Radiation Reaction Force for a Mirror in Vacuurn

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When the surface of a mirror is deformed in a prescribed time–dependent way, it experiences a dissipative force exerted by tlie vacuum field. In order to obtain the dissipative force, we calculate the scattered fields in the long–wavelength approximation. We show that dissipation and fluctuations are related as predicted by linear response theory. The dissipative force is usually interpreted as a radiation reaction force. We confirm such interpretation by explicit evaluation of the radiated energy.

I. Introduction

The most known mechanical effect of vacuum fields is the Casimir attractive force between two mirrors at rest[1]. However, interesting quantum effects may occur even with a single mirror in vacuum. One such effect is the creation of photons out of the vacuum state as a result of non–uniform motion of the mirror. Moore was apparently the first to consider quantum fields with moving boundaries[2]. Following Moore's work, Fulling and Davies derived tlie energy-momentum tensor in the case of a single mirror, showing the effect of emission of radiation[3]. As was done in Ref.[2], they considered a one–dimensional approximation (1D) for the electromagnetic field, thus allowing propagation only normal to the surface of a flat perfectly–reflecting mirror. Since their approach relies on the conforma1 invariance of the one–dimensional wave equation, it cannot be easily generalized to the realistic three–dimensional case (3D).

A completely different approach was employed by Ford and Vilenltin to tacltle the problem for a massless scalar field in 3D[4]. In order to allow for the computation of the motional corrections to the energy–momentum tensor, they took additional approximations, not present in the previous 1D calculations. They considered the non-relativistic limit, and accordingly computed the motional effects to first order in the mirror's velocity and its derivatives. Furthermore, they assumed the mirror's displacement to be much smaller than the field wavelengths relevant for the effect. As shown in Ref.[5], these two assumptions are related in a sense to be explained later.

More recently, another interesting quantum me; chanical effect of the vacuum field was proposed by Barton[6]. He pointed out that the force between two standing mirrors is itself a fluctuating quantity, whose average value is the well ltnown Casimir result. He analysed the fluctuations of the Maxwell stress tensor, not only for tlie case of two mirrors, but also for a single mirror at rest in vacuum. For the latter, he derived the spatio–temporal correlation function of the stress tensor, and tlie corresponding noise spectrum[7]. Partially following his approach, Eberlein calculated time–averaged force fluctuations for the much more difficult problems of spherical[8] and spheroidal[9] perfectly–reflecting mirrors.

The connection between the force fluctuations for mirrors at rest of Refs. [6]-[9], on one hand, and the emission of radiation by moving mirrors of Refs.[2]-[4], on tlie other hand, was suggested by Jaekel and Reynaud[10] (see also Ref.[11] for a review on this subject). As in classical electron theory, the emission of radiation entails the existence of a radiative reaction force which works against the mirror's motion, tlius dissipating its energy. The key point is that such dissipative force on a moving mirror may be immediately related to the force fluctuations upon a standing mirror tlirough linear response theory[12]. For this to apply, however, the dissipative force must be of course computed to first order in the mirror's velocity (and derivatives), ruling out relativistic motions.

The argument goes as follows. The motional effect is described as a perturbation of the free field Hamiltonian taken for the mirror at rest. The work done by the mirror is given by

$$\delta H = -F\ \delta q, \tag{1}$$

where $\delta q(t)$ is the mirror's displacement and $\mathbf{F}$ the force exerted by the field on the mirror. We then take δH as the Hamiltonian operator describing the applied perturbation — note that $\mathbf{F}$ is an operator for the field, whereas $\delta q(t)$ is a (classical) previously defined function of time playing the role of a (small) perturbing parameter. Its Fourier transform $\delta q[\omega]$ is defined as

$$\delta q[\omega] = \int dt\, e^{i\omega t}\, \delta q(t). \tag{2}$$

The force on a moving mirror is then written as

$$\delta F[\omega] = \chi_{FF}[\omega]\, \delta q[\omega], \tag{3}$$

where the susceptibility $\chi_{FF}[w]$represents the response to the applied perturbation. Its imaginary part provides the dissipative force on a moving mirror. We have

$$Im\chi_{FF}[\omega] = \xi_{FF}[\omega], \tag{4}$$

where $\xi_{FF}[\omega]$ is the Fourier transform of the average force commutator

$$\xi_{FF}(t) = \frac{1}{2\hbar} < [F(t), F(0)] >, \tag{5}$$

taken over the *unperturbed system*, i.e., by considering a mirror at *rest*. Eq. *(4)* then connects the force on a moving mirror with the fluctuations of the force on a mirror at rest, which is represented by $\xi_{FF}[\omega]$. Such connection was extensively used in order to obtain the dissipative susceptibilities from the spectrum of fluctuations in a variety of problems, including moving planes[13], pistons[11,14] and spheres[14]. By evaluating independently the fluctuations for a standing mirror and the dissipative force for a moving mirror, Eq. (4) was shown to be correct in 1D[10] as well as in case of a flat moving mirror in 3D space[5].

In this paper, we consider an initially flat mirror, whose surface is deformed during a finite time interval. We assume the mirror to be a perfect reflector. Such approximation corresponds to the low frequency liinit of more consistent models which take into account a finite response time cliaracteristic of the mirror–field coupling (see Refs.[10], [15] and [16]). Moreover, we completely neglect the effect of the dissipative force on the surface time evolution (recoil effect), thus assuming that the mirror's surface is described by a previously defined function of time, which is imposed by some external agent. As discussed in Ref.[17], this assumption (whicli also uiiderlies tlie linear response theory outlined above) is also consistent at low frequencies.

By coinputing the scattered fields in the *long*–wavelength approximation, we derive the dissipative force on the movable part of the rnirror. The problein of a mirror subjected to a rigid motion, considered in Ref.[5], is then re–obtained as a particular case of our general result, which also allows us to evaluate the force on a moving piston. In order to malte a connection with fluctuations, we generalize Eqs. (1)–(5) to the case of non–rigid displacements. This step introduces Barton's stress tensor correlation function for a flat mirror[7]. We show that fluctuations and dissipation are again related as predicted by linear response theory.

For the particular case of rigid motion of the mirror, we analyse in more detail the process of photon creation, by writing a linear transformation between input and output field operators. This result allows us to compute the radiated energy, which is then compared with the dissipative force to show that the latter is indeed the corresponding radiation reaction force. An explicit derivation of the radiated energy in the 1D approximation is also found in' Refs.[18,19]. More recent 3D calculations were performed for dielectrics with index of refraction close to one[20] as well as for collapsing dielectric spheres[21].

The paper is organized in the following way. In Sec. II, we first compute the scattered fields and then the dissipative force. In Sec. III, we show that dissipation and fluctuations are related in the usual way. We also discuss some few particular examples of deforming surfaces. The input–output formalism is developed in Sec. IV. Finally, we summarize the main results of the paper in Sec. V.

II. Dissipative force in vacuum

We suppose that the mirror's right-liancl surface initially occupies the yz plane at tlie position $x = 0$. Each point of coordinates (y, z) at the surface will then move along tlie x-direction according to tlie function

$$x = \delta q(y, z, t), \tag{6}$$

whereas tlie flat left-hand surface stays unperturbed.

Tlie effects relatecl to tlie vector nature of tlie electromagnetic field were discussed in Ref.[5], in tlie case of rigid motion of a flat mirror. One remarkable difference between tlie scalar and electroinagnetic problems is that no mass correction appears for tlie latter, due to an exact cancellation between tlie contributions from tranverse electric and magnetic waves. In this paper, liowever, we will be exclusively concerned witli tlie dissipative force, and thus ignoring dispersive components (and in particular mass corrections). Accordingly, we take for tlie sake of simplicity a scalar field $\Phi(\mathbf{r}, t)$ obeying the boundary condition

$$\Phi(\delta q(y, z, \mathrm{t}), y, z, \mathrm{t}) = 0. \tag{7}$$

We sliall work with tlie Fourier representation defineel by

$$\Phi[x, \mathbf{k}_{\|}, \mathrm{w}] = \int dt \int d^2\mathbf{r}_{\|} e^{\mathrm{i}\omega t} e^{-\mathrm{i}\mathbf{k}_{\|}\cdot\mathbf{r}_{\|}}\ \Phi(x, \mathbf{r}_{\|}, \mathrm{t}), \tag{8}$$

where $\mathbf{r}_{\|} = y\hat{y} + z\hat{z}$. We decompose tlie field as follows:

$$\Phi[x, \mathbf{k}_{\|}, \omega] = \Phi_i[x, \mathbf{k}_{\|}, \omega] + \Phi_s[x, \mathbf{k}_{\|}, \omega]. \tag{9}$$

Φ_i corresponds to waves inciclent on tlie mirror coming from tlie lialf-space corresponding to the positive x-axis (accordingly, we suppose $x \geq 0$ from now on). On the other hand, Φ_s corresponds to scattered waves, including both tlie normally–reflectecl wave, which is of zero orcler in δq, and its first–order correction $\delta\Phi^R$:

$$\Phi_s[x, \mathbf{k}_{\|}, \omega] = -\Phi_i[-x, \mathbf{k}_{\|}, \omega] + \delta\Phi^R[x, \mathbf{k}_{\|}, \omega]. \tag{10}$$

$\delta\Phi^R[x, \mathbf{k}_{\|}, \omega]$ may be calculated through tlie same straightforward method employed in the case of flat moving mirrors[5]. A formal derivation in terms of Green's functions, suitable for tlie input–output formalism cleveloped in Sec. IV, is presented in Appenclix A, wliere tlie meaning of tlie superscript R is explained. We find

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$$\delta\Phi^R[x, \mathbf{k}_{\|}, \mathrm{w}] = -2e^{\mathrm{i}\kappa_x x} \int \frac{d\omega'}{2\pi} \int_{(k'_{\|} \leq |\omega'|)} \frac{d^2k'_{\|}}{(2\pi)^2} \delta q[\mathbf{k}_{\|} - \mathbf{k}'_{\|}, \omega - \omega'] \partial_x \Phi_i[0, \mathbf{k}'_{\|}, \omega'], \tag{11}$$

wliere $\delta q[\mathbf{k}_{\|}, \mathrm{w}]$ is defined in terms of $\delta q(y, z, t)$ as in Eq. (8), and

$$\kappa_x = [(\mathrm{w} + \mathrm{i}\epsilon)^2 - k_{\|}^2]^{1/2},\ \epsilon \to 0^+, \tag{12}$$

is defined as a function in tlie complex plane of w witli a branch cut on tlie real axis between $-k_{\|}$ and $k_{\|}$ (we take c = 1).

Two important approximations are used in tlie derivation of Eq. (11). First, tlie fielcls are supposed to be slowly varying over a distaiice of tlie order of δq (long–wavelengtli approximation). Seconcl, the effect connected to deforming tlie surface is supposed to be a small perturbation ($|\delta\Phi^R| \ll |\Phi|$). For tlie Dirichlet boundary condition considerecl here, such perturbative approach is always valid in the long–wavelength limit. However, this would not be necessarily the case had we taken Neumann boundary conditions (as discussed in Ref.[22], a simple counter–example is provided by tlie Wood's anomalies for diffraction gratings).

Tlie normal–mode expansion of Φ_i includes only wavevectors k witli negative x-components:

$$\Phi_i(\mathbf{r}, t) = \int_{(k_x \leq 0)} \frac{d^3k}{(2\pi)^3} \sqrt{\frac{\hbar}{2k}} a_{\mathbf{k}} e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} e^{-\mathrm{i}kt} + \mathrm{HC}, \tag{13}$$

where HC represents tlie Hermitean conjugate of tlie precelcing expression.

The operators $a_{\mathbf{k}}$ obey the commutation relations

$$[a_{\mathbf{k}}, a_{\mathbf{k}'}] = 0, \tag{14}$$

$$[a_{\mathbf{k}}, a^{\dagger}_{\mathbf{k}'}] = (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}'). \tag{15}$$

Using tlie notation introduced by Eq. (8), we have

$$\Phi_i[x,\mathbf{k}_\parallel,\omega] = \theta(\kappa_x^2)\sqrt{\frac{\hbar|\omega|}{2\kappa_x^2}}e^{-i\kappa_x x}\left[\theta(\omega)a_{-\kappa_x\hat{x}+\mathbf{k}_\parallel} + \theta(-\omega)a^\dagger_{\kappa_x\hat{x}-\mathbf{k}_\parallel}\right], \tag{16}$$

with κ_x given by Eq. (12), and where 0 denotes the step function.

The force on the mirror is obtained from the stress tensor, which is given by

$$S_{ij} = -\partial_i\Phi\partial_j\Phi + \frac{1}{2}\delta_{ij}\left[(\nabla\Phi)^2 - (\partial_t\Phi)^2\right]. \tag{17}$$

In order to compute the the component of the force along tlie x-direction, we need the S_{xx} component near the mirror's surface calculated up to first order in Sq. Since $(\partial_y\Phi(x=0^+))^2, (\partial_z\Phi(x=0^+))^2$ and $(\partial_t\Phi(x=0^+))^2$ are all of second order, Eq. (17) yields

$$S_{xx}[0^+,\mathbf{k}_\parallel,\omega] = -\frac{1}{2}\int\frac{d\omega'}{2\pi}\int\frac{d^2k'_\parallel}{(2\pi)^2}\partial_x\Phi[0^+,\mathbf{k}_\parallel-\mathbf{k}'_\parallel,\omega-\omega']\partial_x\Phi[0^+,\mathbf{k}'_\parallel,\omega']. \tag{18}$$

The zero-th order term of $S_{xx}(x=0^+)$ corresponds to the pressure on a flat mirror at rest at x = Q Its average over the vacuum state is exactly cancelled by the corresponding value talren at the opposite side of the mirror. Therefore, it has no physical effect in connection with the *average* force on tlie mirror. On the other hand, its fluctiiating properties are very important, and will be analysed in the next section.

Accordingly, the average force is entirely due to the first-order correction $< 0_{\rm in}|\delta S_{xx}[0^+,\mathbf{k}_\parallel,\omega]|0_{\rm in} >$, the average being taken over the vacuum state with respect to the operators $a_\mathbf{k}$ and $a^\dagger_\mathbf{k}$. Such state represents the incoming zero–point fluctuations corresponding to the asymptotic limit $t \to -\infty$, when the mirror was still unperturbed. This point is clarified in Sec. IV, where we explicitly build up input and output field operators (the latter corresponding to the limit $t \to \infty$) to calculate the radiated energy.

We calculate the force $\delta F_\mathcal{R}$ upon a given region R of the mirror (probing region), which does not necessarily coincide with the deformed area. Up to first order in δq, we have, in the frequency domain,

$$\delta F_\mathcal{R}[\omega] = \int\frac{d^2k_\parallel}{(2\pi)^2}P_\mathcal{R}[-\mathbf{k}_\parallel] < 0_{\rm in}|\delta S_{xx}[x=0^+,\mathbf{k}_\parallel,\omega]|0_{\rm in} >, \tag{19}$$

where $P_\mathcal{R}[\mathbf{k}_\parallel]$ is the Fourier transform of the step function defining the region R.

We replace the expression for tbe total field Φ given by Eqs. (9)–(1.1) into Eq. (18) and collect the linear terms in Sq. The average over the vacuum state is then calculated by taking the normal mode expansion of Φ_i as given by Eq. (13). We use the shorthand notation $< \ldots >$ to represent the average over $|0_{\rm in} >$. The resulting expression is written as

$$< \delta S_{xx}[x=0^+,\mathbf{k}_\parallel,\omega] >= \chi_{SS}[\mathbf{k}_\parallel,\omega]\delta q[\mathbf{k}_\parallel,\omega], \tag{20}$$

with the susceptibility $\chi_{SS}[\mathbf{k}_\parallel, w]$ given by

$$\chi_{SS}[\mathbf{k}_\parallel,\omega] = i\hbar\int\frac{d\omega_i}{2\pi}\int\frac{d^2k_{i\parallel}}{(2\pi)^2}\theta(k_{ix}^2)|k_{ix}|k_{sx}. \tag{21}$$

In Eq. (21), we have defined

$$k_{ix} = -(\omega_i^2 - k_{i\parallel}^2)^{1/2}$$

and

$$k_{sx} = [(\omega_i + \omega + i\epsilon)^2 - (\mathbf{k}_{i\parallel}+\mathbf{k}_\parallel)^2]^{1/2}.$$

We interpret k_{ix} and h„ as the x-components of the incident and scattered wavevectors,

$$\mathbf{k}_i = k_{ix}\hat{\mathbf{x}} + \mathbf{k}_{i\|},$$

with $|\mathbf{k}_i| = |\omega_i|$, and

$$\mathbf{k}_s = k_{sx}\hat{\mathbf{x}} + \mathbf{k}_{s\|},$$

with $|\mathbf{k}_s| = |\omega_i+\omega|$ and $\mathbf{k}_{s\|} = \mathbf{k}_{i\|}+\mathbf{k}_{\|}$. Eqs. (20) shows that $\chi_{SS}[\mathbf{k}_{\|}, w]$ represents the effect of a given Fourier component (w, $\mathbf{k}_{\|}$) of the time-dependent corrugation. According to Eq. (21), such effect is the outcome of a superposition of all the elementary processes in which a (travelling-wave) wavevector $\mathbf{k}_i$ is scattered into k,.

The dissipative force is related througli Eqs. (19) and (20) to the imaginary part of $\chi_{SS}[\mathbf{k}_{\|}, w]$. Accordingly, we define

$$\chi_{SS}[\mathbf{k}_{\|},\omega] = \chi'_{SS}[\mathbf{k}_{\|},\omega] + \chi''_{SS}[\mathbf{k}_{\|},\omega],$$

where $\chi'_{SS}[\mathbf{k}_{\|}, w]$ and $\chi''_{SS}[\mathbf{k}_{\|}, w]$ are respectively real and pure imaginary. Only tlie values of ω_i and $\mathbf{k}_{i\|}$ that satisfy the innequality

$$|\omega_i + \omega| \geq |\mathbf{k}_{i\|} + \mathbf{k}_{\|}|$$

contribute to $\chi''_{SS}[\mathbf{k}_{\|},\omega]$ in Eq. (21), since they entail real values for the variable L„. It means that the evanescent waves generated in the scattering do not contribute to dissipation, a fact fully consistent with the association (to be proved in Sec. IV) between the dissipative force and the emission of radiation, since tliis latter is related exclusively to the generation of travelling waves.

The region of integration in Eq. (21) may be further simplified because, if w is positive, the contribution from the high-frequency negative part of the spectrum, $\omega_i < -\omega$, is cancelled by the positive part, $\omega_i > 0$ (a similar property occuring in the case of negative w). Accordingly, the dissipative force $\delta\mathcal{F}_{\mathcal{R}}$ may be written as

$$\delta\mathcal{F}_{\mathcal{R}}[\omega] = \int \frac{d^2k_{\|}}{(2\pi)^2} P_{\mathcal{R}}[-\mathbf{k}_{\|}]\chi''_{SS}[\mathbf{k}_{\|},\omega]\ \delta q[\mathbf{k}_{\|},\omega], \quad (22)$$

with $\chi''_{SS}[\mathbf{k}_{\|},\omega]$ given by

$$\chi''_{SS}[\mathbf{k}_{\|},\omega] = i\hbar\int_{-\omega}^{0}\frac{d\omega_i}{2\pi}\int\frac{d^2k_{i\|}}{(2\pi)^2}\theta(k_{ix}^2)\theta(k_{sx}^2)|k_{ix}|k_{sx}. \quad (23)$$

For w $>$ 0, the incident frequencies ω_i appearing in Eq. (23) are such that $\omega_i < 0$ and $\omega_i + \omega > 0$ (and vice-versa if w $<$ O). Since positive and negative frequencies correspond, according to Eq. (16), to annihilation and creation operators, Eq. (23) suggests that the dissipative force is a consequence of photon generation. Such connection will be established in Sec. IV, for tlie particular case of rigid motion. The condition $|\omega_i| \leq |\omega|$ justifies the use of tlie long-wavelength approximation in the derivation of Eq. (11), which becomes thereby closely related to the non-relativistic limit $|\partial_t\delta q(y,z,t)| \ll 1$ (see Ref. [5] for a detailed discussion).

The integral in Eq. (23) may be solved to give a very simple expression for $\chi''_{SS}[\mathbf{k}_{\|}, w]$. Before doing that, however, we will compare this preliminary result with the correlation spectrum of the stress tensor for a flat mirror at rest.

III. Stress tensor fluctuations and their connection with the dissipative force

We begin this session by showing that the result found above, Eq. (23), is in agreement with linear response theory. We work in the frequency domain, and define the spectrum of fluctuations of S_{xx} at x = 0 as

$$C_{SS}[\mathbf{k}_{\|},\omega] = \int dt \int d^2r_{\|}e^{i\omega t}e^{-i\mathbf{k}_{\|}\cdot\mathbf{r}_{\|}} \ < 0|S_{xx}(0\hat{\mathbf{x}} + \mathbf{r}_{\|},t)S_{xx}(\mathbf{0},0)|0>, \quad (24)$$

the average being taken over the vacuum state and for a flat mirror at rest.

We shall use the following representation[14]:

$$C_{SS}[\mathbf{k}_{\|},\omega] = 4\pi^3\int\frac{d^3k_1}{(2\pi)^3}\int\frac{d^3k_2}{(2\pi)^3}\delta(\mathbf{k}_{\|} - \mathbf{k}_{1\|} - \mathbf{k}_{2\|})\delta(\omega - k_1 - k_2) \times |< \mathbf{k}_1, \mathbf{k}_2|S_{xx}(\mathbf{0},0)|0>|^2, \quad (25)$$

where $|\mathbf{k}_1, \mathbf{k}_2>$ is the two-photon state corresponding to the wavevectors $\mathbf{k}_1$ and $\mathbf{k}_2$.

The matrix elements in Eq. (25) may be readily obtained from the normal mode expansion of the field, which follows from Eqs. (9), (10) (taking the motional correction $\delta\Phi = 0$) and (13):

$$| < \mathbf{k}_1, \mathbf{k}_2 | S_{xx}(\mathbf{0},0)|0> |^2 = \frac{(2\hbar k_{1x} k_{2x})^2}{k_1 k_2} \tag{26}$$

By replacing the above expression into Eq. (25) we get

$$C_{SS}[\mathbf{k}_\|, \omega] = \frac{\hbar^2\theta(\omega)}{4\pi^3} \int_0^\omega dk_1 \int_{(|\mathbf{k}_{1\|}| \le k_1)} d^2k_{1\|} \theta((\omega - k_1)^2 - |\mathbf{k}_\| - \mathbf{k}_{1\|}|^2) \times \sqrt{k_1^2 - \mathbf{k}_{1\|}^2}\sqrt{(\omega - k_1)^2 - (\mathbf{k}_\| - \mathbf{k}_{1\|})^2}. \tag{27}$$

We compare Eqs. (23) and (27) by taking $k_1 = -\omega_i$ and $\mathbf{k}_{1\|} = -\mathbf{k}_{i\|}$. It yields

$$\chi''_{SS}[\mathbf{k}_\|, \omega] = \frac{1}{2\hbar}(C_{SS}[\mathbf{k}_\|, \omega] - C_{SS}[-\mathbf{k}_\|, -\omega]), \tag{28}$$

which is in full agreement with linear response theory, if

$$\delta H = -\int d^2\mathbf{r}_\| S_{xx}(\mathbf{r}_\|, t)\, \delta q(\mathbf{r}_\|, t) \tag{29}$$

is taken as the perturbing Hamiltonian (Eq. (1) corresponding to the particular case of rigid motion). However, we cannot make use of dispersion relations in order to relate $\chi'_{SS}[\mathbf{k}_\|, w]$ (which corresponds to the dispersive part of the force) to $\chi''_{SS}[\mathbf{k}_\|, w]$, since our model is not valid at high frequencies. On the other hand, dispersion relations with subtractions are satisfied by the susceptibility function in the case of partially–transmitting frequency–dependent mirrors (which are taken to be transparent at high frequencies)[15].

A simple closed–form expression for $C_{SS}[\mathbf{k}_\|, \omega]$ was obtained in the case of the electromagnetic field[7]. A similar result may of course also be derived for the scalar field considered here by employing a similar approach. It is easier to work with the spectrum taken in the real space, which is defined **as** (see Eq. (24))

$$\tilde{C}_{SS}(\mathbf{r}_\|, \omega) = \int \frac{d^2k_\|}{(2\pi)^2} e^{-i\mathbf{k}_\| \cdot \mathbf{r}_\|} C_{SS}[\mathbf{k}_\|, \omega]. \tag{30}$$

Replacing Eq. (27) into Eq. (30) leads to integrals involving Bessel functions. We refer to Ref. [7] for details of the derivation. The final result is written as

$$\tilde{C}_{SS}(\mathbf{r}_\|, \omega) = \frac{\hbar^2}{3.2^4\pi^3} \frac{\omega^4}{r_\|^3} j_3(\omega r_\|)\theta(\omega), \tag{31}$$

where j_3 is the spherical Bessel function of order three. We then come back to the reciprocal space and use Eq. (28) to find

$$\chi''_{SS}[\mathbf{k}_\|, \omega] = \frac{i\hbar}{2^4.3^2.5\pi^2}\theta(\omega^2 - k_\|^2)(\omega^2 - k_\|^2)^{5/2}. \tag{32}$$

According to Eq. (32), $\chi''_{SS}[\mathbf{k}_\|, w]$ vanishes for $|\omega| < k_\|$. As shown in Ref.[14], this property may also be inferred directly from our starting point, Eq. (25). Here we show that it is related to Lorentz invariance of the vacuum field. For that end, we take the following surface:

$$\delta q[\mathbf{k}_\|, \omega] = 4\pi^3 \delta q_0 [\delta(k_y - K)\delta(k_z)\delta(\omega - \Omega) + \delta(k_y + K)\delta(k_z)\delta(\omega + \Omega)], \tag{33}$$

which corresponds to a diffraction grating of period $2\pi/K$ moving along the y–direction with (uniform) velocity Ω/K. Because of the property mentioned above, the force vanishes if $\Omega/K \le 1$, as expected from Lorentz invariance. Note however that the force is not necessarily zero in the case of accelerated lateral motion of the grating, because the region of space occupied by the reflecting material obviously changes in time, contrary to wliat happens in tlie examples considered in Ref.[20].

More generally, Fourier components of tlie surface corrugation with $k_{\|} > \omega$ do not contribute to the dissipative force. If $\bar{\omega}$ is a typical frequency of the perturbation, then the details of the corrugation in a length scale smaller than $1/\bar{\omega}$ are irrelevant. In the static limit (i.e., a corrugated surface at rest) this typical Iength scale diverges, and hence the force (obviously) vanishes.

Accordingly, the limit correponding to a small area of perturbation is defined by the condition $\bar{\omega}L \ll 1$, where L is a typical dimension of the perturbed region. Then it follows that $k_{\|}L \ll 1$ for all those values that contribute to the force given by Eq. (19), even if the corrugation has larger values of $k_{\|}$ (which would correspond to small spatial periods). As a consequence, we may write (using the notation introduced in Eq. (30))

$$\delta q[\mathbf{k}_{\|},\omega] \approx \int d^2r_{\|}\widetilde{\delta q}(\mathbf{r}_{\|},\omega) = A_{\mathcal{S}} < \widetilde{\delta q}(\omega) >_{\mathcal{S}}, \quad (34)$$

where $A_{\mathcal{S}}$ is the area of the perturbed section S of tlie mirror and $< \widetilde{\delta q}(\omega) >_{\mathcal{S}}$ represents the spatial average of the displacement (or rather its Fourier transform) over the surface S. Furthermore, if the probing surface R is chosen so as to coincide with S, we have

$$P_{\mathcal{R}} \approx A_{\mathcal{S}}. \quad (35)$$

Replacing Eqs. (32), (34) and (35) into Eq. (22) leads to

$$\delta\mathcal{F}_{\mathcal{R}} = \mathrm{i}\frac{\hbar A_{\mathcal{S}}^2\omega^7}{2^5.3^2.5.7\pi^3} < \widetilde{\delta q}(\omega) >_{\mathcal{S}} . \quad (36)$$

We may also obtain Eq. (36) by using the configuration space and taking $\mathbf{r}_{\|} = 0$ in Eq. (31).

Taking tlie opposite limit, we consider the force on tlie entire surface of the mirror (i.e., R is taken to be tlie whole yz plane). In this particular case, we recover the result found by Ford and Vilenkin[4]. We have

$$P_{\mathcal{R}\to\infty}[\mathbf{k}_{\|}] = (2\pi)^2\delta(\mathbf{k}_{\|}), \quad (37)$$

and then from Eq. (22) we obtain

$$\delta\mathcal{F}_{\mathcal{R}\to\infty} = \chi''_{\mathcal{SS}}[\mathbf{k}_{\|} = \mathbf{0},\omega]A_{\mathcal{R}} < \widetilde{\delta q}(\omega) >_{\mathcal{R}}, \quad (38)$$

which may be evaluated from Eq. (32):

$$\delta\mathcal{F}_{\mathcal{R}\to\infty} = \mathrm{i}\frac{\hbar\omega^5}{2^4.3^2.5\pi^2}A_{\mathcal{R}} < \widetilde{\delta q}(\omega) >_{\mathcal{R}} . \quad (39)$$

The force given by Eq. (39) is proportional to the probing surface $A_{\mathcal{R}}$ as it should. Note that Eq. (39) applies even if the perturbed region of the mirror is small. However, $< \widetilde{\delta q}(\omega) >_{\mathcal{R}}$ becomes very small in this case, since the average is taken over the very large surface R.

The particular case of a moving piston may also be considered from Eqs. (22) and (32). If we take the probing surface R to be the surface of the piston, whose position is prescribed by the function $\delta q(t)$, we have

$$\delta q[\mathbf{k}_{\|},\omega] = P_{\mathcal{R}}[\mathbf{k}_{\|}]\,\widetilde{\delta q}(\omega), \quad (40)$$

and from Eqs. (20) and (32)

$$\delta\mathcal{F}_{\mathcal{R}}[\omega] = \frac{\mathrm{i}\hbar}{2^4.3^2.5\pi^2}\int\frac{d^2k_{\|}}{(2\pi)^2}|P_{\mathcal{R}}[\mathbf{k}_{\|}]|^2(\omega^2 - k_{\|}^2)^{5/2}\,\widetilde{\delta q}(\omega). \quad (41)$$

The opposite limits of infinite and very small pistons may be obtained frorn Eq. (41) by taking the suitable approximations for $P_{\mathcal{R}}[\mathbf{k}_{\|}]$. The resulting expressions agree with Eqs. (36) and (39), which are however more general.

Eq. (41) may be compared with the result for a piston moving in the vacuum of the electromagnetic field[14], which was derived from linear response theory based on Eq. (29). The two results are very similar, except for a numerical factor, which is possibly an effect of our approximative scalar model. In fact, the result for a infinite mirror describing a rigid motion, given by

Eq. **(39)** with $< \widetilde{\delta q}(\omega) >_{\mathcal{R}} = \delta q(\omega)$, is exactly equal to the contribution of the TE electromagnetic waves to the dissipative force found in Ref.[5]. Adding the contributions of TE and TM waves leads to an expression with a different numerical factor, but with the same frequency dependence. In the next section, we consider in detail the radiation emitted in this particularly simple example.

IV. Radiation from a moving mirror

Henceforth we consider an infinite flat moving mirror, whose position is given by

$$x = \delta q(t).$$

Furthermore, we assume that the mirror was initially at rest at $x = 0$, then set in motion during a *finite* time interval, and finally placed again at $x = 0$. In order to analyse the process of photon generation, we derive a linear transformation between output and input field operators, which correspond to the limits $t \to \infty$ and $t \to -\infty$.

Accordingly, we decompose the total field Φ not as in Eq. (9), but rather take

$$\Phi(x, \mathbf{r}_{\|}, t) = \Phi_{\rm in}(x, \mathbf{r}_{\|}, t) + \delta\Phi^R(x, \mathbf{r}_{\|}, t). \tag{42}$$

Eqs. (9), (10) and (42) then provide the connection with the approach of Sec. II:

$$\text{i} \qquad \mathbf{r}_{\|}, t) = \Phi_i(x, \mathbf{r}_{\|}, t) - \Phi_i(-x, \mathbf{r}_{\|}\ t). \tag{43}$$

According to Eq. (43), $\Phi_{\rm in}$ corresponds to the total field in the case that the mirror is at rest at $x = 0$:

$$\Phi_{\rm in}(x = 0, \mathbf{r}_{\|}, t) = 0 \tag{44}$$

In Appendix A, we calculate $\delta\Phi^R$ by using *retarded* Green functions. This means that, in a formal sense,

$$\Phi_{\rm in}(x, \mathbf{r}_{\|}, t) = \lim_{t i - 0 3} \Phi(x, \mathbf{r}_{\|}, t), \tag{45}$$

clarifying the meaning of $\Phi_{\rm in}$ as the input field (which must include both incident and normally–reflected waves as shown in Eq. (43), since the mirror already reflects the incident waves at $t \to -\infty$).

We find, for positive values of x,

$$\delta\Phi^R[x, \mathbf{k}_{\|}, \omega] = -e^{i\kappa_x x} \int_{(|\omega'| \geq k_{\|})} \frac{d\omega'}{2\pi} \delta q[\omega - \omega'] \partial_x \Phi_{\rm in}[0, \mathbf{k}_{\|}, \omega'], \tag{46}$$

with κ_x given by Eq. (12). This result should be compared through Eq. **(43)** with our previous result, Eq. (11), which is more general than Eq. (46) since it accounts for an arbitrary (small) perturbation of the mirror's surface (for example, as shown by Eq. (46), the parallel wavevector component $\mathbf{k}_{\|}$ is conserved in the case of an infinite mirror describing a rigid motion, but not in the case of a corrugated surface).

Alternatively, we may make use of *advanced* Green functions to solve the boundary condition of Eq. (7). In this case, the total field is written as

$$\Phi(x, \mathbf{r}_{\|}, t) = \Phi_{\rm out}(x, \mathbf{r}_{\|}, t) + \delta\Phi^A(x, \mathbf{r}_{\|}, t), \tag{47}$$

where the output field $\Phi_{\rm out}$ approximates the total field when $t \to \infty$. We have

$$\Phi_{\rm out}(x = 0, \mathbf{r}_{\|}, t) = 0 \tag{48}$$

as in Eq. (44), and

$$\delta\Phi^A[x, \mathbf{k}_{\|}, \omega] = -e^{-i\kappa_x^* x} \int_{(|\omega'| \geq k_{\|})} \frac{d\omega'}{2\pi} \delta q[\omega - \omega'] \partial_x \Phi_{\rm out}[0, \mathbf{k}_{\|}, \omega'] \tag{49}$$

The first-order input-output transformation is found from Eqs. (42)-(49):

$$\Phi_{\rm out}[x,\mathbf{k}_{\|},\omega] = \Phi_{\rm in}[x,\mathbf{k}_{\|},\omega] -2{\rm i}\sin(\kappa_x x)\int\frac{d\omega'}{2\pi}\theta(\omega'^2-k_{\|}^2)\delta q[\omega-\omega'] \times\partial_x\Phi_{\rm in}[0,\mathbf{k}_{\|},\omega']. \quad (50)$$

We want to derive from Eq. (50) the transformation between the annihilation and creation operators corresponding to the input and output fields. The normal mode expansion of the input field $\Phi_{\rm in}$ is obtained from Eqs. (16) and (43):

$$\Phi_{\rm in}[x,\mathbf{k}_{\|},\omega] = -2{\rm i}\theta(\kappa_x^2)\sqrt{\frac{\hbar|\omega|}{2\kappa_x^2}}\sin(\kappa_x x)\left[\theta(\omega)a_{-\kappa_x\hat{x}+\mathbf{k}_{\|}}+\theta(-\omega)a^{\dagger}_{\kappa_x\hat{x}-\mathbf{k}_{\|}}\right]. \quad (51)$$

The output field $\Phi_{\rm out}$ is expanded in terms of the same eigenfunctions used in Eq. (51) for the input field $\Phi_{\rm in}$, since, according to Eqs. (44) and (48), they satisfy the same boundary condition. Therefore, we have

$$\Phi_{\rm out}[x,\mathbf{k}_{\|},\omega] = -2{\rm i}\theta(\kappa_x^2)\sqrt{\frac{\hbar|\omega|}{2\kappa_x^2}}\sin(\kappa_x x)\left[\theta(\omega)b_{-\kappa_x\hat{x}+\mathbf{k}_{\|}}+\theta(-\omega)b^{\dagger}_{\kappa_x\hat{x}-\mathbf{k}_{\|}}\right]. \quad (52)$$

where the annihilation and creation output operators, $b_{\mathbf{k}}$ and $b^{\dagger}_{\mathbf{k}}$, satisfy the usual free field commutation relations as in Eqs. (14) and (15).

Eqs. (50)-(52) provide the desired input-output transformation:

$$b_{\mathbf{k}} = a_{\mathbf{k}} + \frac{2{\rm i}k_x}{\sqrt{|\mathbf{k}|}}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\,\theta(k_x'^2)\sqrt{|\omega'|}\,\delta q[|\mathbf{k}|-\omega']\left(\theta(\omega')a_{\mathbf{k}'}-\theta(-\omega')a^{\dagger}_{-\mathbf{k}'}\right), \quad (53)$$

where the wavevectors $\mathbf{k}'$ are such that $\mathbf{k}'_{\|}=\mathbf{k}_{\|}$ (expressing the symmetry of translation along the plane of the mirror), $|\mathbf{k}'|=|\omega'|$ and $k_x'<0$. Eq. (53) is the central result in this section (a similar relation was found in Ref.[23] for a 1D resonator with a moving wall). It shows that a given Fourier component of the mirror's motion at $\Omega = \omega-\omega'$ couples input creation operators at negative frequencies ω' to an output annihilation operator at frequency ω (respectively $a^{\dagger}_{-\mathbf{k}'}$ and $b_{\mathbf{k}}$ in Eq. (53)). Therefore, $|0_{\rm in}>$ is a low-frequency excited state with respect to the output operators. In Appendix B, we use this property to compute the total radiated energy δE from Eq. (53). We find

$$\delta E = \frac{\hbar}{720\pi^2}A\int\frac{d\omega}{2\pi}\omega^6\,|\delta q[\omega]|^2, \quad (54)$$

where A is the area of the mirror. Using Eq. (39), we may also write δE as

$$\delta E = -\int\frac{d\omega}{2\pi}\delta\mathcal{F}_{\mathcal{R}\to\infty}[\omega]^*(-{\rm i}\omega)\delta q[\omega]. \quad (55)$$

Eq. (55) shows that the total radiated energy is equal to the total work done by the mirror, thus corroborating the interpretation of $\delta\mathcal{F}_{\mathcal{R}\to\infty}$ as a radiation reaction force.

V. Conclusion

The results found in this paper allow for the computation of the dissipative force on the mirror due to a small but otherwise arbitrary perturbation of its (initially flat) surface. Our model considers the low-frequency scalar scattering by a deformed perfectly-reflecting mirror. We have considered in detail some limiting situations, corresponding to perturbations of large and tiny regions of the mirror. For those cases, as well as for the example of a moving piston, we have found a qualitative agreement with the results obtained for the electromagnetic field by making use of the fluctuation-dissipation theorem.

In the second part of the paper, we have derived the linear transformation between input and output field operators, which provides an explicit representation for the motional emission of radiation. The total radiated energy is then compared with the result for the dissipative force.

Appendix A: Retarded and advanced solutions

In this appendix, we construct retarded and advanced solutions for the boundary condition given by Eq. (7) by making use of suitable Green functions.

In the long–wavelength approximation, Eqs. (7) and (42) yield

$$\delta\Phi^R(0,\mathbf{r}_{\|},t) = -\delta q(t)\partial_x\Phi_{\rm in}(0,\mathbf{r}_{\|},t). \qquad (56)$$

As in Sec. II, we work in the reciprocal space, though keeping the variable x in order to describe the mirror's trajectory. The retarded Green function in free space is then written as

$$G^R_\omega(x-x_0,\mathbf{k}_{\|}) = \frac{\mathrm{i}}{2}\cdot\frac{e^{\mathrm{i}[(\omega+\mathrm{i}\epsilon)^2-k_{\|}^2]^{1/2}\cdot|x-x_0|}}{[(\omega+\mathrm{i}\epsilon)^2-k_{\|}^2]^{1/2}}, \qquad (57)$$

with $\epsilon \rightarrow 0^+$ (see Eq. (12)). The retarded Green function suitable for a Dirichlet boundary condition at x = 0, to be clenoted as $\mathcal{G}^R_\omega(x,x_0,\mathbf{k}_{\|})$, is obtained from $G^R_\omega(x-x_0,\mathbf{k}_{\|})$ by using the method of images:

$$\mathcal{G}^R_\omega(x,x_0,\mathbf{k}_{\|}) = G^R_\omega(x-x_0,\mathbf{k}_{\|}) - G^R_\omega(x+x_0,\mathbf{k}_{\|}). \qquad (58)$$

For x > 0, the retarded solution is then written in terms of $\mathcal{G}^R_\omega(x,x_0,\mathbf{k}_{\|})$ as

$$\delta\Phi^R[x,\mathbf{k}_{\|},\omega] = \delta\Phi^R[x_0=0,\mathbf{k}_{\|},\omega]\frac{\partial}{\partial x_0}\mathcal{G}^R_\omega(x,x_0=0,\mathbf{k}_{\|}). \qquad (59)$$

From Eqs. (57) and (58) we find

$$\partial_{x_0}\mathcal{G}^R_\omega(x,x_0,\mathbf{k}_{\|})\,\big|_{x_0=0} = e^{\mathrm{i}[(\omega+\mathrm{i}\epsilon)^2-k_{\|}^2]^{1/2}x}. \qquad (60)$$

Replacing Eqs. (56) and (60) into Eq. (59) leads to Eq. (46) in Sec. IV.

A similar procedure is employed to derive the advanced solution $\delta\Phi^A$ as given by Eq. (49). Instead of Eq. (57), however, our starting point is the advanced free-space Green function

$$G^A_\omega(x-x_0,\mathbf{k}_{\|}) = -\frac{\mathrm{i}}{2}\cdot\frac{e^{-\mathrm{i}[(\omega-\mathrm{i}\epsilon)^2-k_{\|}^2]^{\frac{1}{2}}\cdot|x-x_0|}}{[(\omega-\mathrm{i}\epsilon)^2-k_{\|}^2]^{\frac{1}{2}}}. \qquad (61)$$

Appendix B: Energy radiated by a moving mirror

In this Appendix, we compute the total radiated energy from the input–output tranformation given by Eq. (53). We take periodic boundary conditions on the yz plane over a square of surface A. The field operators are then renormalized according to

$$b_{k_x,i} = \frac{1}{\sqrt{A}}b_{k_x\hat{x}+\mathbf{k}_{\|_i}}, \qquad (62)$$

where i is the index of the cell in the 2D reciprocal space to which $\mathbf{k}_{\|_i}$ belongs. The commutation relations are as follows:

$$[b_{k_x,i}, b^\dagger_{k'_x,j}] = 2\pi\delta(k_x-k'_x)\delta_{ij}. \qquad (63)$$

The average photon number $\langle 0_{\rm in}|b^\dagger_{k_x,i}b_{k_x,i}|0_{\rm in}\rangle$ is obtained from Eqs. (53) and (63):

$$\langle 0_{\rm in}|b^\dagger_{k_x,i}b_{k_x,i}|0_{\rm in}\rangle = \frac{4k_x^2}{|\mathbf{k}_i|}\int_{-\infty}^{-|\mathbf{k}_{\|,i}|}\frac{d\omega'}{2\pi}\sqrt{\omega'^2-k_{\|_i}^2}\,|\delta q[|\mathbf{k}_i|-\omega']|^2, \qquad (64)$$

where $|\mathbf{k}_i| = \sqrt{k_x^2+k_{\|_i}^2}$.

In the limit t $\rightarrow\infty$, the field dynamics is governed by the Hamiltonian $H_{\rm out}$, which is written in terms of the output operators $b_{k_x,i}$ and $b^\dagger_{k_x,i}$ as follows:

$$H_{\rm out} = \int\frac{dk_x}{2\pi}\sum_i(b^\dagger_{k_x,i}b_{k_x,i}+b_{k_x,i}b^\dagger_{k_x,i})\frac{\hbar\omega_{k_x,i}}{2} \qquad (65)$$

The average value of $H_{\rm out}$ taken over the vacuum state corresponding to the input operators provides the total radiated energy. We have

$$\langle 0_{\rm in}|H_{\rm out}|0_{\rm in}\rangle = E_0+\delta E, \qquad (66)$$

where E_0 and SE are of zero and second order in δq. SE may be consistently calculated from the first–order input–output transformation because second order corrections to Eq. (53) would not contribute to

$\langle 0_{in}|H_{out}|0_{in}\rangle$ up to second order. E_0 represents the (divergent) zero–point energy for a mirror at rest, whereas SE corresponds to the total radiated energy.

We calculate SE by replacing Eqs. (64) and (65) into Eq. (66). Furthermore, we come back to the continuum by replacing

$$\sum_i \rightarrow \frac{A}{(2\pi)^2}\int d^2k_\parallel .$$

We arrive at tlie following expression:

$$\delta E = 4\hbar A \int_0^\infty \frac{dk_x}{2\pi}\int \frac{d^2k_\parallel}{(2\pi)^2}|k_x|^2 \int_{-\infty}^{-k_\parallel}\frac{d\omega'}{2\pi}\sqrt{\omega'^2-k_\parallel^2}\,|\delta q[k-\omega']|^2, \qquad (67)$$

where $k = \sqrt{k_x^2+k_\parallel^2}$. Performing the integrals in Eq. (67) leads to Eq. (54).

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