A Theorem on Space-Times of Class One

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We show a differential necessary condition on the intrinsic geometry of a space-time embedded in E_5.

I. Introduction

The process of embedding has the aim of looking at the space-time from "outside", i. e. studying R_4 from the geometry of a (necessarily pseudo-Euclidean) space of a higher dimension. When such a process is possible, a very valuable tool is available for the analysis of the geometric structure of the 4-space. The class of embedding generates an invariant characterization of the solutions of the Einstein equations, alternative to that of Petrov, which provides a technique to construct solutions: Karmarkar[1], for example, obtained the inner Schwarzschild solution when studying the spherically symmetric class one metrics[2]; also, Singh and Pandey[3] found a generalization of the Lamaitre method.

A disadvantage of this embedding technique is the lack of physical meaning of the new degrees of freedom represented by the Ricci vectors and the second fundamental forms[4]: some authors[5-7] have tried to use them to describe properties of elementary particles in a curved space-time. Barbashov-Nesterenko[8] and Phát[9] have use the embedding process in non-linear models (solitons, strings, etc.) and in the problem of the gravitational energy[10]. These reasons (among others) make us to consider interesting to study R_4 as a hyper surface of E,, n $\geq$ 5.

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A space time is said to be of class one (i.e., accepts embedding into E_5) if and only if there exist the second fundamental form tensor $b_{ac} = b_{ca}$ satisfying the Gauss-Codazzi equations[11-13]

$$R_{ijkc} = \epsilon(b_{ik}b_{jc} - b_{ic}b_{jk}) \ , \tag{1.a}$$

$$b_{ij;k} = b_{ik;j} \tag{1.b}$$

where E $= \pm 1$, R_{abcd} is the R_4 curvature tensor and $;k$ stands for the covariant derivative. In the literature some necessary conditions on the intrinsic geometry (generated by the metric g_{ac}) of the space-time embedded into E_5 can be found: for instance, Collinson[14] proved that[15]:

$$^*R^{jm}_{\ \ it} R_{jmpa} = -\frac{K_2}{12}\eta_{itpa} \ ,$$

$$K_1 \equiv^* R^{abcd}R_{abcd} = 0 \tag{2.a}$$

where K_1, and K_2 are the Lanczos invariants[15]:

$$K_2 =^* R^{*ijkr}R_{ijkr} \tag{2.b}$$

η_{itpa}, $^*R_{atrq}$ and $^*R^{*ijkr}$ being the Levi-Civita tensor, the simple dual and the double dual of the Riemann tensor, respectively. As another example, in Ref. [16] it is shown that in every class one R_4 the following relations holds:

$$R_{abcd}\left(\frac{R}{2}R^{abcd}+\frac{1}{2}R^{cdij}R^{ab}_{ij}-R^{ac}R^{bd}-R^{cdaj}R_{j}^{\ b}-R^{cija}R^{d}_{i\ \ j}{}^{b}\right)=0\ , \tag{3}$$

where $R_{ab} = R^{c}_{abc}$ is Ricci tensor and $R = R^{e}_{\ c}$ is the scalar curvature.

Conditions Eq. (2.a) and Eq. (3) are *necessary* because if an E_4 does not fulfill them it is then impossible for it to be embedded into E_5, and the conditions are *algebraic* because they were derived through the use of Eq. (1.a), which does not contain covariant derivatives. Hence, the problem arises[17-19] of finding a necessary differential condition (i.e., where also Eq. (1.b) is taken into account).

In the next section a necessary condition (including covariant derivatives) is shown for every space-time of class one. Such a condition does not contain the same information as in a purely algebraic condition and, consequently, we hope it proves to be useful to study R_4 embedded into E_5. As an application we prove that Gödel metric[2,13,20-22] is not of class one.

II. Necessary differential condition

Our aim in this section is to show that the intrinsic geometry of any 4-space embedded into E_5 satisfies an identity, generated by the Gauss and Codazzi equations, that is a necessary differential condition in which Eq. (1.b) (and, therefore Bianchi's identities for the curvature tensor) is contained.

Indeed, if $G_{mb} = R_{mb} - \frac{R}{2}g_{mb}$ is Einstein tensor, our result is then:

$$\eta^{tpij}\left[\frac{K_2}{24}R_{;i}g_{jk}+\left(\frac{K_2}{4}-2R_{mc}G^{mc}\right)R_{ki;j}+(R_{;j}R_{kmci}+2R_{ri;k}R^{r}_{\ mcj})G^{mc}\right]+2^{*}R^{tpa}_{\ \ \ k;r}R^{r}_{\ mca}G^{mc}=0\ , \tag{4}$$

and it has not been located in the literature. If a given R_4 violates Eq. (4) its embedding into E_5 is not possible; note that, unlike Eqs. (2.a) and (3), Eq. (4) includes covariant derivatives of several tensor objects. To prove Eq. (4) it is enough to substitute Eq. (1.a) into it and use Eq. (1.b) together with the relevant identity of Goenner[17,23] and González[11] written as[12,13,19,22,24]:

$$pb_{ij}=\frac{K_2}{48}y_{ij}-R_{imnj}G^{mn}\ , \tag{5.a}$$

where

$$p^2=-\frac{\epsilon}{6}\left(\frac{R}{24}K_2+R_{imnj}G^{ij}G^{mn}\right)\geq 0\ . \tag{5.b}$$

Since no empty space ($G_{mc} = 0$) accepts embedding into E_5 [12,13,21], then we will always have $G_{mc} \neq 0$ in Eq. (4).

As an application, Gödel's metric[20] allows to show that Eq. (4) does not posses the same information than Eqs. (2.a) and Eq. (3). Indeed, its line element is

$$ds^2=-(dx^1)^2-2\exp(x^4)dx^1dx^2-\frac{1}{2}\exp(2x^4)(dx^2)^2++(dx^3)^2+(dx^4)^2 \tag{6}$$

and lengthy but straightforward calculations allow to verify the fulfillment of Eq. (2.a) and Eq. (3). However, this cosmological model does not satisfies Eq. (4)

and so it can be concluded that Eq. (6) is not of class one[2,13,19,21,22,25]. The only explicit embedding known so far was published by Rosen[26] where the author embeds the Godel metric into E_{10}. As an original result we give its embedding into E_9. Indeed, let

$$\begin{aligned} z^1 &= \frac{1}{\sqrt{2}} e^{x^4} \cos x^2 \ , \\ z^2 &= \frac{1}{\sqrt{2}} e^{x^4} sin x^2 \ , \\ z^3 &= \sqrt{2} e^{x^4/2} \cos \frac{1}{2}(x^1 + x^2) \ , \\ z^4 &= \sqrt{2} e^{x^4/2} sin \frac{1}{2}(x^1 + x^2) \ , \\ z^5 &= \sqrt{2} e^{x^4/2} \cos \frac{1}{2}(x^1 - x^2) \ , \\ z^6 &= \sqrt{2} e^{x^4/2} sin \frac{1}{2}(x^1 - x^2) \ , \\ z^7 &= x^3 \ , \\ z^8 &= x^1 \ , \\ z^9 &= \rho + \frac{1}{2} \ln \frac{\rho - 1}{\rho + 1} \ , \end{aligned}$$

with

$$\rho \equiv \left(1 + \frac{1}{2} e^{2x^4}\right)^{1/2} \ , \qquad (7.a)$$

then, it is simple to prove that Eq. (6) can be written as:

$$ds^2 = -(dz^1)^2 - (dz^2)^2 - (dz^3)^2 - (dz^4)^2 + (dz^5)^2 + (dz^6)^2 + (dz^7)^2 - (dz^8)^2 + (dz^9)^2 \ , \qquad (7.b)$$

showing that Gödel's solution may be studied as a subspace of E_9. It is known[2] that Eq. (6) admits embedding into E7 and E_8 although nobody has ever tried to construct the corresponding functions z^r. Nevertheless, it is still unknown[19] if Eq. (6) can be embedded into E_6.

III. Comments

We have indicated the result Eq. (2.a), due to Collinson[14], only **as** an example of a algebraic necessary condition for space-times embedded into Eg, but this condition does not participate in the derivation of Eq. (4). This means that, in order to construct Eq. (4), it suffices to consider Eqs. (1.a), (1.b), (5.a) and (5.b). If the Gauss equation (1.a) is introduced into the Lanczos' invariant K_1 [27], it follows immediately that $K_1 = 0$ as indicated in Eq. (2.a) and this is valid for every space-time of class one independently of Einstein-Maxweel fields. Furthermore, the general identity[27]:

$$^{*}R^{*a}_{\ \ bcd} R^{ibcd} = \frac{K_2}{4} g^{ai} \qquad (8)$$

and eq. (1.a) imply Collinson's other relation Eq. (2.a).

In conclusion, using Godel metric, we have shown that our differential neccesary condition, Eq. (4), contains more information that Eq. (2.a) and Eq. (3) since the Codazzi equation (1.b) is involved in its derivation.

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