Acceleration of Quantum Fields

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Received June 27, 1995

We analyze tlie transformation of quantum fields under conforma1 coordinate transformations from inertial to accelerated frames, i11 tlie simple case of scalar massless fields jn a tmo-dimensional spacetime, through tlie transformation of particle number and its spectral density. Particle number is found to be invariant under conforma1 coordinate transformations to uniformly acceleratecl frames, which extends tlie property already known for vacuum. Transformation of spectral density of particle number exhibits a redistribution of particles in the frequency spectrum. This redistribution is determinecl by clerivatives of phase operators with respect to frequency, that is by time ancl position operators clefined in such a manner that the redistribution of particles appears as a Doppler shift which depends on position in spacetime, in conformity with Einstein equivalence principle.

I. Introduction

Lorentz invariance of electromagnetism lies at tlie heart of the theory of relativity[1]. Tliis is true not only for tlie classical tlieory of electromagnetism, but also for the quantum theory. In particular, invariance of vacuum fluctuations under Lorentz transformations is needed to ensure that mechanical effects of these fluctuations preserve tlie relativity of uniform motion in empty space.

In contrast with these universally acceptecl ideas, tlie interplay betweeii quantuin fields and accelerated frames lias been tlie object of much debate. Since Einstein[2], accelerated frames are commonly representecl by using Rindler clianges of coordinates[3] between inertial and accelerated frames. These transformations do not preserve the propagation equations of electromagnetic fields. Light rays appear curved in accelerated frames while frequencies undergo a shift during light propagation[2]. Such a representation of accelerated frames also results in a transformation of vacuum into a thermal bath[4]. Tliis idea has apparently been easily accepted because of its association with tlie most spectacular effect predicted by quantum field theory in curved spacetime, namely thermal particle creation due to curvature[5]. It is nevertlieless clear that accelerated frames and curved spacetime are completely different physical problems, from the point of view of general relativity. Furthermore, the notion of particle number plays a central role in the interpretation of quantum field theory, and the fact that it is not preserved in accelerated frames leads to weighty paradoxes for quanturn theory[6]. These difficulties also raise doubts about the significance of the Einstein equivalence principle in the quantum domain. Such a principle indeed relies on tlie very notion of particle number, as well as on the interpretation of frequency change from inertial to accelerated frames as a Doppler shift depending on position in spacetime[2]. If vacuum or 1-photon states may not be clefinecl in a consistent manner in inertial and accelerated frames, it might appear hopeless to attribute any significance to such a principle in the quantum domain.

In the present paper, we show that the interplay between acceleration ancl quantum fields may be analyzed in a consistent manner which allows to extend the Einstein equivalence principle to the quantum domain.

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Our approach makes use of the conformal symmetry of quantum theory of massless fields, lilte a scalar field in two-dimensional (2D) spacetime or the electromagnetic field in four-dimensional (4D) spacetime. For tlie sake of simplicity, we will restrict our attention here to the 2D case.

It has been known for a long time that the symmetry related to inertial motions, associated witli Lorentz transformations, can be extended for massless fields to a larger group which includes conformal transformations to accelerated frames[7]. Light rays remain straight lines with such a representation of accelerated frames while frequencies are now preserved during liglit propagation. These transformations are known to fit[8] tlie relativistic definition of uniformly accelerated motion[9]. It has also been shown that vacuum remains unchanged under conformal transforrnations to accelerated frames[10]. Here, we extend the latter property by demonstrating the invariance of total particle number under such transformations. This proves the consistency of a point of view which maintains invariance of vacuum and particle number for inertial and accelerated observers.

When a spectral decomposition of particle number is introduced and the transformation of spectral density from inertial to accelerated frames analyzed, field phase operators make an appearance. We will show that the resulting expressions correspond to quantum definitions for position in spacetime[11] which comply with the requirement of the Einstein equivalence principle for tlie interpretation of acceleration on quantum fields in terms of Doppler shifts. It is well known that tlie definition of phase operators, which may be considered as conjugated to tlie number operators[12], leads to ambiguities[13,14]. A lot of work lias been devoted to cure tliese ambiguities (see[15] and references therein). However, the conclusions that we will reach in the present paper will essentially be unaffected by these difficulties.

II. Conforma1 coordinate transformations

In a two-dimensional (2D) spacetime, a free massless scalar field $\phi(t,x)$ is the suin of two counterpropagating components:

$$\phi(t,x) = \varphi(u) + \psi(v) \qquad u = t - x \qquad v = t + x \quad (1)$$

We use natural spacetime units ($c = 1$); t is the time coordinate, x is the space coordinate, u and v are the two light-cone variables.

In the 2D case, conformal coordinate transformations are those transformations which act separately on the two light-cone variables, and they are specified by arbitrary functions f and g describing the relations between sucli variables in the two reference systems:

$$\overline{u} = f(u) \qquad \overline{v} = g(v) \quad (2)$$

The field transformation under conformal coordinate transformations is defined through:

$$\varphi \rightarrow \overline{\varphi} \qquad \varphi(u) = \overline{\varphi}(\overline{u}) \quad (3)$$

From now on, we consider only one (φ) of the two counterpropagating components; the other one (ψ) can be dealt with in exactly the same way.

As well-known in Quantum Field Theory, the field transformation under coordinate transformations may be considered as generated by linear forms of the stress tensor, that is also quadratic forms of the fields[16]. In order to give explicit forms of these generators in the spectral domain, we introduce the Fourier components $\varphi[\omega]$ of the field $\varphi(u)$ according to the general definition:

$$\varphi(u) = \int \frac{d\omega}{2\pi} \varphi[\omega] e^{-i\omega u} \quad (4)$$

These components are related to the standard annihilation and creation operators:

$$\begin{aligned} \varphi[\omega] &= \sqrt{\frac{\hbar}{2|\omega|}} \left(\theta(\omega) a_\omega + \theta(-\omega) a^\dagger_{-\omega}\right) && (5) \\ [a_\omega, a_{\omega'}] &= \left[a^\dagger_\omega, a^\dagger_{\omega'}\right] = 0 && (6) \\ \left[a_\omega ; a^\dagger_{\omega'}\right] &= 2\pi\delta(\omega - \omega') && (7) \end{aligned}$$

wliere θ is the Heaviside function and δ the Dirac distribution. Tlie commutation relations of the Fourier components of the field are given by:

$$[\varphi[\omega], \varphi[\omega']] = \frac{\pi\hbar}{\omega} \delta(\omega + \omega') \quad (8)$$

We now define the generating function $T[\omega]$:

$$T[\omega] = \int \frac{d\omega'}{2\pi} \omega' (\omega + \omega') \varphi[-\omega'] \varphi[\omega + \omega'] \quad (9)$$

as the Fourier transform of the stress tensor[1]:

$$T(u) = (\partial_u \varphi(u))^2 \tag{10}$$

We then introduce the generators T_k as the coefficients of the Taylor expansion of the generating function (k a positive integer):

$$T_k = \left\{(-i\partial_\omega)^k T[\omega]\right\}_{\omega=0} = \int u^k T(u)\, du \tag{11}$$

The commutators of these quantities with the field are obtained as:

$$[T[\omega], \varphi[\omega']] = -\hbar(\omega+\omega')\varphi[\omega+\omega'] \tag{12}$$

$$[T_k, \varphi[\omega]] = -\hbar(-i\partial_\omega)^k \{\omega\varphi[\omega]\} \tag{13}$$

The latter relation precisely fits the action upon the field of an infinitesimal conformal coordinate transformation. Denoting:

$$\delta\varphi[\omega] = \frac{\varepsilon}{i\hbar}[T_k, \varphi[\omega]] \tag{14}$$

with ε an infi:aitesimalreal number, one indeed deduces:

$$\delta\varphi(u) = -\varepsilon u^k \partial_u \varphi(u) \tag{15}$$

This corresponds to equations (2-3) with an infinitesimal coordinate transformation:

$$\overline{u} = u + \delta f(u) \tag{16}$$

$$\delta\varphi(u) \equiv \overline{\varphi} - \varphi = -\delta f(u)\partial_u\varphi(u) \tag{17}$$

$$\delta f(u) = \varepsilon u^k \tag{18}$$

Notice that the generating function $T[\omega]$ may also be associated with an infinitesimal coordinate transformation:

$$\delta f(u) = \varepsilon \exp(i\omega u) \tag{19}$$

This does not correspond to a real coordinate tranformation which would necessarily involve opposite values of the frequency[2].

In order to recover the known commutation relations for the conformal generators[17,18], we write the commutator of the generating function with quadratic forms of the field:

$$[T[\omega], \varphi[\omega]\,\varphi[\omega'']] = -\hbar\{(\omega+\omega')\varphi[\omega+\omega']\varphi[\omega''] + (\omega+\omega'')\varphi[\omega']\varphi[\omega+\omega'']\} \tag{20}$$

We then deduce the commutator of the generating function evaluated at different arguments:

$$[T[\omega], T[\omega']] = \hbar(\omega-\omega')T(\omega+\omega') \tag{21}$$

A Taylor expansion of this relation provides commutators characteristic of the conformal algebra (for positive integers k)[3]:

$$[T_k, T_{k'}] = i\hbar(k'-k)T_{k+k'-1} \tag{22}$$

III. Transformation of vacuum

The conformal coordinate transformations preserve the propagation equation of massless fields, and therefore their commutators[10]. However, not all of them preserve vacuum fluctuations.

The vacuum state is defined by specific correlation functions:

$$\langle\varphi[\omega]\varphi[\omega']\rangle_{\text{vac}} = \theta(\omega)\theta(-\omega')[\varphi[\omega], \varphi[\omega']] \tag{23}$$

$\langle\,\rangle_{\text{vac}}$ represents a mean value in the vacuum state. This means that annihilators vanish when applied to the vacuum state. Using expression (8) of the field commutators, one obtains the correlation function:

$$\langle\varphi[\omega]\varphi[\omega']\rangle_{\text{vac}} = \theta(\omega)\frac{\pi\hbar}{\omega}\delta(\omega+\omega') \tag{24}$$

Using transformation (20) of field quadratic forms, one then deduces the transformation of vacuum correlation functions:

$$\langle[T[\omega], \varphi[\omega']\varphi[\omega'']]\rangle_{\text{vac}} = \pi\hbar^2(\theta(\omega') - \theta(-\omega''))\delta(\omega+\omega'+\omega'') \tag{25}$$

[1] In this definition appear symmetric products of the field operators, rather than normally ordered products. As a consequence, the generating function describes the stress tensor associated with vacuum, as well as the stress tensor associated with particles.

[2] We may emphasize that we are dealing with coordinate transformations of real Minkowski spacetime, here represented by real light-cone variables. The generating function (9), that is the Fourier transfonn of the field stress tensor, is not hermitian but satisfies $T[\omega]^\dagger = T[-\omega]$. The generators T_k are defined in equation (11) as hermitian operators, in contrast with common definitions of generators in Conforma1 Field Theory in 2D spacetime[18].

[3] Note that the original Virasoro algebra is defined with negative and positive order non-hermitian generators. It is generated by $\frac{1}{\hbar\Omega}T[k\Omega]$, where Ω is a scale frequency. It also corresponds to a Laurent expansion in variable u of the function $T(u)$ extended to the complex plane[18].

It is also worth writing the transformation of the vacuum stress tensor, that is of the generating function itself[4]:

$$\langle [T[\omega], T[\omega']] \rangle_{\text{vac}} = \frac{\hbar^2 \omega^3}{12} \delta(\omega + \omega') \tag{26}$$

One then demonstrates, through a Taylor expansion of these relations in the frequency w, that the vacuum correlation function for field derivatives $\partial_u \varphi$ (wliich correspond to Fourier components $-i\omega\varphi[w]$), as well as the vacuum stress tensor, are preserved by the infinitesimal generators T_0, T_1 and T_2 which respectively describe translations, Lorentz boosts and conformal transformations from inertial to accelerated frames. This is no longer the case for the higher-order generators. In particular, the generator T_3 changes the vacuum stress tensor in a manner whicli is consistent with tlie dissipative force felt by a mirror moving in vacuum with a non-uniform acceleration[19,20].

We thus recover the result of reference [10] for a massless scalar field theory in a 2D spacetime: the vacuum is not invariant under the large group of conformal coordinate transformations (equation (2) with an arbitrary function f). It is invariant only under the smaller group of transformations generated by T_0, T_1 and T_2. Those transformations correspond to tlie particular case of liomographic functions[5]:

$$\overline{u} = \frac{au+b}{cu+d} \qquad ad - bc = 1 \tag{27}$$

In the following, we give some results for the large conformal group, but we focus our attention onto the smaller group of transformations which preserve vacuum, and particularly onto the action of the acceleration generator T_2.

IV. Transformation of particle number operators

We will denote n, the spectral density of particle number:

$$n_\omega = a_\omega^\dagger a_\omega = \frac{2\omega}{\hbar}\theta(\text{w})\varphi[-\text{w}]\varphi[\text{w}] \tag{28}$$

The values at different frequencies are commuting quantities:

$$[n_\omega, n_{\omega'}] = 0 \tag{29}$$

and the coinmutators with the field may be written from relations (8):

$$[n_\omega, \varphi[\omega']] = 2\pi \{\delta(\text{w} + \text{w}') - \delta(\text{w} - \text{w}')\} \varphi[\text{w}'] \tag{30}$$

Tliis definition is such that tlie generator T_0, that is the field energy, lias its standard form in terms of number density:

$$T_0 = \langle T_0 \rangle_{\text{vac}} + \int_0^\infty \frac{d\omega}{2\pi} \hbar\omega n_\omega \tag{31}$$

The total number n of particles is defined as the integral of n_ω:

$$n = \int_0^\infty \frac{d\omega}{2\pi} n_\omega \tag{32}$$

The number operators n, are defined for positive frequencies, and vanisli when applied to the vacuum state.

We come now to the main argument of the present paper, that is the transformation of particle numbers under conformal coordinate transformations. As an immediate consequence of transformation (20) of field quadratic forms, we deduce the transformation of the number density:

$$\begin{aligned}[T[\text{w}], n_{\omega'}] &= -2\omega'\theta(\omega')\{(\text{w} - \omega')\varphi[\text{w} - \omega']\varphi[\omega'] \\ &\quad + (\omega + \omega')\varphi[-\omega']\varphi[\omega + \omega']\}\end{aligned} \tag{33}$$

We obtain tlie transformation of tlie total particle number by an integration:

$$[T[\omega], n] = -\int_0^\omega \frac{d\omega'}{\pi} \omega'(\omega - \omega')\varphi[\omega - \omega']\varphi[\omega'] \tag{34}$$

We then derive the effect of the infinitesimal generators by performing a Taylor expansion in tlie frequency w of the previous expressions.

The total particle number n is preserved by tlie generators T_0, T_1 and T_2:

$$\frac{1}{i\hbar}[T_0, n] = \frac{1}{i\hbar}[T_1, n] = \frac{1}{i\hbar}[T_2, n] = 0 \tag{35}$$

This property is well-known for the translations and Lorentz boosts. The new result is that a conformal transformation to an accelerated frame also leads to a redistribution of particles in the frequency domain,

[4] When written in terms of normally ordered products, commutation relations (21) between the generators include a further pure number. This central charge is determined by equation (26).

[5] Note that the modification of the mean vacuum stress tensor $\langle T(u)\rangle_{\text{vac}}$ under a conformal transformation associated with the function f is proportional to the Schwartzian derivative of f, which vanishes for homographic transformations[18,19].

without any cliange of tlie total number of particles. It is consistent with tlie invariance of vacuum in tlie homographic group generated by T_0, T_1 and T_2, as discussed in tlie previous section. It means tliat tlie notion of particle number is tlie same for accelerated observers and for iiiertial ones, provided that acceleratecl frames are defined tlirougli conformal transformations. For the other generators $T_{k\geq 3}$, tlie vacuum is no longer preserved and tlie total particle iiumber n is changed[6].

We now write tlie transforinatioii of tlie spectral density n_ω of particle number under tlie generators T_0, T_1 and T_2 which perserve tlie total number n. As expected, tlie number density is uncliangeed under a translation:

$$\frac{1}{i\hbar}[T_0, n_\omega] = 0 \tag{36}$$

but clianged under a Lorentz boost:

$$\frac{1}{i\hbar}[T_1, n_\omega] = \partial_\omega \{\omega n_\omega\} \tag{37}$$

This latter cliange is a inere mapping in the frequency domain, associated with tlie Doppler shift of tlie field frequency. We tlien write tlie modification of tlie spectral density of particle number in a conformal transformation from an inertial to an accelerated fraine:

$$\frac{1}{i\hbar}[T_2, n_\omega] = 2\partial_\omega \{\omega m_\omega\} \tag{38}$$

$$m_\omega = \frac{\mathrm{w}}{i\hbar}\theta(\omega)\{\varphi[-\omega]\varphi'[\omega] + \varphi'[-\omega]\varphi[\omega]\} \tag{39}$$

The quadratic form m_ω is hermitian. It may not be rewritten in terms of tlie density n_ω or its derivatives. In other words, the modification of n, under T_2 amounts to a redistribution of particles in the frequency domain, without any change of the total particle number, as it was tlie case for tlie modification of n_ω under T_1, but this redistribution is no longer equivalent to a mere mapping of tlie cleiisity n_ω in the frequency spectrum. We will sliow later on that tlie expression (39) may be interpreted as a Doppler shift which depends on position in spacetime, in conformity with Einstein equivalence principle.

V. Quantum phase and phase-time operators

In the present section, we sliow how to obtain quantum operators associated with positions in spacetime.

As a first step in this direction, we introduce operators e, and δ_ω such that:

$$a_\omega = e_\omega \sqrt{n_\omega} \tag{40}$$

$$a_\omega^\dagger = \sqrt{n_\omega} e_\omega^\dagger \tag{41}$$

$$e_\omega = e^{i\delta_\omega} \tag{42}$$

As well-known, these relations are not sufficient to define phase operators since annihilators and creators are not modified hy a redefinition of the phases such tliat:

$$e_\omega \rightarrow e_\omega + \pi_\omega \qquad \pi_\omega n_\omega = 0 \tag{43}$$

Various definitioiis of tlie phase operators, for example tlie Susskind-Glogower definition[14] or the Pegg-Barnett definition[21], are connected through sucli reclefinitions. We sliow below that the properties studied in the present paper may be stated independently of such ambiguities.

We now list some properties which are satisfied for any operators defined from relations (40-42); tliese properties depend only upon the field commutation relations (6-7). First, the exponential operators e, are cominuting variables, like tlie number operators (compare with (29)):

$$[e_\omega, e_{\omega'}] = 0 \tag{44}$$

This is also the case for their adjoint operators $e_\omega^\dagger$:

$$\left[e_\omega^\dagger, e_{\omega'}^\dagger\right] = 0 \tag{45}$$

The commutation relations between operators e, (or $e_\omega^\dagger$) and tlie number operators n_ω satisfy:

$$[n_\omega, e_{„}]\ \sqrt{n_{\omega'}} = -2\pi e_{\omega'}\delta(\mathrm{w} - \omega')\sqrt{n_{\omega'}} \tag{46}$$

$$\sqrt{n_{\omega'}}\left[n_\omega, e_{\omega'}^\dagger\right] = 2\pi\sqrt{n_{\omega'}}e_{\omega'}^\dagger \partial(\mathrm{w} - \mathrm{w}') \tag{47}$$

However, the exponential operators e, do not commute in tlie general case witli their adjoint operators $e_\omega^\dagger$:

$$e_\omega e_\omega^\dagger = 1 \tag{48}$$

$$e_\omega^\dagger e_\omega = 1 - \alpha_\omega \Pi_\omega \tag{49}$$

where Π_ω projects onto vacuuni for field components at frequency w, and a, is a function of w which depends on the specific definitioii of the phase operator.

' ~ ~ that the commutator **(33)** vanishes when applied to the vacuum state, for arbitrar positive frequencies w. However, vacuum and particle numbers are not invariant under generators $T_{k\geq 3}$. These properties are consistent since, as already mentioned, T $[w]$ is not hermitian and real coordinate transformations involve the generating function $T\,[w]$ at negative frequencies as well as positive ones.

It follows that the exponential operators are not necessarily unitary and, hence, that tlie phase operators are not hermitian. One gets for example a, $= 1$ in the Susskind-Glogower definition[14], and a, $= 0$ in the Pegg-Barnett definition, which thus corresponds to hermitian phase operators[21]. For all definitions, one may nevertheless write

$$\sqrt{n_\omega} e^\dagger_\omega e_\omega = e^\dagger_\omega e_\omega \sqrt{n_\omega} = \sqrt{n_\omega} \tag{50}$$

It follows tliat simple relations hold for states orthogonal to vacuum, i.e. states such that tlie probability for having $n_\omega = 0$ vanishes.

We have given definitions of tlie phase operators for a field having a wliole frequency spectruin, and not only for a inonomode field. We are thus able to deal with frequency variation of the phase operators and, in particular, to consider the operators δ'_ω obtained by differentiating phases δ_ω versus frequency, according to the Wigner definition of phase-times[22]. A lot of discussions have been devotecl to the significance of such a definition, and of its relation with time observables whicli can be measured by various techniques[23]. Here, we will emphasize that tlie operators δ'_ω do not commute with number operators and with energy, tlius pioviding quantum phase-times.

Since the exponential opeiators e, coinmute (see relation (44)), the frequency derivative δ'_ω of tlie phase may be defined from tlie frequency derivative e'_ω of the exponential operator:

$$e'_\omega = i\delta'_\omega e_\omega = ie_\omega \delta'_\omega \tag{51}$$

It may be defined as well from tlie adjoint exponential operators $e^\dagger_\omega$:

$$(e')' = -ie^\dagger_\omega (\delta'_\omega)^\dagger = -i(\delta'_\omega)^\dagger e^\dagger_\omega \tag{52}$$

It follows from relation (48) tliat the phase derivative δ'_ω is an hermitian operator, even for non-hermitian definitions of the phase 6,:

$$\delta'_\omega = -ie'_\omega e^\dagger_\omega = ie_\omega \left(e^\dagger_\omega\right)' = (\delta'_\omega)^\dagger \tag{53}$$

Using these properties and definitions (40-42), we may now rewrite the definition (39) of m_ω as:

$$m_\omega = \frac{i}{2}\left\{(a'_\omega)^\dagger a_\omega - a^\dagger_\omega a'_\omega\right\} \tag{54}$$
$$m_\omega = \sqrt{n_\omega}\delta'_\omega\sqrt{n_\omega} \tag{55}$$

The quadratic form m_ω is proportional to the density n,, but also to tlie operator δ'_ω which, as we shall see in tlie next section, has properties of a quantum position in spacetime.

The operators δ'_ω have been defined from phase operators, so that they are expected to have non vanishing commutators witli the number operators[12]. The definition of such commutators is affected by the ambiguities already discussed[14]. We may however state tliem in a rigorous manner by evaluating the commutators between tlie densities m_ω and $n_{\omega'}$ (for $w > 0$ and $\omega' > 0$):

$$[m_\omega, n_{\omega'}] = -2\pi i\delta'(\mathrm{w} - \omega')\, n_\omega \tag{56}$$

These relations are unambiguously defined in any quantum state and they are consistent with Dirac-like commutators in states orthogonal to the vacuum (states such that $n_\omega \neq 0$):

$$\sqrt{n_\omega}\,[\delta'_\omega, n_{\omega'}]\,\sqrt{n_\omega} = -2\pi i\delta'(\mathrm{w} - \mathrm{w}')\; n_\omega \tag{57}$$

To derive this result, we have used relation (55) and the fact that n_ω and $n_{\omega'}$ are commuting variables.

VI. Discussion

A comparisoii between the relations (37) and (38), wliich describe respectively the effect of a Lorentz boost and of a change of acceleration on the number density, shows that the latter is equivalent to a Doppler shift of the field frequency whicli depends on the operator δ'_ω. This property appears to be quite close to a quantum expression of tlie Einstein equivalence principle, provided tliat δ'_ω plays the role of a position in spacetime, in consistency with the Wigner definition of phase-times[22]. The semiclassical character of the Wigner definition makes its extension to the definition of a quantum operator difficult. We show now that it is however possible to write down rigorous quantum statements with δ'_ω used like a position in spacetime.

To this aim, we evaluate commutation relations between δ'_ω and tlie energy operator T_0. Multiplying equation (56) by frequency w' and integrating over ω', we get (see equation (31)):

$$[T_0, m_\omega] = i\hbar n_\omega \tag{58}$$

We may also introduce the integral m of the density m_ω, in the same manner as tlie total particle number n from the density n_ω:

$$m = \int_0^\infty \frac{d\omega}{2\pi} m_\omega \tag{59}$$

We deduce from the commutator (58):

$$[T_0, m] = i\hbar n \tag{60}$$

We notice that the commutation relations between m and the creation and annihilation operators have a simple form:

$$[m, a] := i a'_\omega \qquad [m, a_\omega^\dagger] = i\,(a'_\omega)^\dagger \tag{61}$$

We now discuss these relations from the point of view of the quantum definition of positions in spacetime.

We first discuss the spectral relation (58). Since n_ω is invariant in a translation, we deduce from relation (55):

$$\sqrt{n_\omega}\,[T_0, \delta'_\omega]\,\sqrt{n_\omega} = i\hbar n_\omega \tag{62}$$

For states orthogonal to the vacuum state (n, # O), this has the form of a canonical commutator between T_0 and δ'_ω, thus defining δ'_ω as a quantum phase-time.

More exactly, T_0 is the energy associated with the light-cone variable *u*, so that δ'_ω has to be interpreted **as** a quantum operator U_ω having this variable u as its classical analog. The same manipulations applied to the counterpropagating field component ψ would lead to the definition of a quantum variable V having the light-cone variable v as its classical analog. Combining these two variables, it is therefore possible to define time- and space-like operators:

$$\delta_\omega'^{(\varphi)} \quad U_\omega = \tau_\omega - \xi_\omega \qquad \delta_\omega'^{(\psi)} \equiv V_\omega = \tau_\omega + \xi_\omega \tag{63}$$

which are conjugated to the field energy and momentum:

$$[E, \tau_\omega] = i\hbar \qquad [P, \xi_\omega] = -i\hbar \tag{64}$$

defined through:

$$E = T_0^{(\varphi)} + T_0^{(\psi)} \qquad P = T_0^{(\varphi)} - T_0^{(\psi)} \tag{65}$$

This provides quantum definitions of time and space operators τ_ω and ξ_ω, defined at each frequency *w* like the semiclassical Wigner definitions.

In order to give a more explicit realisation of quantum positions in spacetiine, we now consider the integrated relation (60), in the particular case of a 1-particle state. As already discussed, the notion of a numbei state is preserved in conforma1 transformations to accelerated frames; precisely the total particle number n **is** preserved. In particular, the definition of a 1-particle state (n = 1) is the same for accelerated and inertial observers. For such a state, the commutator (60) now reads as a canonical commutator between the energy T_0 and the operator m:

$$[T_0, m] = i\hbar \qquad n = 1 \tag{66}$$

This relation may be considered as associating a quantum position to tlie 1-particle state, precisely one position for each light-cone variable. Following the same path as from equation (63) to equation (65), we may then obtain time and space operators τ and ξ associated with the state.

In fact, the operator m is a generalization for quantum fields of the Newton-Wigner quantum position[11]. This position, initially defined for a wavefunction, is here extended to 1-particle field states. To make this point explicit, we represent each 1-particle state by a function f of frequency or of position:

$$| f \rangle = \int_0^\infty \frac{d\omega}{2\pi} f[\omega] \mid \omega \rangle = \int_{-\infty}^\infty du\ f(u) \mid u \rangle \tag{67}$$

where we have used Dirac-like ket notations for the basis states:

$$| \omega \rangle = a_\omega^\dagger \mid \text{vac} \rangle \tag{68}$$

$$| u \rangle = \int_0^\infty \frac{d\omega}{2\pi} e^{i\omega u} \mid \omega \rangle \tag{69}$$

Equation (61) thus means that the operator m may be represented in the space of functions f either as the differential operator $(-i\partial_\omega)$ in the frequency domain, or as the multiplication by u in the position domain:

$$m \mid f \rangle = -i \int_0^\infty \frac{d\omega}{2\pi} f'[\omega] \mid \omega \rangle = \int_{-\infty}^\infty du\ u\ f(u) \mid u \rangle \tag{70}$$

Its spectral density m_ω can be shown to be related to its symmetrised product with particle number density n_ω:

$$\frac{1}{2}\{m, n_\omega\} \equiv \frac{1}{2}(m n_\omega + n_\omega m) = m_\omega + : m n_\omega : \tag{71}$$

where : : denotes normal ordering:

$$: mn_\omega := \frac{i}{2}\int_0^\infty \frac{d\omega'}{2\pi}\left\{(a'_{\omega'})^\dagger a^\dagger_\omega a_{\omega'} a_\omega - a^\dagger_{\omega'} a^\dagger_\omega a'_{\omega'} a_\omega\right\} \tag{72}$$

This normal product vanishes when applied to 1-particle field states, so that for such states tlie density m_ω can be identifiecl, as an operator, with the symmetrised product of position m and particle number density n_ω. Using tlie commutation relations (61), it can then be rewritten under the form (55) with position m substituted for δ'_ω:

$$m_\omega = \frac{1}{2}\{m, n_\omega\} = \sqrt{n_\omega} m \sqrt{n_\omega} \qquad n = 1 \tag{73}$$

Finally, transformations of particle number density to inertial or acceleratecl frarnes take tlie simple form of Doppler shifts of tlie frequency ((37) and (38)):

$$\frac{1}{i\hbar}[T_1, n_\omega] = \partial_\omega\{\omega n_\omega\} \tag{74}$$

$$\frac{1}{i\hbar}[T_2, n_\omega] = 2\partial_\omega\{\omega\sqrt{n_\omega}\delta'_\omega\sqrt{n_\omega}\} \tag{75}$$

For 1-particle field states, the last relation can also be written:

$$\frac{1}{i\hbar}[T_2, n_\omega] = 2\partial_\omega\{\omega\sqrt{n_\omega} m \sqrt{n_\omega}\} \qquad n = 1 \tag{76}$$

This Doppler shift is proportional to the acceleration and to the Newton-Wigner position of tlie particle.

We may now summarize the results obtainecl in this paper. In order to talce advantage of the conformal symmetry of massless field theories, we have represented accelerated frames by conformal transformations. Invariance of vacuum under such transformations was alreacly known[10]. We have demonstrated that total particle number was also iiivariant, thus proving the consistency of a point of view where vacuum and number states are tlie same for iiiertial and accelerated observers. In contrast with the cominon Rindler representation of accelerated frames discussed in tlie introduction, this point of view allows to discuss the effect of acceleration on quantum *fields* in terms of a redistribution of particle in the frequency domain. Analyzing the transformation of spectral density of particle nuinber from inertial to acceleratecl frames, we have shown that it may be interpreted in terms of Doppler shifts depending upon position in spacetime, in conformity with the Einstein equivalence principle. This position is defined as the frequency derivative of some phase operators, in analogy with the Wigner definition of phase-times[22]. In the particular case of 1-particle states, it is a generalization to Quantum Field Theory of the Newton-Wigner position operator initially defined for wavefunctions[11]. Consiclered as a whole, these results constitute a step forward in tlie direction of a consistent interpretation of the Einstein equivalence principle in the quantum domain.

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