Stability of Multidimensional Spacetime with Timelike Internal Dirnensions

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We study the stability of a class of cosmological solutions in multidimensional theories where the extra dimensions are supposed to be timelike. We consider more specifically the scalar inodes since they are related to the density perturbations in a later phase of the evolution of the Universe. Analytical solutions for the perturbed equations are found and their general behaviour is studied.

I. Introduction

The main idea of the Kaluza-Klein theories is to consider that the fundamental interactions could be explained by the presence of extra hidden dimensions. Obviously, these extra dimensions have to be compact in order to justify the fact that they cannot be observed. In general, the extra dimensions are considered as spacelike, since it is believed that timelike internal, compact dimensions should bring stability and causality problems[1]. The causality problem must appear due to the presence of closed timelike curves; the unstability would be caused by the presence of tachyons in the spectrum of mass of the compactified configuration. But, if the typical scale of the compactified dimensions are is Planck one, the violation of causality can not be observed in the ordinary phenomenology. Moreover, the presence of tachyons is doubtfull, since even the definition of mass is not a trivial subject in more general geometries. So, we think that the objections quoted above must be seen with caution.

The aim of this article is to study, at classical level, the stability of cosmological solutions where the extra dimensions are supposed to be timelike. The introduction of timelilre extra dimensions is not a new subject in the literature[2-5]. But, a proper investigation of the consequences for the stabifty of the final configuration when such dimensions are introduced has not been carried on, at least to our knowledge. Due to the observations made above, we will not treat the causality problem. The stability study will be carried out by the traditional perturbation analysis of Lifshitz and Khalatnikov[6,7]. More specifically, we will consider just the scalar modes, since they are related to the density perturbations in a later phase of the history of the Universe, which must lead to the formation of galaxies and cluster of galaxies.

This article is organized as follows: in the first part, we look for background solutions with a power law form; in a second section, we introduce small perturbations around these background solutions, and we determine the exact solutions for the perturbed equations as the asymptotic behaviour of these solutions. In the last section, we present a more detailed analysis of the results obtained.

II. The background solutions

We consider a Eagrangian density

$$\mathcal{L} = \sqrt{\tilde{g}}\tilde{R} \quad , \tag{1}$$

representing the gravity in an arbitrary n-dimensional manifold whose metric is taken under the form:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + b(t)^2 d\sigma^2 \tag{2}$$

The indices μ and ν run from one to four and do is the differential section of the d-dimensional compact internal space (d = n − 4) with a constant curvature k. The signature of $g_{\mu\nu}$ is (+ - - -), so that the internal dimensions are timelike. Using (2), we obtain the effective

Lagrangian in four dimensions:

$$\mathcal{L} = \sqrt{-g}\left(uR + \frac{d-1}{d}\frac{u_{;\rho}u^{;\rho}}{u} + d(d-1)u^{-\frac{2}{d}+1}\right) . \quad (3)$$

The corresponding field equations are:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{d-1}{du^2}(u_{;\mu}u_{;\nu} - \frac{1}{2}g_{\mu\nu}u_{;\rho}u^{;\rho}) +$$
$$+\frac{1}{u}(u_{;\mu;\nu} - g_{\mu\nu}\Box u) + \frac{1}{2}d(d-1)g_{\mu\nu}u^{-\frac{2}{d}} \quad ; \quad (4)$$
$$\Box u = d(d-1)u^{\frac{-2}{d}+1} \quad . \quad (5)$$

Here $\Box = \nabla_\mu\nabla^\mu$ and $u = b^d$.

We will look for cosmological solutions under the form of power law. So, first we take the flat Robertson-Walker four dimensional metric,

$$ds^2 = dt^2 - a(t)^2(dx^2 + dy^2 + dz^2) \quad . \quad (6)$$

The field equations reduce to the following differential equations for $a(t)$ and $u(t)$:

$$3(\frac{\dot a}{a})^2 = -\frac{d-1}{2d}(\frac{\dot u}{u})^2 + \frac{\ddot u}{u} - \frac{1}{2}d(d-1)u^{-\frac{2}{d}} \quad (7)$$
$$2\frac{\ddot a}{a} + (\frac{\dot a}{a})^2 = \frac{d-1}{2d}(\frac{\dot u}{u})^2 + \frac{\dot a}{a}\frac{\dot u}{u} + \frac{1}{2}d(d-1)u^{-\frac{2}{d}} \quad (8)$$
$$\ddot u + 3\frac{\dot a}{a}\dot u = d(d-1)u^{\frac{2}{d}+1} \quad . \quad (9)$$

These equations admit a quite simple solution under the form of a power law function:

$$a \propto cte \quad ; \quad u \propto t^d \quad . \quad (10)$$

The expressions *(10)* represent a Minkowskian four-dimensional space-time, where the inverse of the effective gravitational coupling grows linearly with time, following the Dirac's hypothesis. It is remarkable that no power law type solution is possible when the internal dimensions are spacelike.

III. Perturbation Analysis

The stability study of the solutions *(10)* will be carried out through the use of standard techinques. We introduce into the Eq. *(1)* the following quantity:

$$\tilde g_{AB} = g_{AB} + h_{AB} \quad ; \quad A, B = 1, ..., n \quad . \quad (11)$$

Where g_{AB} represents the background solution, and h_{AB} is a small perturbation of this solution. Then, it is possible to linearize Eqs. (4,5), obtaining a system of coupled linear differential equations involving the components of h_{AB}. The coefficients of these equations are determined by the background solution.

The general expressions for this perturbation analysis were given in Ref. [9]. Here, we are interested essentialy in the scalar modes, since they are linked, in a later phase of the Universe, with the density perturbation. More specifically, we look for equations relating the quantities,

$$\frac{1}{a^2}h_{kk} = h \quad ; \quad k = 1, 2, 3 \quad ; \quad (12)$$
$$\frac{1}{b^2}h_{aa} = H \quad ; \quad a = 1, ..., d \quad . \quad (13)$$

In what follows, we will not consider any dependence of these functions on the internal coordinates; the spatial behaviour of the functions h and H will be that of a plane wave. The expressions are *essentially* the same as in Ref. [9]:

$$\ddot H + 2\frac{\dot b}{b}\dot H = -\ddot h - 2\frac{\dot a}{a}\dot h \quad ; (14)$$
$$\ddot H + (2d\frac{\dot b}{b} + 3\frac{\dot a}{a})\dot H + \frac{q^2}{a^2}H = -d\frac{\dot b}{b}\dot h \quad . \quad (15)$$

Where, q is the wavenumber of the perturbation. We can combine these equations in order to obtain a third order differential equation for $H(t)$ which is given by:

$$\dddot H + \left((d+1)\frac{\dot b}{b} - \frac{\ddot b}{\dot b} + 5\frac{\dot a}{a}\right)\ddot H + \left(\frac{q^2}{a^2} - 2\frac{\dot b^2}{b^2} + \frac{\ddot b}{b} + (4d+9)\frac{\dot a\dot b}{ab} +\right.$$
$$\left. +3\frac{\dot a^2}{a^2} - 3\frac{\dot a\ddot b}{a\dot b} + 3\frac{\ddot a}{a}\right)\dot H + \left(\frac{q^2}{a^2}(\frac{\dot b}{b} - \frac{\ddot b}{\dot b}) + 2(d-1)(\frac{\ddot b\dot b}{b^2} - (\frac{\dot b}{b})^3) + 2\frac{\dddot b}{b} +\right.$$
$$\left. +6\frac{\ddot a}{a}\frac{\dot b}{b} + 6(\frac{\dot a}{a})^2\frac{\dot b}{b} + 4\frac{\dot a}{a}\frac{\ddot b}{b} + 4(d-1)(\frac{\dot b}{b})^2\frac{\dot a}{a} - 2\frac{\ddot b^2}{\dot b b}\right)H = 0 \quad . \quad (16)$$

We integrate this equation through the redefinitions $H(t) = \frac{\omega}{t}$ and $w = t^{\frac{3-d}{2}}\lambda$, which lead to Bessel's equation:

$$\ddot{\lambda} + \frac{\dot{\lambda}}{t} + \left(1 - (\frac{d+1}{2t})^2\right)\lambda = 0 \quad . \tag{17}$$

The final solutions for $H(t)$ and $h(t)$ are:

$$d = even:$$

$$H = \frac{1}{t}\left(\int_0^t (C_1\tau^{\frac{3-d}{2}}(J_{\frac{d+1}{2}}(q\tau) + C_2 J_{\frac{-(d+1)}{2}}(q\tau))d\tau + C_3\right) \quad ; \tag{18}$$

$$h = \int_0^t \tau^{\frac{1-d}{2}}(C_1 J_{\frac{d+1}{2}}(q\tau) + C_2 J_{\frac{-(d+1)}{2}}(q\tau))d\tau + + \int_\tau^t \int_0^\tau \theta^{\frac{-(1+d)}{2}}(C_1 J_{\frac{d+1}{2}}(q\theta) + C_2 J_{\frac{-(d+1)}{2}}(q\theta)d\theta)d\tau + \frac{C_3}{d}t \quad ; \tag{19}$$

$$d = odd:$$

$$H = \frac{1}{t}\left(\int (C_1\tau^{\frac{3-d}{2}}(J_{\frac{d+1}{2}}(\tau) + C_2 N_{\frac{(d+1)}{2}}(\tau))d\tau + C_3\right) \quad ; \tag{20}$$

$$h = \int_0^t \tau^{\frac{1-d}{2}}(C_1 J_{\frac{d+1}{2}}(q\tau) + C_2 N_{\frac{(d+1)}{2}}(q\tau)d\tau) + + \int_\tau^t \int_0^\tau \theta^{\frac{-(1+d)}{2}}(C_1 J_{\frac{d+1}{2}}(q\theta) + C_2 N_{\frac{(d+1)}{2}}(q\theta)d\theta)d\tau + \frac{C_3}{d}t \quad . \tag{21}$$

In these expressions, J_ν and N_ν represent Bessel and Neumann functions of order v respectively.

One of the problems of using the synchronous gauge is the existence of a residual gauge freedom. So, it is possible that some of the modes found before are not physical. In fact, if we consider the coordinate transformation $\tilde{x}^A = x^A + \xi^A$, the synchronous gauge is preserved if

$$\xi^0 = \Psi(x^k) \quad , \tag{22}$$

$$\xi^i = \Psi(x^k)_{,i}\int \frac{dt}{a^2} + \chi^{,i}(x^k) \quad , \tag{23}$$

$$\xi^p = 0 \tag{24}$$

where $i, k = 1, 2, 3$ and $p = 1, .., d$. Under this transformation, the quantities h and H transform as,

$$h \rightarrow h - 2\Psi_{;k;k}\int \frac{dt}{a^2} - 2\chi_{;k;k} + 2d_1\frac{\dot{a}}{a}\Psi \quad , \tag{25}$$

$$H \rightarrow 2d_2\frac{\dot{b}}{b}\Psi \quad . \tag{26}$$

By an appropriate choice of the function Ψ we can eliminate the modes represented by the constant C_3 from the solutions (18-21); are gauge modes, and we will not consider them anymore in the rest of our analysis.

We may evaluate the asymptotic behaviour of the above solutions. For $t \rightarrow 0$, the solutions (18-21) assume the form

$$H = C_1 t^3 + C_2 t^{1-d} \quad , \tag{27}$$

$$h = C_1 t^3 + C_2 t^{2-d} \quad , \tag{28}$$

and when $t \rightarrow \infty$, the solutions given by

$$
\begin{aligned}
d &= even: \\
H &= C_1 t^{\frac{-d}{2}} cos(t - \frac{dn}{4}) + C_2 t^{\frac{-d}{2}} sin(t + \frac{dn}{4}) \quad , \qquad (29)\\
h &= C_1\left(a_1 t^{\frac{-d}{2}} cos(qt - \frac{dn}{4}) + a_2 t^{\frac{-(d+2)}{2}} sin(qt - \frac{d\pi}{4})\right) + \\
&\quad + C_2\left(a_3 t^{\frac{-d}{2}} sin(qt + \frac{dn}{4}) + a_4 t^{\frac{-(d+2)}{2}} cos(qt + \frac{d\pi}{4})\right) \quad ;
\end{aligned}
$$

$$
\begin{aligned}
d &= odd: \\
H &= C_1 t^{\frac{-d}{2}} cos(t - \frac{dn}{4}) + \\
&\quad + C_2 t^{\frac{-d}{2}}\left(b_1 cos(qt - \frac{dn}{4}) + b_2 sin(qt - \frac{dn}{4}) + \right. \\
&\quad \left. + b_3 cos(qt + \frac{d\pi}{4}) + b_4 sin(qt + \frac{d\pi}{4})\right) \quad , \qquad (30)\\
h &= C_1\left(c_1 t^{\frac{-d}{2}} cos(qt - \frac{d\pi}{4}) + c_2 t^{\frac{-(d+2)}{2}} sin(qt - \frac{d\pi}{4})\right) + \\
&\quad + (r_1 t^{\frac{-d}{2}} + r_2 t^{\frac{-(d+2)}{2}}) \left(c_3 sin(qt - \frac{dn}{4}) + c_4 cos(qt - \frac{dn}{4}) + \right. \\
&\quad \left. + c_5 sin(qt + \frac{dn}{4}) + c_6 cos(qt + \frac{d\pi}{4})\right) \quad . \qquad (31)
\end{aligned}
$$

In these expressions, the r's, a's, b's and c's are constants.

The main difference with respect to Ref. [9] is that the order of the Bessel functions now depends on the dimension of the spacetime. When the internal dimensions are spacelike and there is no curvature in the internal space, we find Bessel functions of first order. We note, at the same time, that in the asymptotic limit $t \rightarrow \infty$ the modes for $H(t)$ and $h(t)$ are all decreasing modes.

IV. Conclusions

We have analyzed the growth of scalar perturbations when we consider multidimensional theories with extra timelike dimensions and a curvature in the internal space. We have integrated explicitly the equations for the perturbed quantities; the solutions are represented by integrals of Bessel's functions, whose order depends on the dimension of the internal space. So, qualitatively, the solutions found here are quite different of those found in Ref. [9], where the order of the Bessel's functions are always equal to one.

Concerning the stability of the background solutions, our results show very interesting properties. The stability may be more clearly analyzed regarding the asymptotic behaviour when $t \rightarrow \infty$. The modes corresponding to the internal space are, in this limit, always decreasing, and show that this space is stable against small scalar perturbations. In the four-dimensional Minkowskian space-time there is one mode that grows linearly with time, while all others are decreasing. This could be interpreted as a kind of instability. However, this is a gauge mode and it can be eliminated by a coordinate transformation. The background is stable with regard to scalar perturbations.

We have treated this perturbation problem using the synchronous gauge. It is known that this choice does not fix completely the gauge, since there is a residual gauge transformation that preserves the synchronous gauge. This residual gauge freedom can be used to eliminate modes that are, in fact, coordinate artifacts. Another way to cope with this problem is by using the Bardeen's variable[8], which are completely gauge independent. However, when we consider a multidimensional spacetime which divide itself into two spaces (so, a kind of anisotropic spacetime) the generalization cf the Bardeen's formalism is far from being trivial. Attempts of introducing this formalism in multidimensional theories[10] have considered a diiect extension of the four-dimensional problem to an isotropic miiltidimensional space taking into account semi-classical effects. We hope to be able in the future to reanalyse the problem treated here using a general gauge-independent formalism.

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