Ion-Acoustic Eigenmodes in a Collisionless Bounded Plasma

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This work is based on the integral - equation method proposed by S. Kuhn [Phys. Fluids 27, 1321 (1984)] for solving the general linearized perturbation problem for a one dimensional, uniform collisionless plasma with thin sheaths, hounded by two planar electrodes. In a first, predominantly analytical applications this method was used to analyse the Pierce diode with a non-trivial external circuit. Here, on the other hand, we apply the method to ion-acoustic eigenmodes in a one- dimensional, collisionless bounded plasma consisting of non-drifting thermal electrons and a cold ion beam propagating through them. This case is of relevance in the context of both Q- and DP-machines. In this case, the eigenfrequencies can no longer be obtained as solutions of an analytically explicit equation but follows as the eigenvalues of a rather complex, homogeneous system of linear integral equations. Via appropriate basis set expansion of all perturbation functions involved, this system is transformed into a system of linear algebraic equations for the expansion coefficients, from which the eigenfrequencies can be obtained as the zeros of the system determinant. Preliminary numerical results include eigenfrequencies and related eigenmode profiles. For the specific situation considered, these modes turn out to be unstable. We are developing now an analytical approximation to simplify our numerical calculation in a strong cooperation with a team in Innsbruck and we expect to apply it to other kind of instabilities occuring in these plasma configurations.

I. The collisionless plane-diode model: linearized basic equations

We consider a one-dimensional diode (Fig. 1) where the surfaces of the (ideally conducting) electrodes are located at x = 0 ("left-hand electrode") and x = L ("right-hand electrode"), and the far ends of the electrodes are connected through an external circuit with specified properties. The intervening space ("interelectrode region", "diode gap") is filled with a collisionless plasma consisting of n, particle species. The particle charge and mass of species $\sigma(\sigma = 1, ...n_s)$ are denoted by e^{σ} and m^{σ}, respectively. Fig. 1 shows the model geometry, with a monotonically decreasing equilibrium distribution as an example. Each physical quantity Q involved is decomposed in the form

$$Q(x,v,t) = \bar{Q}(v) + \tilde{Q}(x,v,t) , \qquad (1)$$

where Q is the (given) time-independent, or "equilibrium" part, and $\tilde{Q}$ is the small amplitude perturbation, which is to be calculated. For the equilibrium state we assume a uniform plasma with two thin sheaths adjacent to the electrodes. This "uniform-plasma, thin-sheath approximation" is of relevance, e.g., for longitudinal modes in a single-ended Q-machine at "moderate" values of the interelectrode bias, whereas for "very high" values the sheath widths may no longer be negligible[4,5]. The small amplitude longitudinal perturbations in the collisionless plane diode are governed by the following set of equations[1]:

$$\frac{\partial \tilde{f}^{\sigma}}{\partial t} + v\frac{\partial \tilde{f}^{\sigma}}{\partial x} = -\frac{e^{\sigma}}{m^{\sigma}}\bar{f}^{\sigma}_{v}\tilde{E} , \qquad (2)$$

linearized Vlasov equations with $\sigma = 1, ...n_s$,

$$\frac{\partial \tilde{E}}{\partial x} = 4\pi \sum_{\sigma=1}^{n_s} e^\sigma \int_{-\infty}^{\infty} \tilde{f}^\sigma dv \qquad (3)$$

(Poisson's equation)

$$\frac{1}{4\pi}\frac{\partial \tilde{E}(x,t}{\partial t} + \sum_{\sigma=1}^{n_s} e^\sigma \int_{-\infty}^{\infty} v\tilde{f}^\sigma dv = \tilde{j}_e(t) , \qquad (4)$$

(equation of total current conservation).

$$-\int_{0_+}^{L_-} \tilde{E}(x,t)dx = \hat{Z}(\tilde{j}_e(t)) , \qquad (5)$$

(external-circuit equation).

$$\tilde{f}_l^\sigma(v>0,t) = \tilde{f}_{lg}^\sigma(v,t) + \sum_{\sigma'=1}^{n_s}\int_{-\infty}^{0} b_l^{\sigma\sigma'}(v,v')\tilde{f}_l^{\sigma'}(v',t)dv' , \qquad (6)$$

(left-hand particle boundary condition)

$$\tilde{f}_r^\sigma(v<0,t) = \tilde{f}_{rg}^\sigma(v,t) + \sum_{\sigma'=1}^{n_s}\int_{0}^{\infty} b_r^{\sigma\sigma'}(v,v')\tilde{f}_r^{\sigma'}(v',t)dv' , \qquad (7)$$

(right-hand particle boundary condition), where $E(x,t) = \tilde{E}(x,t)$ is the electrostatic field, $f^\sigma(x,v,t) = \bar{f}^\sigma(v) + \tilde{f}^\sigma(x,v,t)$ is the velocity distribution function of species **a**, $\bar{f}_v^\sigma(v) \equiv d\bar{f}^\sigma/dv$; $\tilde{j}_e(t)$ is the perturbation of the external circuit current density (i.e., of the external circuit current per unit electrode area), $\hat{Z}$ is the (linear) "impedance operator" of tlie "extended external circuit" (by which we mean the "real" external circuit plus the two electrodes sheaths); $\tilde{f}_{lg}^\sigma$ and $\tilde{f}_{rg}^\sigma$ are externally generated (and, hence, explicitly given) perturbations, and the "generalized reflection- coefficient functions" $b_l^{\sigma\sigma'}(v>0, v'<0)$ and $b_r^{\sigma\sigma'}(v<0, v'>0)$ essentially represent the probabilities for a sheath-bound particle of species a' with velocity v' to "produce" a plasma-bound particle of species a with velocity v. Clearly, $\tilde{f}_l^\sigma(v>0,t)$ and $\tilde{f}_r^\sigma(v<0,t)$ are the perturbations of the distribution functions of the plasma-bound particles at the sheath-plasma boundaries, and lience may be sloppily referred to as "injection distribution functions". Eqs. (1) to (7) constitute a complete system of evolution equations (including boundary conditions) for the perturbations. In (1) they have been transformed into $(2+2n_s)$ time Laplace transforms $\tilde{j}(w)$, $\tilde{E}(x,w)$, $\tilde{f}_l^\sigma(v>0,w)$, and $\tilde{f}_r^\sigma(v<0,w)$,. These "Laplace-transformed integral equations" (Eqs. (37)-(40)) of Ref. [1] are tlie basis of our analysis, and their specific form appropriate to the physical situation considered here (Sec.II) will be given in Sec. III.

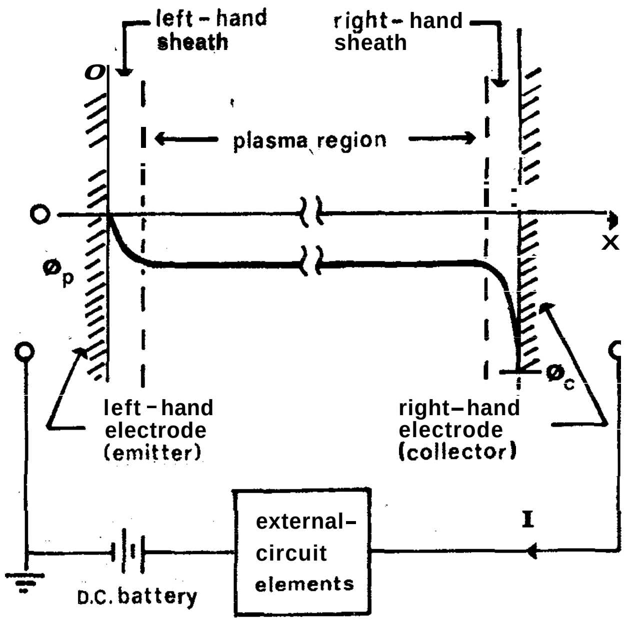

Figure 1: Geonietry of the one-dimensional diode, witli one-minimum equilibrium potential distribution.

II. Special case: ion acoustic oscillations in the negatively biased single cnded Q-machine

For Q-inachines under a wide range of operating conditions, the sheath regions are iisually far less extended than tlie plasma region, so that the uniform plasma, thin-sheath approximation is applicable. Assuming half-Maxwellian emission from the hot plate (whose temperature is T), the injection distribution functions at the left hand plasma houndary are cut-off Maxwellians for both the electrons and the ions[5]. At the right hand houndary plane (x = L−), on the other hand, all ions are absorbed, while all electrons are specularly reflected due to the cold plate sheath (Fig. 1). A quantitative analysis of this model has heen given in Ref. [5]. In this first approach to the problem, we try to simplify our model as much as possible, while still keeping the essential physics. In particular, we approxiinate the electron velocity distrihution function by the "waterbag" function given by

$$\bar{f}^e(v) = \bar{n}_p[U(v+\bar{v}_{cw}^e) - U(v-\bar{v}_{cw}^e)]/(2\bar{v}_{cw}^e) , \qquad (8)$$

where n_p is the equilibrium plasma density, $\bar{v}^e_{cw} = \sqrt{(3kT/m)}$ s the waterbag cutoff velocity, and the distribution by the cold beam is given by

$$\bar{f}^i(v) = \bar{n}_p\delta(v - \bar{v}^i) \; , \qquad (9)$$

with $\bar{v}^i$ as the ion average velocity. For the initial perturbations of the distribution functions we assume,

$$\tilde{f}^e_i(x, v) = 0$$

and

$$\tilde{f}^i_i(x,v) = \delta(x-\xi)\delta(v-\bar{v}^i) \; , \qquad (10)$$

which, although the simplest possihle perturbation, is sufficient to excite all eigenmodes of the system. Since we do not allow for any externally imposed modulation of particle injection througli the boundary planes, we have that $\tilde{f}^e_{lg} = \tilde{f}^i_{lg} = \tilde{f}^e_{ra} = \tilde{f}^i_{ra} = 0$. Most of the generalized reflect on-coefficient functions introduced in Eqs. (6) and (7) vanish ($b^{ei}_l = b^{ie}_l = b^{ei}_r = b^{ie}_r = b^{ee}_l = b^{ii}_l = b^{ii}_r = 0$), and the only non-trivial one is

$$b^{ee}_r(v < 0, v' > 0) = \delta(v + v') \; . \qquad (11)$$

Finally, the external circuit is taken to be a short-circuit, which corresponds to $Z \equiv 0$.

III. Solving the eigenmode problem by means of tlie integral-equation method

With the specifications of Sec. II, the Laplace-transformed integral equations yield the explicit relations

$$\tilde{f}^e_l(v > 0, w) = \tilde{f}^i_l(v > 0, w) = \tilde{f}^i_r(v < 0, w) = 0 \; , \qquad (12)$$

and the "reduced" system of coupled integral equations:

$$\int_{0_+}^{L_-} \tilde{E}(x', w)dx' = 0 \; , \qquad (13)$$

(external-circuit equation)

$$- k_5(x,w)\tilde{j}_e(w) + \tilde{E}(x,w) + \mathcal{S}_0(x,[x'],w)\tilde{E}([x'],w) + \nu^e_{0r}(x,[v<0],w)\tilde{f}^e_r(v,w) = \tilde{k}_8(x,w) \; , \qquad (14)$$

(Poisson's equation)

$$\mathcal{S}^e_r(v<0,[x'],w)\tilde{E}([x'],w) + \tilde{f}^e_r(v,w) = 0 \; , \qquad (15)$$

(right hand electron boundary condition)
from which tlie remaining unknowns functions $\tilde{j}_e(w)$, $\tilde{E}(x,w)$ and $\tilde{f}^e_r(v,w)$ must be determined. Here, k_5 and $\tilde{k}_8$ are known functions, $\mathcal{S}_0$ and $\mathcal{S}_r$ are known x-space operators, and ν^e_{0r} is a known v-space operator. While for tlie full details of these functions and operators the reader has to be referred to Refs. [1] and [3], we present here, as an example, the operator $\mathcal{S}^e_r$:

$$\mathcal{S}^e_r(v<0,[x'],w)\tilde{E}([x'],w) = 2\frac{en_p}{m(\bar{v}^e_{cw})^2}\delta(v+\bar{v}^e_{cw})\int_{0_+}^{L_-} \exp\left(iw\frac{L-x'}{\bar{v}^e_{cw}}\right)\tilde{E}(x',w)dx' \; . \qquad (16)$$

Using appropriate basis-set expansions of all functions and operators involved, Eqs. (13-15) are then transformed into a system of linear algehraic equations for the w-dependent expansion coefficients, whicli can be written as the matrix equation

$$\mathcal{D}(w) \, . \, \tilde{u}(w) = \tilde{k}(w) \; , \qquad (17)$$

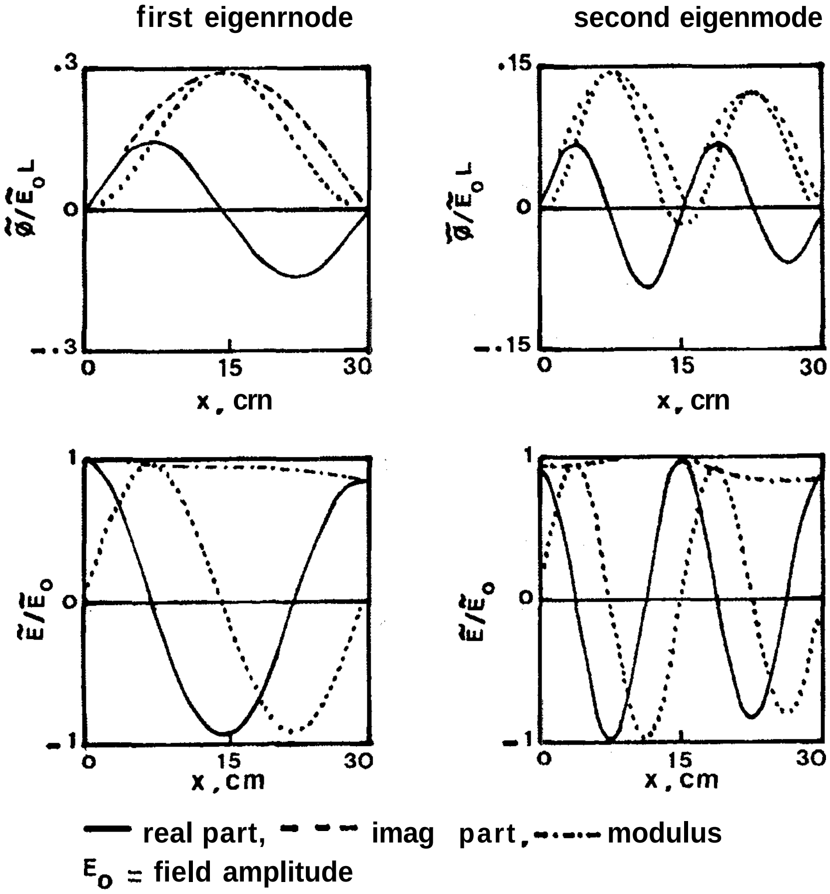

Figure 2: First and second eigenmodes for a negatively biased single-ended Q-macliine.

with

$$\mathcal{D}(w) \equiv \begin{pmatrix} 0 & L\varphi^T & 0^T \\ -k_5 & 1+\mathcal{S}_0 & \nu^e_{0r} \\ 0 & \mathcal{S}^e_r & 1 \end{pmatrix} \tag{18}$$

$$\tilde{k}(w) \equiv \begin{pmatrix} 0 \\ \tilde{k}_s \\ 0 \end{pmatrix} \tag{19}$$

$$\tilde{u}(w) \equiv \begin{pmatrix} \tilde{j}_e \\ \tilde{E} \\ \tilde{f}^e_r \end{pmatrix} \tag{20}$$

Here, $\mathcal{D}(w)$ is the system matrix, $\tilde{k}(w)$ is the vector of known expansion coefficients. The formal solution to our perturbation problem is obtained by inverting **Eq.** (17) and performing the inverse Laplace transformation:

$$\tilde{u}(t) = \int \frac{dw}{2\pi} e^{-iwt}\tilde{u}(w) = \int \frac{dw}{2\pi} e^{-iwt}\mathcal{D}^{-1}(w)\cdot\tilde{k}(w) = \int \frac{dw}{2\pi} e^{-iwt}\frac{\mathcal{A}(w)}{D(w)}\cdot\tilde{k}(w)\ , \tag{21}$$

where $\mathcal{D}^{-1}$ is the inverse matrix, $\mathcal{A}(w)$ is the adjoint matrix, and D is the determinant of $\mathcal{D}(w)$. More specifically, one can show that the n-th eigenmode is defined by

$$\tilde{u}_n(t) \;=\; -iRes_n[\mathcal{D}^{-1}(w)\cdot\tilde{k}(w)e^{-iwt}]$$

$$= \quad -i[\mathcal{A}(w_n)/D'(w_n)] \cdot \tilde{k}(w_n)e^{-iw_nt} \quad (22)$$

where w_n is the n-th solution of the eigenfrequency equation

$$D(w_n) = 0 \quad (23)$$

and Res_n denotes the residue at the pole w = w_n and $D'(w) \equiv dD(w)/d(w)$.

IV. Resulte and discussions

Fig. 2 shows the first and second eigenmodes for a negatively biased single- ended Q-machine characterized by the following parameters: electron-K^+ plasma; interelectrode distance L = 30 cm; hot-plate temperature T = 2200 K; electron emission density $n_{e0}^+ = 10^{10}cm^{-3}$; ion emission density $n_{i0}^+ = 10^8$ cm^{-3} (with neutra ization parameter $\alpha \equiv n_{i0}^+/n_{e0}^+ = 10^{-2}$); plasma density $n_p = 2.10^7$ cm^{-3}; plasmapotential $\phi_p = -1.3$V; electron waterbag cutoff velocity $v_{cw}^e = 2.7.10^5$ cm/s; electron plasma frequency w_{pe} = 2.5.10' s^{-1}; ion beam velocity $V^i = 2.7.10^5$ cm/s; ion plasma frequency $9.4.10^5$ s^{-1}; and externa1 short-circuit. Each of the relevant variable spaces (O ≤ x ≤ L; 0 ≤ V ≤ V and $v_{min} \le v \le 0$) was discretized by a one dimensional mesh consisting of 21 grid-points, and the basis functions were chosen to be square functions whose localization interval essentially coincides with one meshwidth. The eigenfrequencies corresponding to the first and second eigenmodes shown in Fig. 2 have been found to be $w_1 = (3.2204.10^4 + 1.5235.10^3 i)s^{-1}$ and $w_2 = (6.3512.10^4 + 6.6399.10^2 i)s^{-1}$, respectively. The positive imaginary parts mean that these modes are unstable, so that ion acoustic turbulence should be expected in the negatively biased single-ended Q-machine for the parameters considered here. These results are the first concerning unstable modes in this system of operation[3]. We are trying to simplify our numerical calculations, and we expect to include this in a future work, showing how the eigenfrequencies depend on external circuit parameters and to study other instabilities of interest in these plasmas as the Buneman instability.

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