Im(d\xi_n)Im(d\xi_m)] = \delta_{nm}\frac{\delta t}{2}\ , \\
&M(d\xi_m) = 0 \ \ , \ \ M(Re(d\xi_n)Im(d\xi_m)) = 0
\end{aligned}
\tag{10}
$$

where M represents the mean over the probability distribution, where the Ito calculus is implied here, thus keeping **differentials up** to second order. Once more, it is straightforward to prove that this Stochastic Schrodinger Equation, when averaged over many such "quantum trajectories", gives back the master equation (1).

In both methods described above, one has the distinctive advantage of solving N equations, instead of N^2 equations, for the density matrix, where N is the state space dimension.

IV. Application. The Raman laser

We consider a doubly resonant optical cavity, where the ground state, intermediate and excited states are labeled by Q 1 and 2 respectively. (Figure 1).

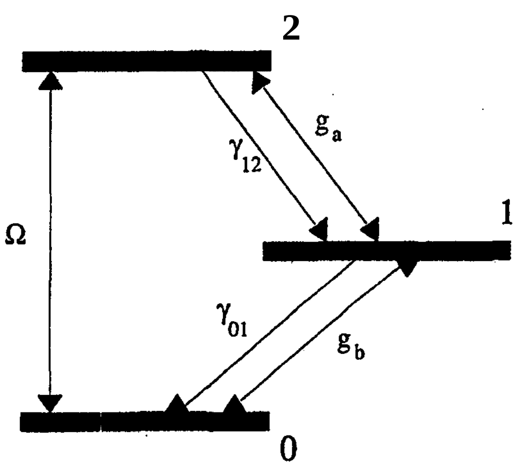

Figure 1. The level atom inside tlie double cavity.

Lasing takes place in the $|1> -|2>$ transition coupled to the first mode with a coupling constant g_a. On the other hand tlie second cavity mode is resonant with the $|0> -|1>$ transition with coupling constaiit g_b. We will assume that the cavity is "bad" for tlie mode b, so that it contains a very low photon number and acts as a recycling mode. Finally, Ω is the pump that drives the $|0> -|2>$ transition.

The Hamiltonian of the system is:

$$
H = i\hbar g_a(a\sigma_{21} - a^\dagger\sigma_{12}) + i\hbar g_b(b\sigma_{10} - b^\dagger\sigma_{01}) + i\hbar\Omega(\sigma_{20} - \sigma_{02})\ , \tag{11}
$$

and the damping terms are characterized by:

$$
\begin{aligned}
L_{\text{relax } p} \ = \ & \frac{\gamma_{12}}{2}(2\sigma_{12}\rho\sigma_{21} - \sigma_{21}\sigma_{12}\rho - \rho\sigma_{21}\sigma_{12}) \\
+ \ & \frac{\gamma_{01}}{2}(2\sigma_{01}\rho\sigma_{10} - \sigma_{10}\sigma_{01}\rho - \rho\sigma_{10}\sigma_{01}) \\
+ \ & \frac{\gamma_a}{2}(a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a) \\
+ \ & \frac{\gamma_b}{2}(b\rho b^\dagger - b^\dagger b\rho - \rho b^\dagger b)\ .
\end{aligned}
\tag{12}
$$

For the numerical analysis of this problem, we have used the state diffusion picture, Eq. (19).

In the reference 11, a superradiant laser is described, where the collective behaviour is, at least in part, described by the "passive" mode b. Physically, this is a low finesse mode thus justifiying the adiabatic eliminitation. It is simple to show[11] that if one writes the Heisenberg equations for the system, this adiabatic elimination is mathematically equivalent to a collective decay from the level 1 to the ground state. On the other hand, previous workers[12,13] found that in a one Atom Raman Laser (without b-field) and with a detuning of the order of the atom-field coupling constant, one could prepare a low plioton number eigenstate of a single damped cavity mode coupled to a single three level atom.

We could in principle, monitor the photon number of tlie b-field that will provide information or influence the atomic populations and hence, the lasing mode. If we write the state vector of the system as:

$$
|\psi> = \sum_{i,n_a,n_b} C_{i,n_a,n_b}|i>|n_a>|n_b>\ , \tag{13}
$$

from eq. (8), one can write:

$$
\begin{aligned}
d|\psi\rangle &= [\{g_a(a\sigma_{21} - a^+\sigma_{12}) + g_b(b\sigma_{10} - b^+\sigma_{01}) \\
&+ \Omega(\sigma_{20} - \sigma_{02}) - \frac{\gamma_a}{2}a^+a + \gamma_a < a^+ > a - \frac{\gamma_b}{2}b^+b \\
&+ \gamma_b < b^+ > b - \frac{\gamma_{12}}{2}\sigma_{22} + \gamma_{12} < \sigma_{21} > \sigma_{12} - \frac{\gamma_{01}}{2}\sigma_{11} \\
&+ \gamma_{01} < \sigma_{10} > \sigma_{01}\}\Delta t + \sqrt{\gamma_a}ad\xi_a + \sqrt{\gamma_b}bd\xi_b + \sqrt{\gamma_{12}}\sigma_{12}d\xi_{12} + \sqrt{\gamma_{01}}\sigma_{01}d\xi_{01}]|\psi > \qquad (14)
\end{aligned}
$$

In equation (14) we have omitted the terms in the direction of $|\psi >$. This is justified for complex fluctuations satisfying eq. (10).[6] Introducing the expression of $|\psi >$ (eq. 13) into (14) and after some simple algebra, one gets the following differential recursion relation for the expression coefficientes:

$$
\begin{aligned}
dC_{i,n_a,n_b}(t) &= \Delta t[\delta_{1,2}g_a\sqrt{n_a+1}C_{1,n_a+1,n_a} \\
&- g_a\sqrt{n_a}\delta_{i,1}C_{2,n_a-1,n_a} + g_b\delta_{i,q}\sqrt{n_b+1}C_{0,n_a,n_b+1} \\
&- g_b\sqrt{n_b}\delta_{i,0}C_{1,n_a,n_b-1} + \Omega(\delta_{i,2}C_{0,n_a,n_b} - \delta_{i,0}C_{2,n_a,n_b}) \\
&- \frac{1}{2}(\gamma_a n_a + \gamma_b n_b)C_{i,n_a,n_b} - \frac{\gamma_{12}}{2}\delta_{i,2}C_{2,n_a,n_b}) \\
&- \frac{\gamma_{01}}{2}\delta_{i,1}C_{1,n_a,n_b} + \gamma_a < a^+ > \sqrt{n_a}C_{i,n_a+1,n_2} \\
&+ \gamma_b < b^+ > \sqrt{n_b}C_{i,n_a,n_{b+1}} + \gamma_{12} < \sigma_{21} > \delta_{i,1}C_{2,n_a,n_b}] \\
&+ \gamma_{01} < \sigma_{10} > \delta_{i,1}C_{0,n_a,n_b}] \\
&+ [\sqrt{\gamma_a}\sqrt{n_{a+1}}d\xi_a C_{i,n_a+1,n_b} + \sqrt{\gamma_b}\sqrt{n_{b+1}}d\xi_b C_{i,n_a,n_b+1} \\
&+ \sqrt{\gamma_{12}}d\xi_{12}\delta_{i,1}C_{2,n_a,n_b} + \sqrt{\gamma_{01}}d\xi_{01}\delta_{i,0}C_{i,n_a,n_b}] \qquad (15)
\end{aligned}
$$

We take as initial condition for eq. (15)

$$C_{i,n_a,n_b}(t=0) = \delta_{i,0}\delta_{n_a,0}\delta_{n_b,0}$$

The stochastic terms $d\xi_a$, $d\xi_b$, $d\xi_{12}$ and $d\xi_{01}$ are complex numbers and are obtained through a usual normally (gaussian) distributed independent random sequence for both the real and the imaginary parts, with á variance $\delta t/2$ and zero mean value, according to eq. (10).

According to the above discussion, we iterate numerically eq. (15), doing it several times and averaging the result. Since we wanted to monitor the system with the field, we took some fixed parametres $g_a = 1$, $\Omega = 1$, $\gamma_a = .05$, $\gamma_b = 5.$, $\gamma_{12} = 0.$, $\gamma_{01} = 0.4$ and taking $g_b = .5$ (Figure 2), $g_b = 2.$ (Figure 3) and $g_b = 2.5$ (Figure 4).

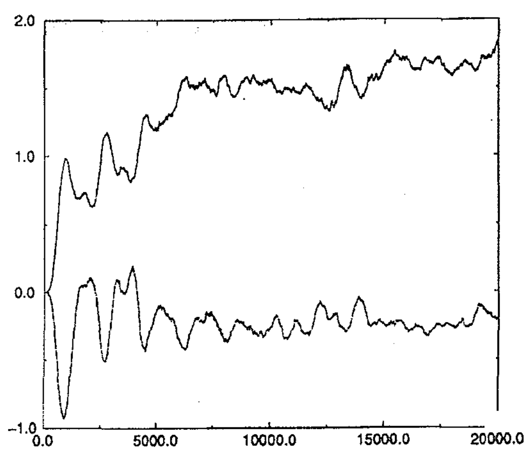

Figure 2. Average photon number (upper curve) and Mandel's Q-factor lower curve for $g_a = 1$, $g_b = 0.5$, $\Omega = 1$, $\gamma_a = .05$, $\gamma_b = 5$; $\gamma_{12} = 0$, $\gamma_{01} = .4$. The horizontal axis shows the number of steps, with $\delta t = .0025$. 10 loops were taken.

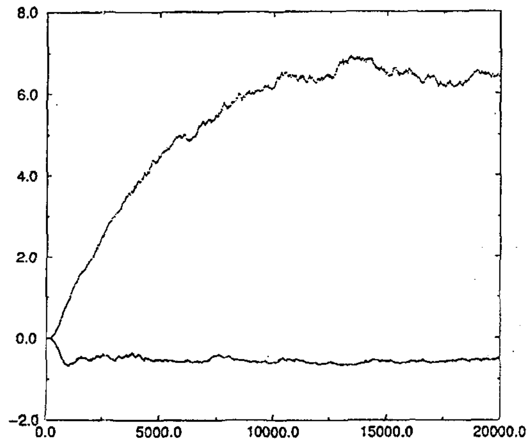

Figure 3. Average photon number (upper curve) and Mandei's Q-factor (lower curve) for the same parameters as in figure 2, except $g_b = 2$.

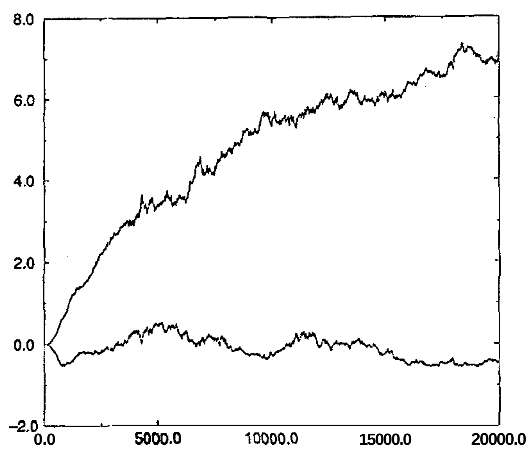

Figure 4. Average photon number (upper curves) and Mandei's Q-(factor/lower) curve, for the same parameters as in figure 2, except $g_b = 2.5$.

The upper curve in each case is the average photon number and the lower curve is the Mandel Q_a-parameter, defined as:

$$Q_a = \frac{< n_a^2 > - < n_a >^2}{< n_a >} - 1 \qquad (16)$$

We notice that for increasing g_a, Q_a goes down, thus enhancing the non classical behaviour, reaching a minimum value of $Q_a \approx -0.5$ for $g_b = 2.5$. For larger g_b values, Q_a increases again. It is interesting to notice that this field state with low photon number fluctuations has a non negligible intensity (photon number) as compared with previous results where the photon number is the order of unity.[12]

V. Summary and discussion

The method of the quantum trajectories is a powerful tool to solve either analytically or numerically problems that otherwise would be very difficult to handle. We reviewed two different ways of attaclting this problem, namely via quantum jumps or stochastic Schrodinger equation. These two methods when averaged over many trajectories, give back the master equation with the irreversible or bath terms.

The first way (quantum jump) can be interpreted, if viewed from the measurement theory viewpoint, as photodetection and the second way (Stochastic Schrodinger) as heterodyne field measurement scheme. Finally, we analyze the example of the Raman Laser, using quantum trajectories generated numerically with the Stochastic Schrodinger approach.

We find, that by introducing an auxiliary b-field in the model, one can generate sub Poissonian fields with a non negligible number of photons, which represents an advantage over previous work.[12]

Finally, we think it is of interest to explore a Raman or Superradiant Laser with several atoms to test important properties, such as N^2 emission, bright squeezing, etc. This will be the subject of a future publication.

Acknowledgements

This research was funded by FONDECYT projects #1950790 and #1950801.

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