Antipodal Universes in the Topology *113* x *R* and S^3 x *R*

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On the Lie groups H^3 x *R* and S^3 x R, we construct solutions of Einstein equations by introducing left-invariant metrics *g* ~and right-invariant metrics *g* ~on tlie group manifold, and examine their differences and similarities from a local and global point of view. *g* ~ and g~ are related isometrically to each other by the Lie group inverse map, which induces an anti-isomorphism between the Lie algebra. of the left-invariant vector fields and that of right-invariant vector fields. We introduce a coordinate system where the inverse map group operation is expressed by a coordinate inversion. We denote these universes as antipodal. The resulting spacetimes admit by construction a Gs group of motions. In the family with topology *H* *R* there occurs the particular case *g* ~= *g* ~ which admits a *G7* group of motions with the 3-dim sections *H* maximally symmetric. The existence of acausal curves in these spacetimes is examined. Some of them are avoided by modifying the connectivity-in-the-large of the manifold, while others can he avoided by tlie introduction of a line of singularities in tlie spacetime manifold. We show that *g* ~and g~ correspond to rotating universes with matter vorticity of opposite sign. Massless neutrinos (produced via weak interaction processes) can be used as a. probe to distingnish physically the antipodal universes, because the nctive transformation changing *g* ~into g~ also transforms neutrinos into antineutrinos (or vice versa).

I. Introduction and motivation

In the geometrical study of cosmological models, powerful methods were developed and extensively used to construct invariant Lorentzian metrics on spacetime manifolds which are Lie groups[1-3]. As it is well known the action of the Lie group on itself can be divided into two independent subgroups, namely the left(L) and right (R) action of the group on itself. Two sets of invariant Lorentzian metrics g_L and g_R can be introduced, denoted respectively L-invariant and R-invariant geometries, constructecl with vector fields/forms which are invariant under the left, or right action of the group. The inverse map of the Lie group on itself induces an anti-isomorphism between the Lie algebra of L-invariant vector fields/forms and the Lie algebra of R-invariant vector field/forms. In this context we construct two families of models with Lorentzian metrics g_L and g_R, over the Lie groups S^3 x R and H^3 x R. We denote these families antipodal universes because they are isometrically related to each other by the Lie group inverse map. We introduce a coordinate system globally defined over the group manifold (except for one point) where the Lie group inverse map is described by a coordinate inversion. Symmetries of the models are easily characterized. A special case occurs when $g_L = g_R$. Global properties of the models are also examined. We discuss the causality problems arising in these models from a global aspect and some possible modifications in the topology of the manifolds to circumvent these pathologies. For simplicity we use the algebra of quaternions to characterize the semi-simple Lie groups S^3 and H^3, and all necessary concepts in Lie group theory is given in terms of quaternions[4,5]. We also examine the geometrical and physical relation between the L-invariant and R-invariant geometries. They are shown to correspond to rotating universes with opposite vorticity. A possible physical distinction between antipodal universes g_L and g_R is discussed by introducing neutrinos as test particles in these universes.

II. The Spacetimes $H^3 \times R$ and $S^3 \times R$

The methods used in this section are in part borrowed from Ref. [1], and are presented here concisely for completeness. Calculations are not given in detail but they can be checked without difficulty.

Let E_4 be the four dimensional Euclidean space with Cartesian coordinates $a = (a^0, a^1, a^2, a^3)$. We define the hypersurface M of E_4 as the set of points of E_4 which satisfy

$$(a^0)^2 + (a^1)^2 - (\varepsilon a^2)^2 - (\varepsilon a^3)^2 = 1 \qquad (2.1)$$

where $\varepsilon = i, 1$ whether M is S^3 or H^3, respectively. For every $a = (a^0, a^1, a^2, a^3)$ and $b = (b^0, b^1, b^2, b^3) \in M$ we define the multiplication law[6]

$$ab = (a^0b^0 - a^1b^1 + (\varepsilon)^2a^2b^2 + (\varepsilon)^2a^3b^3\ ,\ a^0b^1 + a^1b^0 - (\varepsilon)^2a^2b^3 + (\varepsilon)^2a^3b^2\ ,\ a^0b^2 + a^2b^0 + a^3b^1 - a^1b^3\ ,\ a^0b^3 + a^3b^0 + a^1b^2 - a^2b^1) \qquad (2.2)$$

Under (2.2) M becomes a group, acting on itself by left multiplication; namely, for a given $v \in M$, a left motion of M into itself is expressed as

$$a' = va \qquad (2.3)$$

and we have that $a' \in M$ for all $a \in M$. M is said simply transitivity since for each $a \in M$ there exists only one left motion v from a to a given $a' \in M$.

M acting on itself by left multiplication (2.3) is a Lie group with the three independent left-invariant vector fields on M

$$\begin{aligned} e^{\mu}_{(1)}(a) &= (-a^1, a^0, a^3, -a^2) \\ e^{\mu}_{(2)}(a) &= ((\varepsilon)^2a^2, (\varepsilon)^2a^3, a^0, a^1) \\ e^{\mu}_{(3)}(a) &= ((\varepsilon)^2a^3, -(\varepsilon)^2a^2, -a^1, a^0) \end{aligned} \qquad (2.4)$$

They are obtained by an arbitrary left motion a of the three independent unit vectors $(0,1,0,0)$, $(0,0,1,0)$ and $(0,0,0,1)$ which define the tangent space of M at the identity $(1,0,0,0)$. We remark that a left invariant vector field $\chi(a)$ on the Lie group M is defined by $v\chi(a) = \chi(va)$, for all $v, a \in M$.

We have an analogous picture for right motions of M on itself, namely

$$a' = av \qquad (2.5)$$

with the corresponding independent right-invariant vector fields

$$\begin{aligned} d^{\mu}_{(1)} &= (-a^1, a^0, -a^3, a^2) \\ d^{\mu}_{(2)} &= ((\varepsilon)^2a^2, -(\varepsilon)^2a^3, a^0, -a^1) \\ d^{\mu}_{(3)} &= ((\varepsilon)^2a^3, (\varepsilon)^2a^2, a^1, a^0) \end{aligned} \qquad (2.6)$$

In what follows we express the fields (2.4) and (2.6) in the coordinate independent form

$$e_{(i)}(a) = \frac{1}{2}\, e^{\mu}_{(i)} \frac{\partial}{\partial a^{\mu}} \quad , \quad d_{(i)}(a) = \frac{1}{2}\, d^{\mu}_{(i)} \frac{\partial}{\partial a^{\mu}} .$$

The fields $\{e_{(i)}\}$ and $\{d_{(i)}\}$ constitute two distinct representations of the algebra of the Lie group M, which are related by an anti-isomorphism induced by the inverse map of M on itself, as we shall show. Explicitly[1]

$$[e_{(i)}, e_{(j)}] = C^k_{\ ij} e_{(k)} \qquad (2.7a)$$

$$[d_{(i)}, d_{(j)}] = -C^k_{\ ij} d_{(k)} \qquad (2.7b)$$

where $C^k_{ij} = \varepsilon_{\ell ij} n^{\ell k}$ with $n^{\ell k} = diag(-\varepsilon^2, 1, 1)$. We obviously have

$$[e_{(i)}, d_{(j)}] = 0 \quad , \quad i,j = 1,2,3 \qquad (2.8)$$

The inverse map group operation is the application $\psi : M \rightarrow M$ such that

$$a \rightarrow \psi(a) = a^{-1} = (a^0, -a^1, -a^2, -a^3) \qquad (2.9)$$

for all $a \in M$. The effect of (2.9) on the fields (2.4) and (2.6) is easy to obtain. The Jacobian matrix of the transformation (2.9) changes the sign of the spatial components of the fields, while the argument of

[1] The square brackets denote the commutator of the two fields.

the fields must obviously be transformed according to $a^0 \to a^0$, $a^i \to -a^i$, $i = 1,2,3$. We obtain that

$$\begin{aligned} e_{(i)}(a) &\to -d_{(i)}(a) \\ d_{(i)}(a) &\to -e_{(i)}(a) \end{aligned} \qquad (2.10)$$

under (2.9). In other words, the inverse map (2.9) induces the anti-isomorphism (2.10) between the Lie algebra of L-invariant and R-invariant vector fields. We note that the L-invariant vector field and the R-invariant vector field related by the inverse map coincide at the identity of the group.

On the $1-$ dim manifold R we introduce the coordinate $z(-\infty < z < \infty)$, with vector field expressed as $\partial/\partial z$.

The Lie group $M \times R$ can be characterized by the invariant vector fields[7] $e_{(A)} = \{e_{(i)}, e_{(4)} = \frac{\partial}{\partial z}\}$ and $d_{(A)} = \{d_{(i)}, d_{(4)} = \frac{\partial}{\partial z}\}$, $A = 1,2,3,4$. They satisfy the algebra (2.7) and

$$[e_{(i)}, e_{(4)}] = 0 \quad , \quad [d_{(i)}, d_{(4)}] = 0 \qquad (2.11)$$

and constitute bases for vector fields in $M \times R$. It follows

$$[e_{(A)}, d_{(B)}] = 0 \qquad , \qquad A, B = 1,2,3,4 \qquad (2.12)$$

We may now introduce invariant Lorentz metrics on $M \times R$. We make a particular choice by prescribing the following scalar product rules

$$g_L(e_{(A)}, e_{(B)}) = g_{AB} \quad , \quad g_R(d_{(A)}, d_{(B)}) = g_{AB} \qquad (2.13)$$

where

$$g_{AB} = diag(\alpha^2, -\beta^2, -\beta^2, -1) \; , \qquad (2.14)$$

and α and β parameters. We will refer to g_L and g_R as left- and right-invariant metrics on $M \times R$.

III. Symmetries and Coordinates Systems

Let us now introduce on M the coordinate system (χ, η, ρ) by the transformations

$$\begin{aligned} a^0 &= \cosh\left(\frac{\varepsilon\rho}{2}\right)\cos\chi \\ a^1 &= \cosh\left(\frac{\varepsilon\rho}{2}\right)\sin\chi \\ a^2 &= \frac{1}{\varepsilon}\sinh\left(\frac{\varepsilon\rho}{2}\right)\cos\eta \\ a^3 &= \frac{1}{\varepsilon}\sinh\left(\frac{\varepsilon\rho}{2}\right)\sin\eta \end{aligned} \qquad (3.1)$$

For $\varepsilon = i$ we obtain a chart on S^3, the coordinates being a linear combination of Euler angles with range $0 \le \chi, \eta \le 2\pi$, $0 \le \rho/2 \le \phi$. For $\varepsilon = 1$ a chart is defined on H^3, with $0 \le \chi, \eta \le 2\pi$, $-\infty < \rho < \infty$. These coordinates have a singularity at the identity of the group $a = (1,0,0,0)$ or $(\chi = 0,\ \rho = 0)$ which is the fixed point of the inversion operation (2.9). The L-invariant vector fields $e_{(i)}$ are expressed

$$\begin{aligned} e_{(1)} &= \frac{\partial}{\partial\chi} - \frac{\partial}{\partial\eta} \\ e_{(2)} &= -2\cos(\chi-\eta)\frac{\partial}{\partial\rho} - \frac{2\varepsilon\sin(\chi-\eta)}{\sinh(\varepsilon\rho)}\left[-\sinh^2\left(\frac{\varepsilon\rho}{2}\right)\frac{\partial}{\partial\chi} + \cosh^2\left(\frac{\varepsilon\rho}{2}\right)\frac{\partial}{\partial\eta}\right] \\ e_{(3)} &= -2\sin(\chi-\eta)\frac{\partial}{\partial\rho} + \frac{2\varepsilon\cos(\chi-\eta)}{\sinh(\varepsilon\rho)}\left[-\sinh^2\left(\frac{\varepsilon\rho}{2}\right)\frac{\partial}{\partial\chi} + \cosh^2\left(\frac{\varepsilon\rho}{2}\right)\frac{\partial}{\partial\eta}\right] \end{aligned} \qquad (3.2)$$

The inverse map gropup operation is described globally by the transformation $(\chi \to -\chi, \rho \to -\rho)$, which operating on (3.2) yields

$$\begin{aligned} -d_{(1)} &= -\left(\frac{\partial}{\partial\chi} + \frac{\partial}{\partial\eta}\right) \\ -d_{(2)} &= 2\cos(\chi+\eta)\frac{\partial}{\partial\rho} - \frac{2\varepsilon\sin(\chi+\eta)}{\sinh(\varepsilon\rho)}\left[\sinh^2\left(\frac{\varepsilon\rho}{2}\right)\frac{\partial}{\partial\chi} + \cosh^2\left(\frac{\varepsilon\rho}{2}\right)\frac{\partial}{\partial\eta}\right] \\ -d_{(3)} &= -2\sin(\chi+\eta)\frac{\partial}{\partial\rho} - \frac{2\varepsilon\cos(\chi+\eta)}{\sinh(\varepsilon\rho)}\left[\sinh^2\left(\frac{\varepsilon\rho}{2}\right)\frac{\partial}{\partial\chi} + \cosh^2\left(\frac{\varepsilon\rho}{2}\right)\frac{\partial}{\partial\eta}\right] \end{aligned} \qquad (3.3)$$

as expected from (2.10).

The invariant metrics g_L and g_R defined in (2.13) have the expression

$$g(e) = A(\rho)d\chi^2 + B(\rho)d\eta^2 + \frac{1}{2}e\left(\alpha^2 - \frac{\beta^2}{\varepsilon^2}\right)\sinh^2(\varepsilon\rho)d\chi d\eta - \frac{\beta^2}{4}d\rho^2 - dz^2 \tag{3.4}$$

where

$$\begin{aligned} A(\rho) &= \alpha^2\cosh^4\left(\frac{\varepsilon\rho}{2}\right) - \frac{\beta^2}{4\varepsilon^2}\sinh^2(\varepsilon\rho) \\ B(\rho) &= \alpha^2\sinh^4\left(\frac{\varepsilon\rho}{2}\right) - \frac{\beta^2}{4\varepsilon^2}\sinh^2(\varepsilon\rho) \end{aligned}$$

and $e = \pm 1$ for g_L and g_R, respectively. We note that g_L and g_R differ only by the sign of the cross term $d\chi d\eta$. They are connected by the coordinate inversion $(\chi \to -\chi, \rho \to -\rho)$ which is the inverse map in the coordinates (3.1). We denote these spacetimes antipodal. In the realm of pure gravitational interaction the two metrics are indistinguishable if the covariance group of the theory includes improper transformations. The possibility of physical distinction between the two spacetimes is discussed in Çection 5, where the nature of these transformations is analysed when we include neutrinos as test particles.

From (2.12) and (2.13) it follows that – by construction – the left(right)-invariant geometries $g_L(g_R)$ have the four right(left)-invariant vector fields $d_{(A)}(e_{(A)})$ as Killing vectors. A direct inspection of (3.4) shows that $\partial/\partial\eta$ is an additional independent Killing vector. Summing up, we have in general

(a) five independent Killing vectors associated to g_L

$$\left\{d_{(A)}, \frac{\partial}{\partial\eta}\right\} \tag{3.5}$$

(b) five independent Killing vectors associated to g_R

$$\left\{e_{(A)}, \frac{\partial}{\partial\eta}\right\} \tag{3.6}$$

The spacetimes with metric (3.4) are then endowed with a G_5 group of isometries acting transitively on the spacetime manifolds.

In the hyperbolic family $\varepsilon = 1$, an exceptional case occurs for $\alpha^2 = \beta^2$: inspection of (3.4) yields immediately $g_L = g_R$. It then follows from a trivial counting in (3.5)-(3.6) that this particular geometry has seven independent Killing vectors, for instance $(e_{(1)}, e_{(2)}, e_{(3)}, e_{(4)}, d_{(1)}, d_{(2)}, d_{(3)})$. The sections $z = const.$ are maximally symmetric with a group of motion generated by $(e_{(1)}, e_{(2)}, e_{(3)}, d_{(1)}, d_{(2)}, d_{(3)})$. This case corresponds to the metric of the $z = const.$ sections, namely the metric on M, being conformal to the Cartan-Killing metric on M, $g_{ij} = C^{\ell}_{\ ik}C_j^{\ k}{}_{\ell}$ (cf. (2.7) and (2.13)).

The class of L$-$ and R-invariant metrics (3.4) for the hyperbolic case has the structure of the reflection group of hyperbolae through their asymptotes[8]. In fact the function (cf. (3.4)) $y = e(\alpha^2 - \beta^2)$ defined in the plane of the metric parameters (α,β) describes a congruence of hyperbolae which are reflected through the asymptotes $y = 0$ when we change the sign of e, that is, when we go from g_L to g_R and vice-versa. The asymptotes correspond to the exceptional geometry discussed above with a G_7 isometry group.

IV. Causality and Topological Defects in Sections

The spacetimes introduced here present some pathological properties like the existence of time-like or null-like closed curves. As we shall see, this is connected to the fact that the sections $z = const.$ have, by construction, the structure of S^3 or H^3, and the restriction of the invariant geometries (3.4) to S^3 or H^3 has signature $(+--)$. In some cases, by a legitimate alteration of the topology we can eliminate the acausal curves, but they are in general inevitably present.

We examine separately the two cases

(i) $H^3 \times R$

The sections $z = $ const. with the topology of H^3 are described, in terms of Cartesian coordinates of the embedding Euclidean spaces E_4, by (2.1) with $\varepsilon = 1$. Let us consider the 2$-$dimsections of H^3 which we shall describe in the coordinate system (ρ, χ, η) defined in (3.1), with $-\infty < \rho < \infty$, $0 \leq \chi, \eta \leq 2\pi$. Taking firstly the sections $\eta =$ const. (we choose for convenience $\eta = 0$) we obtain

$$\begin{aligned} a^0 &= \cosh\left(\frac{\rho}{2}\right)\cos\chi \\ a^1 &= \cosh\left(\frac{\rho}{2}\right)\sin\chi \\ a^2 &= \sinh\left(\frac{\rho}{2}\right) \\ a^3 &= 0 \end{aligned} \tag{4.1}$$

describing the points of the one-leaf hyperboloid of Fig.1.

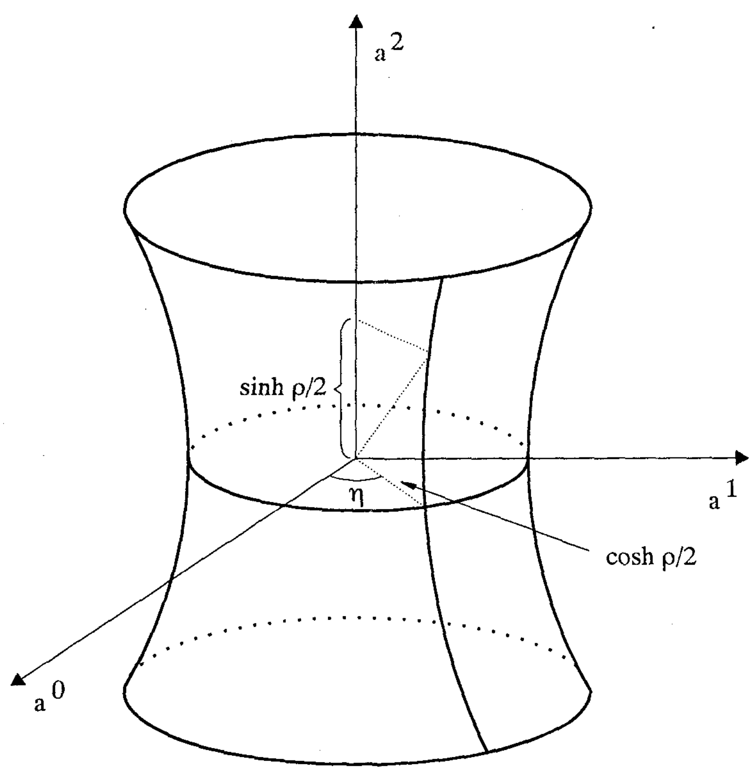

Figure 1: The one-leaf hyperboloid embedded in E_4 corresponding to the section $\eta, z = const.$ of the manifold $H^3 \times R$.

For the sections $\chi = const.$ (we choose for simplicity $\chi = 0$) we have

$$\begin{aligned} a^0 &= \cosh\left(\frac{\rho}{2}\right) \\ a^1 &= 0 \\ a^2 &= \sinh\left(\frac{\rho}{2}\right)\text{cosq} \\ a^3 &= \sinh\left(\frac{\rho}{2}\right)\text{sinhq} \end{aligned} \tag{4.2}$$

which describe in E_4 the points of the two-leaves hyperboloid of Fig. 2

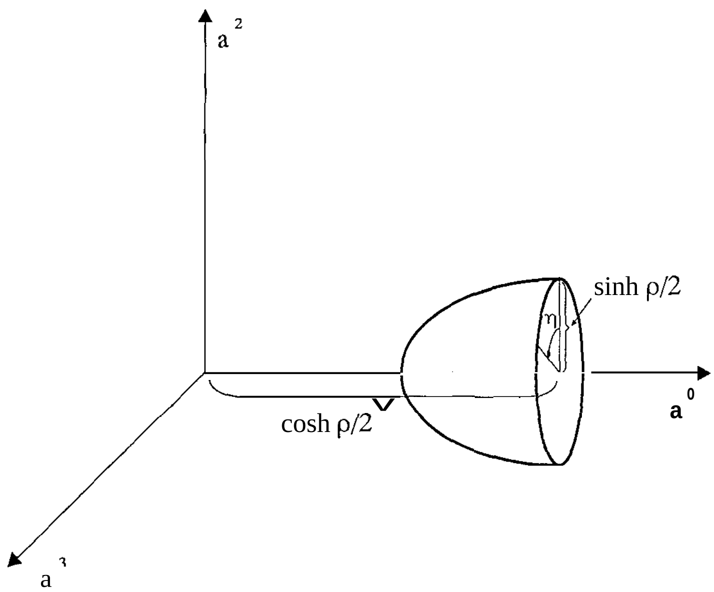

Figure 2: Two-leaves hyperboloid corresponding to the sections $\chi, z =$ const. of the manifold $H^3 \times$ R.

We are now ready to examine how the topology of the sections is related to the causality problem in these spacetimes. To this end let us consider the invariant Lorentzian geometry (3.4) with $\varepsilon = 1$ defined on $H^3 x$ R. We distinguish

$(i.a)$ case $a^2 > \beta^2$:

In the sections $\eta, z =$ const. we have $A(\rho) > 0$ for all p. Therefore the integral curves of the vector field $\partial/\partial\chi$ are time-like closed curves on the hyperboloid of Fig. 1. The violation of causality implied by this can however be circumvented – in fact the hyperboloid of Fig. 1 is homeomorphic to the cylinder and can be continuously developed into the plane. The curves defined by $\partial/\partial\chi$ can thus be extended into infinite lines and the causality problem avoided. This procedure corresponds to modifying the connectivity-in-the-large properties of the manifold – namely the manifold $H^3 x$ R differs from the above extended manifold by the identification of the point set $(\chi + 2n\pi, \eta, \mathrm{r}, z)$, n = integer[9]. However the extended manifold still contains closed time-lilte curves, as will see.

In the sections $\chi, z =$ const. the space-like, time-like or null-like character of the vector field $\partial/\partial\eta$ depends on the sign of the function $B(\rho)$ (cf. (3.4) for $\varepsilon = 1$). The closed curves defined by the field 8/87 on the two-leaves hyperboloid of Fig. 2 are time-like or null-like for values of p such that $tgh^2(\rho/2) \geq \beta^2/\alpha^2$. Contrary to the case of the sections $\eta, z =$ const. however, the presence of the closed time-like lines cannot be circumvented by modifying the topology without introducing singularities in the space-time manifold. This is the case since a branch of the two-leaves hyperboloid of Fig. 2 is homeomorphic to the cylinder only by the extraction of one point. In other words, to eliminate the causality problems in these sections – by developing the closed coordinate Iines η into open infinite Iines – we must introduce a line of singularities in the spacetime manifold. A possible way to implement the extraction of points is the introduction of a string in the sense of Ref. [11], transforming the point r = 0 into a conical singularity, which contributes to the curvature tensor of the spacetime with a δ-type term having support on the singular line r = 0

(ib) Case $a^2 < \beta^2$

In the sections $\eta, z = const.$ the integral curves of $\partial/\partial\chi$ on the hyperboloid of Fig. 1 are time-like or null whether $tgh^2(\rho/2) \geq \alpha^2/\beta^2$. As in case (a), they can be open into infinite lines by an appropriate modification of the topology, and the causality problem avoided without the introduction of singularities in the space-time. In the sections $\chi, z =$ const. we have $B(\rho) < 0$ for all p, so that no acausal curves occur in these sections. By adopting the topology R^4 and using Maitra's method[12], it is straighforward to show that no closed time-like or null-like curves are present in these space-times. In this sense the spacetimes of case (ib) are said to be causal. The case $\alpha^2 = \beta^2$ can be considered a limiting case of (ib).

(ii) S^3 x R

The geometry here is giveii by (3.4) with $\epsilon = i$. The closed lines defined by the vector fields $\partial/\partial\chi$ or $\partial/\partial\eta$ are time-lilte, depencling on the sign of the functions $A(\rho)$ or $B(\rho)$, respectively. Analogous analysis as in case (i) allows us to conclude that tlie spacetimes with topology S^3 x R can be made free of the above-mentioned acausal curves only by the introcluction of at least two lines of singularities in tlie manifold.

V. Neutrinos and the physical distinction of antipodal universes

As we have discussed, in the coordinate system introducecl in (3.1) the inverse group map operation connectiong g_L and g_R is expressed by a coorclinate inversion. In the realm of pure gravitational interaction the geometries characterizing antipodal spacetimes are indistinguishable if the covariance group of the theory includes improper transformations. The physical distinction between these two spacetimes is possible when we include neutrinos as test particles, because improper transformations are no longer symmetries of tlie system universe-plus-neutrinos of a given helicity. An intuitive argument can be given in this direction. To the passive transformation $\eta \to -\eta$ (or $\chi \to -\chi, \rho \to -\rho$) there corresponds the active transformation $g(e) \to g(-e)$ or equivalently, the change of the sign of the vorticity associated to the velocity field of the matter content of the universe (cf. Appendix B). On the other hand, neutrinos can be an absolute standarcl for the sign of rotation of the universe because, as prescribed by weak interaction processes, a massless neutrinos is an absolute left-handed screw and can be used to define a.n absolute sense of rotation about a given direction. Therefore the active transformation $g(e) \to g(-e)$, or equivalently the change of the sign of vorticity, is not a symmetry of the system universe-plus-neutrino because – to preserve the symmetry – left-lianded neutrinos should then be transformed into right-handed neutrinos, which is a forbidden configuration.

To describe neutrinos we use Dirac spinors from tlie point of view of the tetrad formalism[13]. We restrict our analysis to the cases $\alpha^2 \geq \beta^2$ for the hyperbolic family. In other cases the analysis still apllies, with a proper reinterpretation of neutrinos global modes in regions of the spacetime where the Killing vector fields $\partial/\partial\chi$ and $\partial/\partial\eta$ have space-like and time-like character, respectively.

We consider neutrinos fields in the invariant modes

$$\psi(\chi,\eta,\rho,z) = \phi(\rho,L)e^{-iE\chi-ik\eta-i\nu z} \qquad (5.1)$$

where $\phi(\rho, L)$ is a four-spinor, eigenstate of γ^5,

$$\gamma^5\phi(\rho,L) = L\phi(\rho,L) \quad , \quad L^2 = 1 \qquad (5.2)$$

These modes may be interpreted as eigenstates of energy E, momenta k and ν, and helicity L. The operator γ^5 is proportional to the helicity operator for neutrinos, in the local Lorentz frames of the geometries (3.4).

To proceed let us now define a symmetry transformation of a system from the point of view of passive and active transformations: we say that a coordinate transformation (passive transformation) is a symmetry of the system if there exists a corresponding *active* transformation of the system into another system equivalent (physically indistinguishable) to it. In the present case the passive transformation $\eta \to -\eta$ (or $\chi \to -\chi, \rho \to -\rho$) is a symmetry of the gravitational field (as far as gravitational interaction is concerned) in the sense that it is equivalent to inverting the rotation of the universe. On the system *universe* with geometry $g_R = g(e = 1)$ plus neutrinos described in the modes (5.1) let us perform the following active transformation **P**:

(i) inversion of the rotation of the matter content of the universe (cf. Appendix B).

(ii) inversion of the momentum of neutrinos, associatecl to the Killing symmetry 8/87.

We note that (i) has the effect of changing $g_R = g(e = -1)$ into $g_L = g(e = 1)$. A straightforward computation shows that if $\phi(L)$ is a neutrino solution of Dirac's equation in g_R, then after the operatiori **P** the resrtlting spinor solution $\phi'(L')$ is related to $\phi(L)$ by

$$\phi'(L',k) = \gamma^1\phi(L,-k) \qquad (5.3)$$

The neutrinos $\phi'(L')$ and $\phi(L)$ have obviously opposite helicity. Therefore the physical (active) transformation **P** (which takes g_L into g_R and vice-versa) also transforms neutrino states in g_R into neutrinos states with opposite helicity in g_L. This active transformation corresponds to the passive transformation $\eta \to -\eta$ over the system universe with metric g_L or g_R plus neutrinos.

Analogous analysis applies to tlie transformation $(\chi \longrightarrow -\chi, \rho \longrightarrow -\rho)$ where the corresponding nctive transformation **P'** is given by

(i) above, ancl

(ii) inversion of the sign of the energy of neutrinos, associated to the Killing symmetry $\partial/\partial\chi$.

The resulting transformation of $\phi(L)$ is analogous to (5.3), relating states with opposite sign of energy ancl opposite helicity.

If neutrinos are assumed to have only one type of helicity (as prescribed by weak interaction experiments) the transformations discussed above are no longer a symmetry of the system. Indeed the configuration g_L plus left-handed neutrinos is led under the active transformations **P** or **P'** into the configuration g_R plus right-handed anti-iieutrinos. The system g_L plus left-hancled neutrinos and g_R plus left-handed neutrinos are therefore distinguishable. Of course a pure passive coordinate transformation $\eta \longrightarrow -\eta$ is a mere change of labels ancl can always be performed (even in the presence of weak interaction processes); it corresponds to just changing conventions as, for instance, the definition of the sign of the neutrino helicity, and cannot produce any physically distinct situation.

Finally we must comment tlie following possibility. Suppose that the gravitational field of the interior of two rotating dense bodies with opposite rotation in the same physical universe, can be approximatecl by g_L and g_R. If neutrinos are produced in their interior via weak interaction processes then (i) the emission of neutrinos polarized along a given direction, in one of the bodies, would imply tlie emission of antineutrinos polarized along tlie same direction, in the otlier; (ii) a process producing only neutrinos (or only antineutrinos) in one the bodies would be forbidden in the otlier.

VI. Final conclusions

In this paper we have constructed spacetimes with the topology of the Lie groups $H^3 \times R$ and $S^3 \times R$, by introducing left-invariant metrics g_L and right-invariant metrics g_R on the group manifold. g_L and g_R are related isometrically to each other by the Lie group inverse map, which induces an anti-isomorphism between the Lie algebra of tlie left-invariant vector fields and that of riglit-invariant vector fields. We introduce a coordinate system where the inverse map group operation is expressed by a coordinate inversion. We denote these universes antipodal. Each of these spacetirnes admit by construction a G_5 group of motions. In the family with topology H^3 x R there occurs a particular case $g_L = g_R$, which admits a G_7 group of motions with the 3-dim sections corresponding to H^3 rnaximally symmetric.

The knowledge of the topology allows us to identify immediately the nature of the coordinates used, ancl global causality problems associated to the existence of closed time-like lines of coordinates are easily characterized. The basic result is that in the family with topology H^3 x R some of the acausal curves can be avoided simply by the operation of developing one-leaf liyperboloids on the plane, while other acausal curves can only be avoided by extracting points of the spacetime. The latter procedure corresponds to introducing a line of singularities in the spacetime. In the family with topology S^3 x R, the procedure involves the introduction of two lines of singularities to avoid the acausal curves. These procedures do not guarantee the complete elimination of acausal curves in the spacetimes. However for tlie cases $\alpha^2 \leq \beta^2$ of the farnily H^3 x R, it can be proved that no acausal curves exist.

In appendix B we discuss the several possible physical sources for the universes ancl exhibit coordinate systems in which the metrics g_L and g_R assume the forin of a Gödel-type geometry[14]. The physical sources correspond to a rotating perfect fluid, except for the case $\alpha^2 = p^2$.

A rotating universe and its antipodal are relatecl by tlie inversion of the sign of matter vorticity. We show that massless neutrinos produced via weak interaction processes can be used as a probe to distinguish physically the antipodal universes. This is so because the nctive transformation $g_L \longrightarrow g_R$ also transforms neutrinos in antineutrinos (or vice versa). This result leacls to the following possiblity. If neutrinos are produced in tlie interior of two rotating dense bodies with opposite rotation in the same physical universe (the gravitational field of the interior of the bodies being approximated by g_L and g_R), then a process producing only neutrinos in one of the bodies would be forbidden in the other. The corresponding permitted configuration would be a process producing only antineutrinos.

Appendix A

The class of invariant metrics (3.4) introduced on S^3 x R and H^3 x R can be cast in a simpler form by the use of cylindrical coordinates defined below.

In the coordinate system $(t, \phi, \bar{\rho}, \bar{z})$ determined by the transformation equations

$$\rho = \frac{m}{\varepsilon}\,\bar{\rho} \qquad \eta = \frac{m^2}{4\Omega}\,t - \phi \qquad (A.1)$$

$$\chi = \frac{m^2}{4\Omega}\,t \qquad z = \bar{z}\,,$$

where the new parameters m and R are given by

$$\Omega = \frac{\varepsilon^2\alpha}{\beta} \quad , \quad m = 2\varepsilon/\beta \qquad (A.2)$$

g_R can be expressecl

$$g_R = (dt + H(\bar{\rho})d\phi)^2 - D^2(\bar{\rho})d\phi^2 - d\bar{z}^2 - d\rho^2 \quad (A.3)$$

where

$$H(\bar{\rho}) = \frac{4\Omega}{m^2}\sinh^2\frac{m\bar{\rho}}{2} \qquad D(\bar{\rho}) = \frac{1}{m}\sinh m\bar{\rho} \quad (A.4)$$

In terms of the new parameters (m, Ω) the family of geometries g_R over S^3 x R and H^3 x R are obtained by taking $m^2 < 0$ or $m^2 > 0$, respectively.

In the coordinate system (A.1), g_L does not assume a form symmetric to g_D, as it does in coordinates (χ, η, ρ, z) (cf. (3.4)). However an analogous coordinate system can be introduced where g_L takes the form (A.3) - (A-4). The transformation equations are obtained from (A.1) by the substitution $\eta \to -\eta$ or $(\chi \to -\chi, \rho \to -\rho)$, for instance

$$\rho = -\frac{m}{\varepsilon}\,\bar{\rho} \qquad \eta = \frac{m^2}{4\Omega}\,t - \phi \qquad (A.5)$$

$$\chi = -\frac{m^2}{4\Omega}\,t \qquad z = \bar{z}\,,$$

The spacetimes characterized by a line element of the form (A.3) are denoted in the literature as Gödel-type spacetimes. We have obtained the functions (A.4) by construction, starting from the Lie group structure of the spacetimes, plus the choice (2.13). These functions could be equivalently derived if spacetime homogeneity is assumed for the geometry (A.3), that is, if (A.3) is restricted to admit a simply transitive isometry group. This was the point of view of Ref. [14]. The solution originally proposed by Godel can be recognized in the cylindrical coordinate system to correspond to the particular case $m^2 = 2\Omega^2$. In the coordinates (χ, η, ρ, z) the spacetime (3.4) with $\varepsilon = 1$ $\alpha^2 = 2\beta^2$ is locally isometric to the Gödel solution[10] but with topology H^3 x R. As we have mentioned already, Godel adopted for his solution the topology R^4.

We finally comment on the possible sources of curvature compatible with the metrics (3.4) via the field equations. We start with the model proposed by Gödel[10] which is locally isometric to (3.4), with $\varepsilon = 1$ and $\alpha^2 = 2\beta^2$. The metric is a solution of Einstein field equations with the cosmological constant term, and incoherent matter. For $\varepsilon = 1$, $\alpha^2 \geq 2\beta^2$ (or $0 \leq m^2 \leq 2\Omega^2$ in cylindrical coordinates), and for $\varepsilon = i$ and arbitrary α, β (or $-\infty < m^2 < 0$), the corresponding classes of metrics are solutions of Einstein-Maxwell equations with charged dust[15,16], or neutral dust plus a free electromagnetic field[16]. The admissible range of parameters can be extended by adding to the energy-moment tensor of dust and electromagnetic fields the energy momentum-tensor of a scalar field. For $\alpha^2 = \beta^2$ (or $m^2 = 4\Omega^2$) we have a massless scalar field as the source. The spectrum of Godel type homogeneous solutions was further extended to $\beta > \alpha^2$, $\varepsilon = 1$ (or $m^2 > 4\Omega^2$) in the context of Einstein-Cartan theory. The latter have as source a perfect fluid with spin in rigid rotation. For a general review, see Refs. [14,17].

Except for the case $\alpha^2 = \beta^2$, a perfect fluid is always present as source of curvature. The kinematical parameters of the models are then unambigously defined, associated to the four-velocity of the fluid. The models have zero acceleration, expansion and shear, and a non-null vorticity

$$\omega = \left(\epsilon\varepsilon^2\frac{\alpha}{\beta^2}\right)\frac{\partial}{\partial z}$$

relative to the local compass of inertia (cf. Appendix B). The infinitesinal elements of the perfect fluid in these models are therefore in geodesic motion with constant rigid rotation.

Appendix B

We assume that the metrics g_L and g_R are associated to distinct spacetimes, both defined in the same manifold. We shall prove that the perfect fluid content of these spacetirnes have opposite rotation, relative to the local compass of inertial[1]. For our calculations we consider the vector field basis

$$\begin{aligned}
X_0 &= \frac{1}{\alpha}\left(\frac{\partial}{\partial\chi} - e\frac{\partial}{\partial\eta}\right)\\
X_1 &= \frac{1}{\beta}\left[2\cos(\chi - e\eta)\frac{\partial}{\partial\rho} + \frac{2\varepsilon\sin(\chi - e\eta)}{\sinh\varepsilon\rho}\left(-\sinh^2\frac{\varepsilon\rho}{2}\frac{\partial}{\partial\chi} + e\cosh^2\frac{\varepsilon\rho}{2}\frac{\partial}{\partial\eta}\right)\right]\\
X_1 &= \frac{1}{\beta}\left[-2\sin(\chi - e\eta)\frac{\partial}{\partial\rho} + \frac{2\varepsilon\cos(\chi - e\eta)}{\sinh\varepsilon\rho}\left(-\sinh^2\frac{\varepsilon\rho}{2}\frac{\partial}{\partial\chi} + e\cosh^2\frac{\varepsilon\rho}{2}\frac{\partial}{\partial\eta}\right)\right]\\
X_3 &= \frac{d}{\partial z}
\end{aligned}$$

which define local Lorentz frames in the spacetimes (3.4) (cf. (3.3),(3.3) and (2.13)).

The four velocity field of matter relative to this basis is given is

$$V = V^A X_A = X_0 \ , \qquad (B.1)$$

and the motion of the matter with respect to the frame $\{X_A\}$ is described by the equation

$$\mathop{\mathcal{L}}_{X_0}\mu = 0 \qquad (B.2)$$

where $\mu = \sum_{a=1}^{3}\mu^a X_a$ is a vector orthogonal to X_0 and connecting two neighbouring Auid particles, one of them located at the origin of the frame. C denotes the Lie derivative. Denoting $\dot\mu^a = X_0 p^a$ we obtain from (B.2)

$$\dot\mu^1 = \frac{2e}{\alpha}\,\mu^2 \quad , \quad \dot\mu^2 = \frac{-2e}{\alpha}\,\mu^1 \quad , \quad \dot\mu^3 = 0 \ ,$$

which corresponds to a rotation of period $\pi\alpha/e$ with respect to the inertial frame χ_A. The motion of the frame $\{X_A\}$ along a material world-line determined by X_0 can be calculated as

$$\dot X_A = \mathop{\nabla}_{X_0} X_A \ ,$$

and results in

$$\begin{aligned}
\dot X_1 &= e\left(\varepsilon^2\frac{\alpha}{\beta^2} - \frac{2}{\alpha}\right)X_2\\
\dot X_2 &= -e\left(\varepsilon^2\frac{\alpha}{\beta^2} - \frac{2}{\alpha}\right)X_1\\
\dot X_3 &= 0 = \dot X_0 \ ,
\end{aligned}$$

that is, the plane 1-2 of the frame $\{X_A\}$ rotates with respect to the local compass of inertia with a circular frequency $e\left(\frac{\varepsilon^2\alpha}{\beta^2} - \frac{2}{\alpha}\right)$ (the axes of the local compass of inertia being determined for instance by gyroscopes). Since the shear of the velocity field (A.1) is zero, the rotation of matter relative to the local compass of inertia is given by the angular velocity

$$\omega = e\varepsilon^2\frac{\alpha}{\beta^2}\frac{\partial}{\partial z} \qquad (B.3)$$

which changes sign as $e \to -e$, that is, as we change g_L into g_R and vice-versa.

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