Complexity and Extrema1 Dynamics

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Often, the dynamics of large systems in Nature is governed by extrema1 events rather than typical events. This type of dynamics causes the system to self-organize into a critical, complex state. The approach to the complex attractor can be related to the behavior in the complex state itself. This behavior may be characterized as a fractal in d spatial plus one temporal dimension. The temporal l/f-like signal, and the spatial fractal structure appear along different cuts in this fractal.

I. Introduction

Many large dynamical systems in Nature are complex. This means that they have information on a wide range of length and time scales. Fractal spatial structures[1] and a temporal signal with a $1/f$ type noise[2] are signatures of complexity.

Complex systems evolve in an intermittent way rather than in a smooth, gradual manner. Fluctuations in economics, as for instance the variations in cotton prices[3], appear to follow Levy distributions, with power-law tails describing intermittent large events. The distribution of earthquake magnitudes obeys the Gutenberg-Richter law[4], which is essentially a power-law with respect to the energy release. Biological evolution takes place in terms of punctuations[5], where many species become extinct and new species emerge, interrupting quiet periods of apparent equilibrium, known as stasis. Power-law distributions are quite different from Gaussian distributions, which have exponential tails and a vanishing probability for large fluctuations.

Both Gaussian and Levy distributions appear as limiting distributions when many independent random variables are added together. Roughly, if the distribution of individual events falls off sufficiently rapidly, with a non-diverging second moment, the limiting distribution is Gaussian. The largest fluctuations come about because many individual events happen to pull in the same direction. If the individual events have a diverging second moment, or, perhaps, even a diverging average size, the limiting distribution could be a Levy distribution. This may happen when the probability distribution for individual events has a powerlaw distribution, $\mathcal{P}(E) = E^{-\alpha}$, with $\alpha < 3$.

In contrast to the Gaussian case, the tails of a Levy distribution are formed by individual events rather than the sum of many events. Thus, when studying large, catastrophic events in a large system (which could for example be an economy with many interacting agents) one can identify the individual event, and therefore the particular source and mechanism for this event. As pointed out by Mandelbrot, this might cause an observer to discard the event as "atypical" when studying the statistics of fluctuations, and thus throw the baby out with the bath water. The remaining events trivially follow Gaussian statistics.

How can single identifiable events possibly have a statistical explanation? As a result of this almost universal reaction to find reasons for large, specific events, economists tend to look for specific mechanisms for large fluctuations; geophysicists look for specific configurations of fault zones etc. leading to catastrophic earthquakes. Biologists look for externa1 sources, such

*Department of Physics, SUNY at Stony Brook, Stony Brook, New York 11794 as a meteor hitting the earth, in order to explain large extinction events; scientists across the board are reluctant to view large events as statistical phenomena. Even physicists may view the large scale structure of the universe as the consequence of some particular dynamics, rather than as one accidental outcome of random dynamics.

II. Self-organized criticality

There is another explanation. It has been argued that large dynamical systems tend to self organize into a critical state with events of all sizes[6]. In principle, each large event has a specific source that can be blamed. For example a particular grain of sand can land on a specific spot on a sand-pile, causing a large avalanche or sandslide, or a particular tree in a forest fire can be the first to catch fire, or a particular fault segment can be the first to rupture during an earthquake that might some day destroy California, or a particular car can be the first to slow down on a highway and cause a humongous traffic jam. The important point, however, is that even if this particular initiating event were prevented, large events would eventually start for some other "reason" at some other place in the system. No local attempt to control large fluctuations can be successful. Of course, from a selfish viewpoint one might be able to shift the disaster to neighbors!

What is the origin of the process of self-organization to the complex state? Self-organized critica1 systems appear to have one feature in common: the dynamics is governed by sites with extremal values of the signal[7], such as a force acting on the crust of the earth, or the slope of a sand-pile, or the oldest tree in the forest, rather than by some average property of the field. Nothing happens before some threshold is reached; the system is frozen. But when the least stable part reaches its threshhold, this may trigger a burst of activity in the system. Thus, one might argue that complexity is due to the fact that the dynamics of Nature is driven by atypical, extremal features[8]. Biological evolution is driven by exceptional mutations leading to a species with superior ability to proliferate; similarly, a new invention at the leading edge of technology may lead to a breakthrough causing ripples throughout the economy. It has been speculated that the introduction of program trading might have caused the "crash" of stock prices in October 1987.

For some models, the randomness of the driving mechanism may hide the fact that the dynamics is driven by extremal events. For a forest-fire model[9] of self-organized criticality, it can be proven that the dynamics is entirely driven by burning the largest forests, or the oldest trees, despite the fact that both tree growth and fires occur randomly. The growth of diffusion-limited aggregation clusters (DLA) takes place on a fractal structure with extremal values of the growth potential field[10]. For the early models of SOC there is, still, little understanding of the process of self-organization. Even though, there are many analytical results for sand pile models[11], there are none which elucidate the approach to the critica1 attractor, and which describe the avalanches in the critical state. The same goes for the earthquake models[12], where there are no analytical results whatsoever, and all insight stems from numerical simulations. One exception is the Drossel-Schwabl version of a forest fire model of SOC which has been solved exactly in one dimension[13]. It is not yet clear, though, how this solution fits into the general scheme of self-organized criticality.

Recently there has been a breakthrough in our understanding of how extremal dynamics leads to complex behavior[14–18]. This is due to the invention of some particularly simple models representing phenomena as diverse as surface growth[19], traffic jams[20], and possibly biological evolution[21]. One outstanding virtue of these models is that they can be attacked by analytical methods rather than relying on computer simulations alone. We have found that the complex state can be viewed as a fractal in d spatial plus one temporal dimension. The approach to the critical attractor is governed by a "gap-equation" for the divergence of avalanches. The exponent for the resulting algebraic approach to the complex state can be derived from the geometrical properties of the space-time fractal. So can the power spectrum of the local signal. For some models, the critical behavior can be mapped to known models. In addition, new ways of measuring well known exponents, such as β and χ for surface growth models have been derived by relating these exponents to the space time fractal. In the following we shall describe our results, for one particular model, namely, the Bak-Sneppen (BS) toy model of evolution, but many of the results can be straightforwardly generalized to a large class of models, including the Sneppen model[19] of interface depinning and invasion percolation[22].

III. Complexity in the BS model

III.I The stationary complex state

The BS model was proposed as a coarse grained description of evolution[21]. For a discussion of its possible relevance to biological evolution, see Refs. [21] and [23] and N. Jan's article in this volume. Related ideas can be found in Kauffman's book[24]. The BS model is, perhaps, the simplest model with extremal dynamics that exhibits self-organized criticality.

The model is defined as follows: random numbers, f_i, are assigned independently to sites on a d-dimensional lattice. They are chosen from a uniform distribution between zero and one, $\mathcal{P}(f)$. At each step, the site with the lowest random number, $f_{\min}$ is chosen. Then that site and its 2d nearest neighbors are assigned new random numbers which are also drawn from $\mathcal{P}$. The dynamics is extremal because the global minimum is selected at each time step. After many updates have occurred, the system reaches a statistically stationary state in which the density of random numbers in the system vanishes for f below f_c and is uniform above f_c. The activity pattern in the BS model, shown in Fig. 1, is a fractal in both space and time. This fractal exhibits spatio-temporal complexity. The activity at any particular site is recurrent in time; it is a "fractal renewal process"[25].

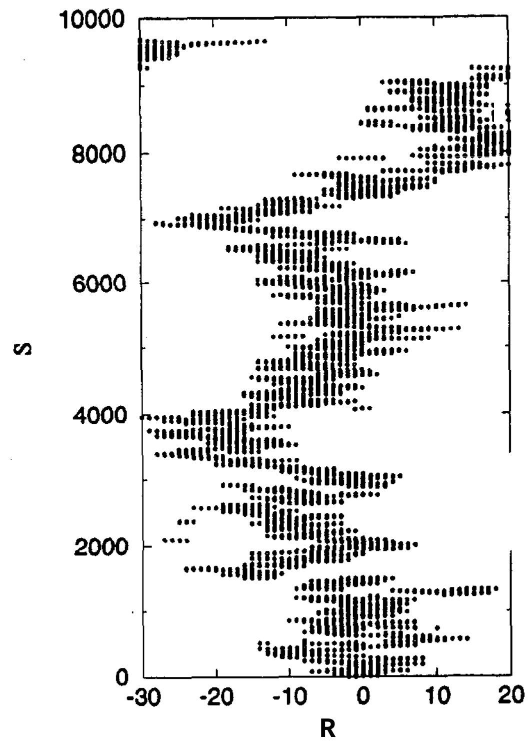

Figure 1: Fractal cluster of activity in the d = 1 BS evolution model. The horizontal axis is a row of lattice sites and the vertical axis is sequential time. Note the appearance of holes of all sizes between subsequent returns of activity to a given site.

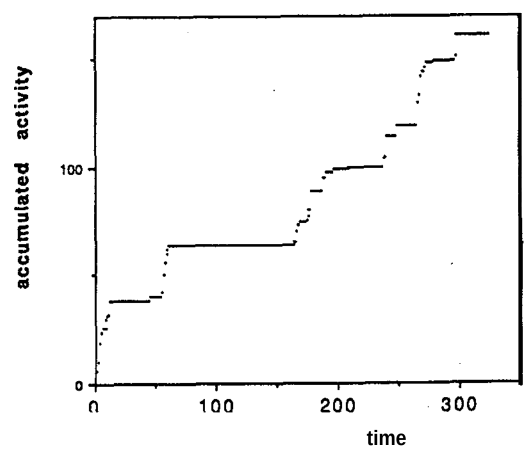

Figure 2: Accumulated number of active events at a single site. The curve exibits punctuated equilibrium behavior, with large periods of stasis interrupted by intermittent bursts.

The fractal has a dimension D which has been measured to be D = 2.32 [14] for the one dimensional BS model. Fig. 2 shows the integrated activity along the time axis. It represents the accumulated number of changes of a single site vs time. It exhibits punctuated equilibrium[5] behavior, with large periods of relatively small activity interrupted by intermittent bursts. The bursts are due to many changes taking place in a small time interval. Fig. **3** shows a snapshot of f vs. species space X during a burst.

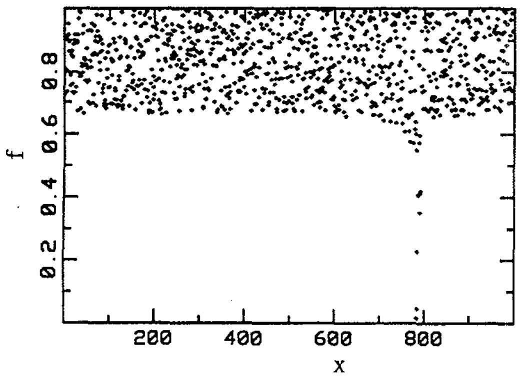

Figure **3:** Snapshot of f vs X during an avalanche. Most of f_s are above the critical value. The active sites with f $< f_c$ are clustered around X = 800.

The scaling behavior of recurrent activity in time can be described by the fractal dimension, $\tilde{d}$, of the return points on the one-dimensional time axis. This is related to the exponent τ_{first} for the first return time, i. e. the size distribution of holes between active points, by the relation $\tilde{d} = \tau_{\text{first}} - 1$ [16]. The power spectrum, $S(f)$, of the local activity is the Fourier transform of the autocorrelation function of the activity. This is another power law, $S(f) \sim f^{-\tilde{d}}$ [16]. A signal with non-trivial power law correlations is often called "one over f noise" even if the exponent is not one. The exponent $\tilde{d}$, and consequently the power spectrum, can be related to the fractal dimension D of the cluster. The total size of the cluster, S, scales with the spatial extent, R, as S $\sim R^D$. We consider the cluster to be a composition of R^d one-dimensional fractals in time. Consequently

$$S \sim R^D \sim R^d R^{D\tilde{d}} \qquad (1)$$

and the power spectrum becomes $S(f) \sim f^{-(1-d/D)} \sim f^{-0.57}$ in one dimension.

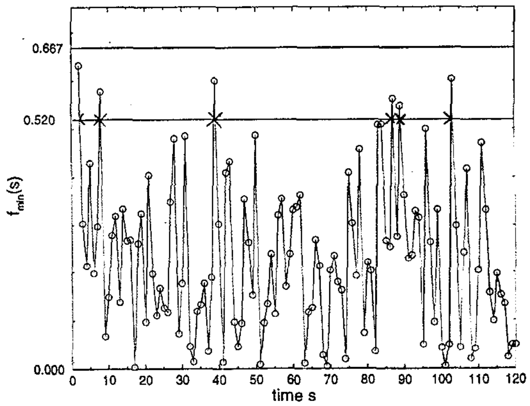

Figure 4: f_{min} vs time in the stationary critical state. New f avalanches start whenever f_{min} exceeds f. Avalanches are indicated by arrows.

Fig. 4 shows the value of f_{min} vs time in the stationary SOC state. By definition, f_{min} is always less than f_c. We define "f_0 - avalanches", representing bursts of activity in the following way. f_0 - avalanches start whenever f_{min} exceeds a parameter f_0; they are defined for any $0 < f_0 < f_c$. For a number of time steps the value of f_{min} will be less than f_0. The avalanche ends, after a total of S time steps, when f_{min} again exceeds f_0. As f_0 approaches the critical value, f_c the size distribution of avalanches approaches a power law

$$N(S) \sim S^{-\tau} \qquad (2)$$

The exponent r has been measured to be $\tau \approx 1.1$ for d = 1; r $\approx$ 1.27 for d = 2. Fig. 5 shows $N(S)$ in one dimension. These value for r and D are identical to tliose for directed percolation within numerical accuracy[14]. Below criticality the avalanches have a cutoff,

$$S_{co} \sim (f_c - f_0)^{-1/\sigma} , \qquad (3)$$

with $a \approx 0.34$ in one dimension. The average duration of avalanches below criticality scales as

$$< S > \sim (f_c - f_0)^{\frac{2-\tau}{\sigma}} \qquad (4)$$

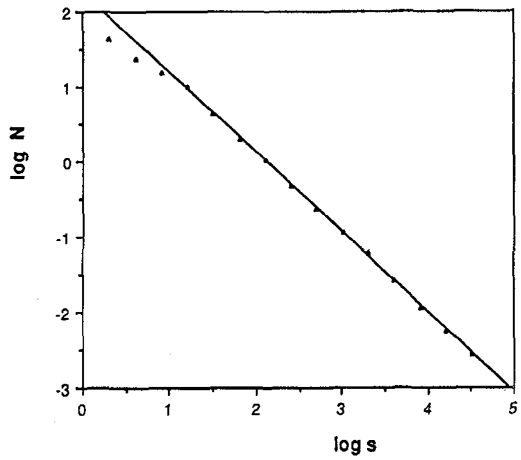

Figure 5: Distribution of avalanches for the BS branching process at the critical point, $f_c = 0.667$.

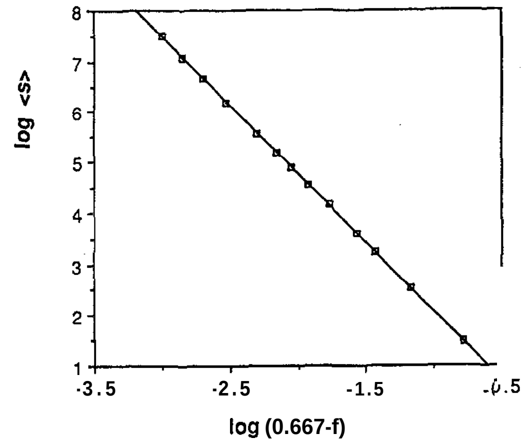

Figure 6: Average size of avalanches vs $f_c - f$. The slope yields $\gamma = 2.70$.

Note that while the exponents, $\gamma = (2-\tau)/\sigma$, σ, etc. describe properties of the system off criticality, the exponents τ and D describe properties of the stationary state at criticality. The f_0- avalanches can be studied directly through an equivalent BS branching process[14]. An initial site, say $x = 0$, is chosen and updated according to the rule for the BS model, that is the site and its two neighbors are assigned new random numbers. Then the next site with the smallest ramlom number is found and updated, and so on. All sites with $f_i < f_0$ are called "active". The values of f_i on the passive vacuum sites are irrelevant and not stored. The process continues until all f's exceed f_0. A single fo avalanche is identical to a single BS branching process at f_0. Thus, the properties of the f_0-avalanches for the BS model approaching its critical attractor are mapped to the properties of the BS branching process off criticality. This speeds up the numerical simulations while completely eliminating finite-size effects. Fig. 6 shows $< S >$ vs $(f_c - f_0)$, with $f_c = 0.667$. The slope of the curve yields an exponent $\gamma \approx 2.7$. Superficially, one might suspect that since the exponents τ and D are given by directed percolation, the exponent γ should also be identical to the exponent γ for directed percolation, or whatever the appropriate model at criticality turns out to be. In the following we shall see that this is not so.

III.2. The self-organization process

In contrast to earlier models of SOC, the process of self- organization to the critical point in the BS model is completely understood[14]. The critical stationary state is approached algebraically, through transient states. Let us consider the situation where the distribution of f initially is uniform in the interval (0,1) in a d-dimensional system of linear size L. The first value of f to be chosen for updating is $\mathcal{O}(L^{-d})$. Eventually, after s time steps, a gap $G(s)$ opens up in the distribution of f's. We define the current gap $G(s)$ as the maximum of all the $f_{\min}(s')$ for all $0 \leq s' \leq s$. By definition $G(s)$-avalanches (as defined above) separate instances when the gap $G(s)$ jumps to its next higher value. The average size of the jump in the gap is $(1-G(s))/L^d$. Consequently, the growth of the gap versus time s is described by the following equation:

$$\frac{\partial G(s)}{\partial s} = \frac{1 - G(s)}{L^d < S >_{G(s)}} \qquad (5)$$

As the gap increases, so does the average avalanche size, which eventually diverges as $G(s) \rightarrow f_c$ whereupon the model is critical and the process achieves stationarity. In the limit $L \rightarrow \infty$, the density of sites with $f < f_c$ vanishes, and the distribution of f is uniform above f_c. The gap equation (5) defines the mechanism of approach to the self-organized critical attractor. In order to solve it we need to determine precisely how the average avalanche size $< S >_{G(s)}$ diverges as the critical state is approached. Inserting Eqn. (4) into Eqn. (5) and integrating, we find

$$\Delta f = f_c - G(s) \sim \left(\frac{s}{L^d}\right)^{-\frac{1}{\gamma-1}} , \qquad (6)$$

which shows that the critical point (Af = 0) is approached algebraically with an exponent 0.58 in one dimension.

Consider the stationary SOC state, and let $\mathcal{P}(f)$ be the probability to have a f-avalanche separating consecutive points in time where the minimum random number chosen is greater than f. An avalanche of spatial extent r leaves on average $< r^d >$ sites with new uncorrelated random numbers between f and 1. If f is increased by a small amount df, the differential in the probability that an f $+ df$-avalanche will not end at the same time as the f-avalanche is determined by the probability that any of the new random numbers generated by the avalanche fall within df of f. This probability is $< r^d > df/(1-f)$. We thus obtain the rigorous result:

$$\frac{d \ln \mathcal{P}}{df} = \frac{< r^d > f}{1-f} , \quad \text{for} \quad f < f_c . \qquad (7)$$

Since close to f_c, $\mathcal{P}(f) \sim \Delta f^{\gamma}$, where Af $= f_c - f$, the fundamental relation

$$\gamma = \frac{< r^d > \Delta f}{1-f} \qquad (8)$$

holds for Af $\to 0$. Surprisingly, the quantity γ appears as a constant rather than a critical exponent. It is the number of random numbers between f and f_c left behind by a f—avalanche that has died. Thus $< r^d > \sim$ A$f^{-\gamma_\perp}$ where $\gamma_\perp = 1$. Using r $\sim s^{1/D}$, where s is the time extent of the avalanche, gives

$$< r^d > \sim \int^{\Delta f^{-1/\sigma}} s^{d/D} s^{-\tau} ds \sim \Delta f^{\frac{\tau - d/D - 1}{\sigma}} . \qquad (9)$$

As a result, o $= 1 + d/D - \tau$ and y $= (2-\tau)/\sigma = \frac{2-\tau}{1+d/D-\tau}$. Also, the spatial correlation length exponent $\nu_\perp = \frac{1}{D\sigma} = \frac{1}{D+d-D\tau}$.

Inserting the Padé approximate values $\tau = 1.108$ and D = 2.327 for d = 1 directed percolation[26], and the Monte Carlo values $\tau = 1.270$ and D = 2.951 for $d = 2$ RFT [27], we find $\sigma = 0.32$, y = 2.77, and $\nu_\perp = 1.34$ for d = 1. For $d = 2$, $\sigma = 0.41$, y = 1.79, and $\nu_\perp = 0.83$. These values agree with the measured values.

The "gamma" equation, (8), gives a convenient way to determine the critical point accurately. If $(1-f)/< r^d >$ is plotted vs. f, the slope is equal to $1/\gamma$ and the intersection with the f-axis is f_c. We find $\gamma = 2.7$ and $f_c = 0.66695 \pm 00005$ (Fig. 7). Thus, contrary to early speculations it is highly unlikely that the critical f_c is exactly 2/3.

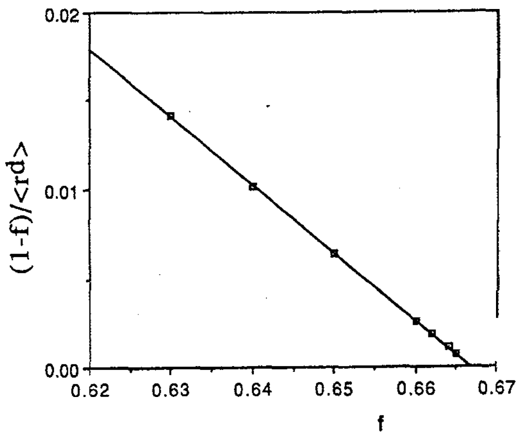

Figure 7: Plot of $(1-f)/< r^d >$ (f) vs f. The slope gives the inverse value of the exponent γ. The intersection with the f axis gives f_c.

In conclusion, we have a rather complete description of the self-organization process, and the resulting critical properties in the stationary attractor of the BS model. In particular, the model exhibits punctuated equilibrium behavior with a power law distribution of intervals between active events, and a 1/f power spectrum.

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