Kinetic Theory for a Dense Gas of Rigid Disks

Wilson Marques Junior
Departamento de Física, Universidade Federal do Paraná
Caixa Postal 19091, 81531-970 Curitiba, Brazil

Received March 14,1994; revised manuscript received June 9, 1994

Eight-field and a four-field kinetic theories are developed for a dense gas of rigid disks based on Enskog's dense gas theory and on Grad's metliod of moments. The constitutive relations for the pressure tensor and for the heat flux are obtained from the transition of the eiglit-field theory to the four-field theory through an iteration method akin to the so called Maxwellian procedure.

I. Introduction

In 1922 Enskog[1] proposed a kinetic theory for a dense gas of hard spherical particles based on a generalization of tlie Boltzmann equation. In this theory we consider only two-body collisions but take into account the difference in position of the colliding particles. In addition, the influence of triple and higher-order collisicns is approximatted by scaling the Boltzmann collision integral with the local equilibrium radial distribu tion function at contact. Comparison of the transport coefficients obtained by the Chapman-Enskog solution[2,3] with those from experimental data shows a reasonable agreement at moderate densities.[4]

The aim of tliis paper is the determination of the laws of Navier-Stokes and Fourier for a moderately dense gas of hard disks using the method of moments of Grad.[5,6] In this method the macroscopic state of the gas is characterized by the fields of mass density, velocity, pressure tensor and heat flux. The corresponding balance equations for tlie basic fields are obtained from a transfer equation derived from tlie Enskog equation. The moments of tlie distribution function and the production terms are calculated by using the distribution function of Grad for a two-dimensional space. The transition from the eight-field theory to the four-field theory (mass density, velocity and temperature) is obtained by means of an iterative scheme akin to the Maxwellian iteration method.[7] As a consequence the constitutive relations for the pressure tensor and heat flux are derived and expressions for tlie transport coefficients of shear viscosity, volume viscosity and thermal conductivity are obtained.

Cartesian notation for tensors is used and two indices between angular parentheses denote the symmetric and traceless part of a tensor.

II. The equation of transfer

The theory of Enskog for a moderately dense gas of rigid disks is based on the two-dimensional Enskog equation

$$\begin{aligned}\frac{\partial f}{\partial t}+c_i\frac{\partial f}{\partial x_i} &= \int\Big[\ \chi(\mathbf{x}+\frac{a}{2}\mathbf{k},t)f(\mathbf{x},\mathbf{c}',t)f(\mathbf{x}+a\mathbf{k},\mathbf{c}_1',t)\\ &- \chi(\mathbf{x}-\frac{a}{2}\mathbf{k},t)f(\mathbf{x},\mathbf{c},t)f(\mathbf{x}-a\mathbf{k},\mathbf{c}_1,t)\ \Big]\,a\,(\mathbf{g}\cdot\mathbf{k})\,d\mathbf{k}\,d\mathbf{c}_1\ ,\end{aligned} \tag{1}$$

which is a generalization of the Boltzmann equation for the single-particle distribution function f $(\mathbf{x},\mathbf{c},t)$. In the above equation $(\mathbf{c},\mathbf{c}_1)$ and $(\mathbf{c}',\mathbf{c}_1')$ are the velocities of two particles before and after the collision, $\mathbf{g}=\mathbf{c}_1-\mathbf{c}$ is the relative linear velocity, a is tlie diameter of the particle and k is the unit vector in the direction of the line which joins the two particles centers at collision, pointing from the particle labeled 1 to the other. Furthermore, external body forces are neglected.

The collision term on the right-hand side of the Enskog Eq. (1) has the two modifications introduced by Enskog in the Boltzmann equation, namely:

(i) the two distribution functions should be evaluated at different points, since the centers of the two particles are separated by a distance $\pm a\mathbf{k}$ at collision (the plus and minus sign refer to collisions that take $(\mathbf{c}, \mathbf{c}_1)$ or $(\mathbf{c}', \mathbf{c}_1')$ as initial velocities);

(ii) the product of the two distribution functions should be multiplied by a factor χ, since the probability of a collision increases for a dense gas. The factor χ may be a function of the density, i.e. of position and time, and should be evaluated at the point of contact of the two disks at collision $\mathbf{x} \pm \frac{a}{2}\mathbf{k}$.

Assuming that the conditions in the gas are sufficiently smooth we expand the functions $\chi(\mathbf{x} \pm \frac{a}{2}\mathbf{k}, t)$, $f(\mathbf{x} + a\mathbf{k}, \mathbf{c}_1', t)$ and $f(\mathbf{x} - a\mathbf{k}, \mathbf{c}_1, t)$ in a Taylor series near $\mathbf{x}$ and neglect the third- and higher-order terms. Then, Enskog's equation may be written as

$$\frac{\partial f}{\partial t} + c_i \frac{\partial f}{\partial x_i} = J_O(ff) + J_I(ff) + J_{II}(ff) \ , \tag{2}$$

where

$$J_O(ff) = \chi \int (f_1' f' - f_1 f)\, a\, (\mathbf{g}\cdot\mathbf{k})\, d\mathbf{k}\, d\mathbf{c}_1 \ , \tag{3}$$

$$J_I(ff) = a \int \left\{ \chi \left[f' \frac{\partial f_1'}{\partial x_i} + f \frac{\partial f_1}{\partial x_i} \right] + \frac{1}{2} \frac{\partial \chi}{\partial x_i} (f_1' f' + f_1 f) \right\} k_i\, a\, (\mathbf{g}\cdot\mathbf{k})\, d\mathbf{k}\, d\mathbf{c}_1 \ , \tag{4}$$

$$\begin{aligned} J_{II}(ff) &= \frac{a^2}{2} \int \left\{ \chi \left[f' \frac{\partial^2 f_1'}{\partial x_i \partial x_j} - f \frac{\partial^2 f_1}{\partial x_i \partial x_j} \right] + \frac{\partial \chi}{\partial x_i} \left[f' \frac{\partial f_1'}{\partial x_j} - f \frac{\partial f_1}{\partial x_j} \right] \right. \\ &+ \left. \frac{1}{4} \frac{\partial^2 \chi}{\partial x_i \partial x_j} (f_1' f' - f_1 f) \right\} k_i k_j\, a\, (\mathbf{g}\cdot\mathbf{k})\, d\mathbf{k}\, d\mathbf{c}_1 \ . \end{aligned} \tag{5}$$

In Eqs. (3)-(5) we have introduced the abbreviations $f = f(\mathbf{x}, \mathbf{c}, t)$, $f_1 = f(\mathbf{x}, \mathbf{c}_1, t)$, $f' = f(\mathbf{x}, \mathbf{c}', t)$, $f_1' = f(\mathbf{x}, \mathbf{c}_1', t)$ and $\chi = \chi(\mathbf{x}, t)$. $J_O(ff)$ with $\chi = 1$ is the usual collision term of the Boltzmann equation for a rarefied gas. $J_I(ff)$ includes only the gradients of first order while $J_{II}(ff)$ contains the gradients of second order and the products of gradients.

The transfer equation follows through the multiplication of Eq. (2) by an arbitrary function $\psi(\mathbf{x}, \mathbf{c}, t)$ and integration over all values of $\mathbf{c}$. This equation of transfer can be written in a simplified form as

$$\frac{\partial \Psi}{\partial t} + \frac{\partial}{\partial x_i} (\phi_i^O + \phi_i^I + \phi_i^{II}) = P_O + P_I + P_{II}. \tag{6}$$

In the above equation Ψ is the density of an arbitrary additive quantity; ϕ_i^O, ϕ_i^I and ϕ_i^{II} are flux densities which correspond to contributions due to the flow of the particles (the kinetic part ϕ_i^O) and the collisional transfer (the potential parts ϕ_i^I and ϕ_i^{II}); and P_O, P_I and P_{II} are the production terms. The quantities that appear in the transfer Eq.(6) are defined by

$$\Psi = \int \psi f\, d\mathbf{c}, \tag{7}$$

$$\phi_i^O = \int \psi\, c_i\, f\, d\mathbf{c}, \tag{8}$$

$$\phi_i^I = \frac{a}{2}\int \chi(\psi' - \psi) f f_1\, k_i\, \mathrm{d}\mathbf{\Gamma}, \tag{9}$$

$$\phi_i^{II} = \frac{a^2}{4}\int \chi\left[(\psi' - \psi)\frac{\partial}{\partial x_j}\left(\ln\frac{f}{f_1}\right) + \frac{\partial(\psi' - \psi)}{\partial x_j}\right] f f_1\, k_i k_j\, \mathrm{d}\mathbf{\Gamma} \tag{10}$$

$$-\frac{a^2}{8}\frac{\partial}{\partial x_j}\int \chi(\psi' - \psi) f f_1\, k_i k_j\, \mathrm{d}\mathbf{\Gamma}, \tag{11}$$

$$P_O = \int\left[\frac{\partial\psi}{\partial t} + c_i\frac{\partial\psi}{\partial x_i}\right] f\,\mathrm{d}\mathbf{c} + \int \chi(\psi' - \psi) f f_1\, \mathrm{d}\mathbf{\Gamma}, \tag{12}$$

$$P_I = \frac{a}{2}\int \chi\left[(\psi' - \psi)\frac{\partial}{\partial x_i}\left(\ln\frac{f}{f_1}\right) + \frac{\partial(\psi' - \psi)}{\partial x_i}\right] f f_1\, k_i\, \mathrm{d}\mathbf{\Gamma}, \tag{13}$$

$$\begin{aligned} P_{II} = {} & \frac{a^2}{4}\int \chi\left[\frac{\partial(\psi' - \psi)}{\partial x_i}\frac{\partial}{\partial x_j}\left(\ln\frac{f}{f_1}\right) + \frac{1}{2}\frac{\partial^2(\psi' - \psi)}{\partial x_i \partial x_j}\right] f f_1\, k_i k_j\, \mathrm{d}\mathbf{\Gamma} \\ & - \frac{a^2}{4}\int \chi(\psi' - \psi)\frac{\partial f}{\partial x_i}\frac{\partial f_1}{\partial x_j}\, k_i k_j\, \mathrm{d}\mathbf{\Gamma} \\ & + \frac{a^2}{8}\int \chi(\psi' - \psi)\left[f_1\frac{\partial^2 f}{\partial x_i \partial x_j} + f\frac{\partial^2 f_1}{\partial x_i \partial x_j}\right] k_i k_j\, \mathrm{d}\mathbf{\Gamma}. \end{aligned} \tag{14}$$

In order to get the transfer Eq. (6) we have transformed all giadients of χ into gradients of integrals and the latter wrre defined as gradients of fluxes as indicated by equations (9) and (10). The introduction of ψ' follows from standard transformations of the unprimed into primed velocities. Moreover, we have introduced the abbreviation $\mathrm{d}\mathbf{\Gamma} = a\,(\mathbf{g}\cdot\mathbf{k})\,\mathrm{d}\mathbf{k}\,\mathrm{d}\mathbf{c}_1\,\mathrm{d}\mathbf{c}$.

III. The eight-field theory

The macroscopic state of a dense gas of rigid disks can be characterized by the eight scalar fields of density ϱ, velocity v_i, kinetic pressure tensor p_{ij} and kinetic heat flux q_i defined by

$$\varrho = \int m\, f\,\mathrm{d}\mathbf{c}, \tag{14}$$

$$v_i = \frac{1}{\varrho}\int m\, c_i\, f\,\mathrm{d}\mathbf{c}, \tag{15}$$

$$p_{ij} = \int m\, C_i C_j\, f\,\mathrm{d}\mathbf{c}, \tag{16}$$

$$q_i = \frac{1}{2}\int m\, C^2 C_i\, f\,\mathrm{d}\mathbf{c}, \tag{17}$$

wliere m is the molecular mass and $C_i = c_i - v_i$ is the peculiar velocity.

In tliis section we are interested in a linearized theory witli first order gradients. Accordingly we disregard the terms ϕ_i^{II} and P_{II} in the transfer Eq. (6) in order to get the balance equations for the basic fields (14)-(17). These balance equations are obtained by choosing ψ in Eq.(6) equal to m, mc_i, mC_iC_j and $mC^2C_i/2$:

(i) Balance of mass:

$$\frac{\partial\varrho}{\partial t} + \frac{\partial\varrho v_i}{\partial x_i} = 0, \tag{18}$$

(ii) Balance of linear momentum:

$$\frac{\partial\varrho v_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\varrho v_i v_j + p_{ij} + p_{ij}^I\right) = 0, \tag{19}$$

(iii) Balance of kinetic pressure tensor:

$$\frac{\partial p_{ij}}{\partial t} + \frac{\partial}{\partial x_k}\left(p_{ij}v_k + p_{ijk} + p_{ijk}^I\right) + \left(p_{jk} + p_{jk}^I\right)\frac{\partial v_i}{\partial x_k} + \left(p_{ik} + p_{ik}^I\right)\frac{\partial v_j}{\partial x_k} = P_{ij} + P_{ij}^I, \tag{20}$$

(iv) Balance of kinetic heat flux:

$$\frac{\partial q_i}{\partial t} + \frac{\partial}{\partial x_j}\left(q_i v_j + q_{ij} + q_{ij}^I\right) + \left(q_j + q_j^I\right)\frac{\partial v_i}{\partial x_j} + \left(p_{ijk} + p_{ijk}^I\right)\frac{\partial v_j}{\partial x_k}$$
$$- \frac{p_{ij}}{\varrho}\frac{\partial}{\partial x_k}\left(p_{jk} + p_{jk}^I\right) - \frac{p_{rr}}{2\varrho}\frac{\partial}{\partial x_k}\left(p_{ik} + p_{ik}^I\right) = Q_i + Q_i^I. \tag{21}$$

In the derivation of Eq. (21) we have used Eq. (19). Moreover, we have introduced in the above balance equations:

$$p_{ij}^I = \frac{a}{2}\int \chi\, m\,(C_i' - C_i)\, f f_1\, k_j\, d\mathbf{\Gamma}, \tag{22}$$

$$p_{ijk} = \int m\, C_i C_j C_k\, f\, d\mathbf{c}, \tag{23}$$

$$p_{ijk}^I = \frac{a}{2}\int \chi\, m\,(C_i' C_j' - C_i C_j)\, f f_1\, k_k\, d\mathbf{\Gamma}, \tag{24}$$

$$q_{ij} = \frac{1}{2}\int m\, C^2 C_i C_j\, f\, d\mathbf{c}, \tag{25}$$

$$q_{ij}^I = \frac{a}{4}\int \chi\, m\,(C'^2 C_i' - C^2 C_i)\, f f_1\, k_j\, d\mathbf{\Gamma}, \tag{26}$$

$$q_k^I = \frac{a}{4}\int \chi\, m\,(C'^2 - C^2)\, f f_1\, k_k\, d\mathbf{\Gamma}, \tag{27}$$

$$P_{ij} = \int \chi\, m\,(C_i' C_j' - C_i C_j)\, f f_1\, d\mathbf{\Gamma}, \tag{28}$$

$$Q_i = \frac{1}{2}\int \chi\, m\,(C'^2 C_i' - C^2 C_i)\, f f_1\, d\mathbf{\Gamma}, \tag{29}$$

$$P_{ij}^I = \frac{a}{2}\int \chi\, m\,(C_i' C_j' - C_i C_j)\, f f_1\, \frac{\partial}{\partial x_k}\left(\ln\frac{f}{f_1}\right) k_k\, d\mathbf{\Gamma}, \tag{30}$$

$$Q_i^I = \frac{a}{4}\int \chi\, m\,(C'^2 C_i' - C^2 C_i)\, f f_1\, \frac{\partial}{\partial x_k}\left(\ln\frac{f}{f_1}\right) k_k\, d\mathbf{\Gamma}. \tag{31}$$

The quantities p_{ijk} and p_{ijk}^I are the kinetic and potential part of the flux of the pressure tensor, respectively, while q_{ij} and q_{ij}^I are the corresponding parts of the heat flux. The potential parts of the pressure tensor and of the heat flux are, respectively, p_{ij}^I and q_i^I, while P_{ij}, P_{ij}^I, Q_i and Q_i^I are production terms.

If a relationsliip can be established between the quantities (22)-(31) (henceforth called constitutive quantities) and the basic fields (14)-(17), the system of balance equations (18) through (21) becomes a system of field equations for ϱ, v_i, p_{ij} and q_i. The objective of the next section is the determination of such relations.

IV. Evaluation of the constitutive quantities

The dependence of the constitutive quantities upon tlie basic fields is attained if we know the distribution function f as a function of ϱ, vi, p_{ij} and q_i. We assume that tlie distribution function may be expressed in tlie following form

$$f = \left(1 - a_i\frac{\partial}{\partial c_i} + \frac{a_{ij}}{2!}\frac{\partial^2}{\partial c_i \partial c_j} - \frac{a_{ijk}}{3!}\frac{\partial^3}{\partial c_i \partial c_j \partial c_k} + \ldots\right) f_0, \tag{32}$$

where

$$f_\circ = \frac{\varrho}{m}\left(\frac{m}{2\pi k_B T}\right)\exp\left(-\frac{mC^2}{2k_B T}\right) \tag{33}$$

is the two-dimensioiial Maxwell distribution function, T is the absolute temperature and k_B is the Boltzmann constant. In the kinetic theory the specific interna1 energy of the gas ε is defined directly in terms of the peculiar velocities by the relation

$$\varrho\varepsilon = \frac{1}{2}\int mC^2 f \mathrm{d}c. \qquad (34)$$

According to the principle of equipartion of energy, the specific internal energy ε of a monatomic gas consisted of rigid disks in a two-dimensional space is equal to $k_B T/m$, so that the absolute temperature is defined by

$$T = \frac{m}{2\varrho k_B}\int m C^2 f \,\mathrm{d}c. \qquad (35)$$

The coefficients ai, a_{ij}, a_{ijk}, ..., which depend on x and t but not on c, are first, second and higher order symmetric tensor functions. The determination of these coefficients follows from the use of the definitions of the basic fields (14)-(17). By neglecting tlie contribution of the traceless part of a_{ijk} and all higher terms of the expansion, we obtain after some calculations that ai $= 0$, a,, $= 0$, $a_{(ij)} = p_{(ij)}/\varrho$ and ai,, $= 2\, q_i/\varrho$, where $p_{(ij)} = p_{ij} - (p_{rr}/2)\,\delta_{ij}$ is tlie pressure deviator, i.e., the traceless part of tlie pressure tensor. Insertion of these coefficients irito Eq. (32) leads to the expression

$$f = f_0\left[1 + \frac{p_{(ij)}}{2\varrho}\left(\frac{m}{k_B T}\right)^2 C_i C_j + \frac{q_i}{\varrho}\left(\frac{m}{k_B T}\right)^2\left(\frac{mC^2}{4k_B T} - 1\right) C_i\right]. \qquad (36)$$

Insertion of the distribution function (36) into Eq. (23) and Eq. (25) leads after integration over all values of c to

$$p_{ijk} = \frac{1}{2}\left(q_i\delta_{jk} + q_j\delta_{ik} + q_k\delta_{ij}\right), \qquad (37)$$

$$q_{ij} = 2\,\varrho\left(\frac{k_B T}{m}\right)^2 \delta_{ij} + 3\,\frac{k_B T}{m}\, p_{(ij)}. \qquad (38)$$

The evaluation of the fluxes p^I_{ij}, q^I_i, p^I_{ijk} and q^I_{ij} and of the production terms P_{ij}, Q_i, P^I_{ii} and Q^I_i is more involved. By neglecting all nonlinear terms we get

$$p^I_{ij} = \varrho b\chi\left(\varrho\frac{k_B}{m} T\,\delta_{ij} + \frac{1}{2}\,p_{(ij)}\right), \qquad (39)$$

$$q^I_i = \frac{3}{4}\,\varrho b\chi q_i, \qquad (40)$$

$$p^I_{ijk} = \frac{1}{2}\,\varrho b\chi\left(q_i\delta_{jk} + q_j\delta_{ik} + \frac{1}{2}\,q_k\delta_{ij}\right), \qquad (41)$$

$$q^I_{ij} = 2\,\varrho b\chi\,\frac{k_B T}{m}\left(\varrho\frac{k_B}{m} T\,\delta_{ij} + \frac{7}{8}\,p_{(ij)}\right), \qquad (42)$$

$$P_{ij} = -2\,a\chi\,\frac{\varrho}{m}\left(\frac{\pi k_B T}{m}\right)^{1/2} p_{(ij)}, \qquad (43)$$

$$Q_i = -\,a\chi\,\frac{\varrho}{m}\left(\frac{\pi k_B T}{m}\right)^{1/2} q_i, \qquad (44)$$

$$P^I_{ij} = \varrho b\chi\left(\varrho\frac{k_B}{m} T\,\frac{\partial v_{\langle i}}{\partial x_{j\rangle}} + \frac{1}{4}\,\frac{\partial q_{\langle i}}{\partial x_{j\rangle}}\right) \qquad (45)$$

$$Q^I_i = \frac{-}{2}\,\varrho^2 b\chi\left(\frac{k_B}{m}\right) \qquad \partial x_i, \qquad (46)$$

where b $= (\pi a^2/2m)$. The quantity ϱb is called the co-area of the particles.

V. The linearized field equations

We insert the calculated values of the constitutive quantities (37)-(46) into tlie balance Eqs. (18)-(21) to get a system of field equations for the basic fields (14)-(17) or, equivalently, for ϱ, v_i, T, $p_{(ij)}$ and q_i. Disregarding all nonlinear terms in v_i, $p_{(ij)}$, q_i, $\partial\varrho/\partial x_i$, $\partial T/\partial x_i$ and their derivatives it follows a system of linearized field equations which reads

$$\frac{\partial\varrho}{\partial t} + \varrho\frac{\partial v_i}{\partial x_i} = 0, \qquad (47)$$

$$\varrho\frac{\partial v_i}{\partial t} + \varrho\frac{k_B}{m}(1 + \varrho b\chi)\frac{\partial T}{\partial x_i} + \frac{k_B T}{m}\left(1 + 2\varrho b\chi + \varrho^2 b\chi\frac{\partial\chi}{\partial\varrho}\right)\frac{\partial\varrho}{\partial x_i} + \left(1 + \frac{1}{2}\varrho b\chi\right)\frac{\partial p_{(ij)}}{\partial x_j} = 0, \qquad (48)$$

$$\varrho\frac{k_B}{m}\frac{\partial T}{\partial t} + \left(1 + \frac{3}{4}\varrho b\chi\right)\frac{\partial q_i}{\partial x_i} + \varrho\frac{k_B}{m}T\,(1 + \varrho b\chi)\,\frac{\partial v_i}{\partial x_i} = 0, \qquad (49)$$

$$\begin{aligned}\frac{\partial p_{\langle ij\rangle}}{\partial t} &+ \left(1+\frac{3}{4}\varrho b\chi\right)\frac{\partial q_{\langle i}}{\partial x_{j\rangle}} + 2\varrho\frac{k_B}{m}T\left(1+\frac{1}{2}\varrho b\chi\right)\frac{\partial v_{\langle i}}{\partial x_{j\rangle}} \\ &= -2a\chi\frac{\varrho}{m}\left(\frac{\pi k_B T}{m}\right)^{1/2} p_{\langle ij\rangle},\end{aligned} \tag{50}$$

$$\begin{aligned}\frac{\partial q_i}{\partial t} &+ 2\varrho\left(\frac{k_B}{m}\right)^2\left(1+\frac{3}{4}\varrho b\chi\right)\frac{\partial T}{\partial x_i} + \frac{k_B T}{m}\left(1+\frac{3}{4}\varrho b\chi\right)\frac{\partial p_{\langle ij\rangle}}{\partial x_j} \\ &= -a\chi\frac{\varrho}{m}\left(\frac{\pi k_B T}{m}\right)^{1/2} q_i.\end{aligned} \tag{51}$$

Eq. **(49)** and **Eq. (50)** represent tlie trace and the traceless part of Eq. (20), respectively.

VI. The four-field theory

We characterize now a macroscopic state of a dense gas of rigid disks by the four scalar fields of density ϱ, velocity v_i and temperature T. For this purpose, the kinetic pressure tensor p_{ij} and the kinetic heat flux q_i must be expressed in terms of the basic fields ϱ, v_i and T. In a linearized theory the constitutive relations for p_{ij} and q_i are given in terms of tlie basic fields ϱ, vi, T and of its gradients. Hence in the derivation of the balance equations for ϱ, vi and T we must consider the terms ϕ_i^{II} and P_{II} of the transfer **Eq. (G),** since ϕ_i^{II} liolds a gradient of the distribution function.

The balance equations for the basic fields ϱ, v_i and T are obtained by choosing ψ in tlie transfer Eq. (6) equal to m, mc_i and mc^2/2:

(i) Balance of mass:

$$\frac{\partial \varrho}{\partial t} + \frac{\partial \varrho v_i}{\partial x_i} = 0, \tag{52}$$

(ii) Balance of linear momentum:

$$\frac{\partial \varrho v_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\varrho v_i v_j + p^*_{ij}\right) = 0, \tag{53}$$

(iii) Balance of energy:

$$\frac{\partial}{\partial t}\left(\varrho\frac{k_B}{m}T + \frac{\varrho v^2}{2}\right) + \frac{\partial}{\partial x_i}\left[\left(\varrho\frac{k_B}{m}T + \frac{\varrho v^2}{2}\right)v_i + q^*_i + p^*_{ij}v_j\right] = 0. \tag{54}$$

In the above equations p^*_{ij} and q^*_i are tlie total pressure tensor and the total heat flux, respectively. They are defined as

$$p^*_{ij} = p_{ij} + p^I_{ij} + p^{II}_{ij} \qquad \text{and} \qquad q^*_i = q_i + q^I_i + q^{II}_i, \tag{55}$$

where

$$p^{II}_{ij} = \frac{a^2}{4}\int \chi m\left(C'_i - C_i\right) f f_1 \frac{\partial}{\partial x_k}\left(\ln\frac{f}{f_1}\right) k_j k_k \,\mathrm{d}\boldsymbol{\Gamma}, \tag{56}$$

$$q^{II}_i = \frac{a^2}{8}\int \chi m\left(C'^2 - C^2\right) f f_1 \frac{\partial}{\partial x_j}\left(\ln\frac{f}{f_1}\right) k_i k_j \,\mathrm{d}\boldsymbol{\Gamma}. \tag{57}$$

In tlie four-field theory the constitutive quantities are tlie total pressure tensor p^*_{ij} and the total heat flux qf. If we know the dependence of them on the basic fields ϱ, vi and T, the system of balance Eq. (52) through Eq. (54) becomes a system of field equations for ϱ, v_i and T.

Let us first evaluate the quantities p_{ij}^{II} and q_i^{II} which are defined by Eq. (56) and Eq. (57). Since we are looking for a linearized theory with gradients of first order in ϱ, v_i and T, we substitute the Maxwellian distribution function (33) into Eq. (56) to Eq. (57) to get after integration:

$$p_{ij}^{II} = -\frac{1}{2}\varrho^2\chi\frac{a^3}{m}\left(\frac{\pi k_B T}{m}\right)^{1/2}\left(\frac{\partial v_r}{\partial x_r}\delta_{ij}+\frac{\partial v_{(i}}{\partial x_{j)}}\right), \quad (58)$$

$$q_i^{II} = -\frac{1}{2}\varrho^2\chi\frac{a^3}{m^2}\left(\frac{\pi k_B^3 T}{m}\right)^{1/2}\frac{\partial T}{\partial x_i}. \quad (59)$$

Since the constitutive quantities p_{ij}^I and q_i^I are known functions of ϱ, T, $p_{\langle ij\rangle}$ and q_i (see Eq. (39) and Eq. (40)), we need only to evaluate $p_{\langle ij\rangle}$ and q_i as functions of ϱ, v_i and T. In order to achieve this objective, we use the Eqs. (50) and (51) and a method akin with the Maxwellian iteration procedure.[7] For the first iteration step we insert the equilibrium values $p_{\langle ij\rangle}=0$ and $q_i = 0$ in the left-hand side of Eq. (50) and Eq. (51) and get the first iterated values of $p_{\langle ij\rangle}$ and q_i on the riglit-hand side, i.e.,

$$p_{\langle ij\rangle} = -2\frac{\mu_0}{\chi}\left(1+\frac{1}{2}\varrho b\chi\right)\frac{\partial v_{\langle i}}{\partial x_{j\rangle}} \quad \text{and} \quad q_i = -\frac{\lambda_0}{\chi}\left(1+\frac{3}{4}\varrho b\chi\right)\frac{\partial T}{\partial x_i}, \quad (60)$$

where

$$\mu_0 = \frac{1}{2a}\left(\frac{mk_BT}{\pi}\right)^{1/2} \quad \text{and} \quad \lambda_0 = \frac{2}{a}\left(\frac{k_B^3T}{m\pi}\right)^{1/2} \quad (61)$$

are, respectively, the coefficients of shear viscosity and thermal conductivity for an ideal gas of hard disks. Hence the insertion of Eq. (58), Eq. (59) and Eq. $(60)_{1,2}$ into Eq. $(55)_{1,2}$ yields

$$p_{ij} = \left(p-\eta\frac{\partial v_r}{\partial x_r}\right)\delta_{ij} - 2\mu\frac{\partial v_{\langle i}}{\partial x_{j\rangle}} \quad \text{and} \quad q_i = -\lambda\frac{\partial T}{\partial x_i}. \quad (62)$$

Eq. $(62)_{1,2}$ are the mathematical expressions of the laws of Navier-Stokes and Fourier, respectively. In these equations the pressure p and the coefficients of volume viscosity η, shear viscosity μ and tliermal conductivity λ are given by

$$p = \varrho\frac{k_B}{m}T(1+\varrho b\chi), \quad (63)$$

$$\eta = -\frac{\mu_0}{\chi}(\varrho b\chi)^2, \quad (64)$$

$$\mu = \frac{\mu_0}{\chi}\left[1+\varrho b\chi+\left(\frac{1}{4}+\frac{2}{\pi}\right)(\varrho b\chi)^2\right], \quad (65)$$

$$\lambda = \frac{\lambda_0}{\chi}\left[1+\frac{3}{2}\varrho b\chi+\left(\frac{9}{16}+\frac{1}{\pi}\right)(\varrho b\chi)^2\right]. \quad (66)$$

The above expressions for the pressure p and for the coefficients of volume viscosity η, shear viscosity μ and thermal conductivity λ are the same as those obtained by Gass[8] using tlie Chapman-Enskog procedure.

The variation of tlie various transport coefficients with the co-area of the particles ϱb is shown graphically in Figs. 1, 2 and 3. This is achieved by using for tlie factor χ the following virial expansion[9]

$$\begin{aligned}\chi &= 1+0.7820\varrho b+0.5322(\varrho b)^2+0.3338(\varrho b)^3\\ &+ 0.1992(\varrho b)^4+0.1141(\varrho b)^5+\ldots \quad . \end{aligned} \quad (67)$$

We note that tlie theory of Enskog predicts a uniform behavior of μ/μ_0 and λ/λ_0 as a function of the co-area of the particles ϱb. Moreover, the transport coefficients for a rarefied gas of hard disks are recovered, if put in the above equations ϱb equal to zero.

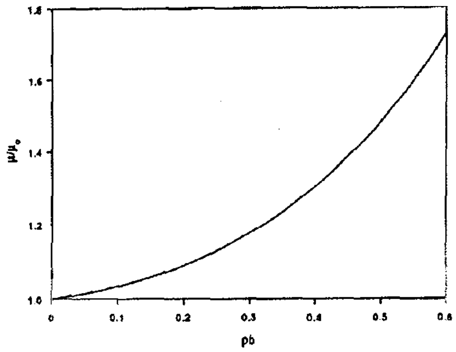

Figure 1: μ/μ_0 as **a** function of ϱb.

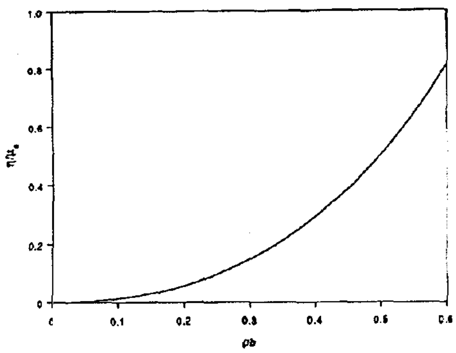

Figure 2: η/μ_0 as **a** function of ϱb.

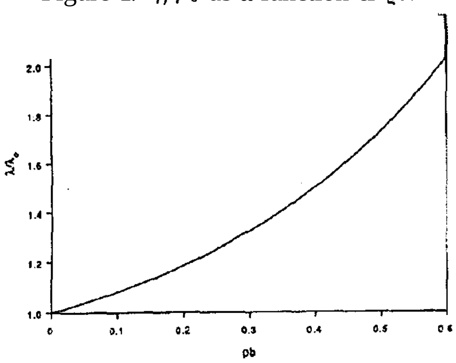

Figure *3:* λ/λ_0 *as* ***a*** function of ϱb.

Acknowledgment

The author would like to express his appreciation to G.M. Kremer for helpful discussions.

References

1. D. Enskog, *Kinetic Theory of Heat Conduction, Viscosity and Self-Diffusion in Compressed Gases and Liquids.* English translation in Kinetic Theory, edited by S.G. Brush, (Pergamon, Oxford, 1972) vol. III, p. 226.
2. S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases,* (Cambridge University Press, Cambridge, 1970).
3. J. H. Ferziger and H. G. Kaper, *Mathematical Theory of Transpor2 Processes in Gases,* (North-Holland, Amsterdam, 1972).
4. J. V. Sengers, Int. J. Heat Mass Transfer **8,** 1103 (1965).
5. H. Grad, Commun. Pure **Appl.** Math. **2,** 331 (1949).
6. G. M. Kremer and E. Rosa Jr., J. Chem. Phys. 89, 3240 (1988).
7. E. Ikenberry and C. Truesdell, J. Rat. Mech. Anal. 5, 1 (1956).
8. D. M. Gass, J. Chem. Phys. 54, 1898 (1971).
9. F. H. Ree and W. G. Hoover, J. Chem. Phys. 46, 4181 (1969).