Spin Glass: an Unfinished Story

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In this work a short overview of the development of spin glass theories, mainly long and short range Ising models, are presented.

I. Prologue

It has been a long and hard way to unravel the fascinating subtleties involved in the physics of spin glasses since the pioneering experimental work of Cannella and Mydosh[1] in the dilute metallic alloy $CuMn$ with 0.9% Mn. Nowadays these systems comprise a large variety of distinct materials emboding the two basic ingredients: frozen disorder and antagonistic interactions (frustration). For a review see for instance Rammal and Souletie[2], Binder and Young[3], Chowdury and Mookerjee[4], Mèzard et al [5], Bray[6] and Fisher and Hertz[7].

The most successful and popular theoretical model to describe the physical properties of spin glasses is the one introduced by Edwards and Anderson[8] whose mean field version was proposed by Sherrington and Kirkpatrick (SK)[9]. Even today some aspects of the SK model remains elusive such as the structure of the free energy barriers[10], the ordering field of the condensed phase[11,12] and its dynamical properties[13].

In this work a short account of the theoretical development in spin glasses mainly the Ising model, is presented. There is no intention of completeness in this work nor to give any details of the model calculations or a complete list of references but only of providing useful informations concerning some results up to now. In section II the mean field SK model and the picture arising from its solution are discussed. In Section III the counterpart of some special short range models where exact solutions were obtained are considered as well as some scaling and renormalization group theories. Some concluding remarks are presented.

II. Mean field theory: the unfolding of complexity

The much referenced mean field theory of ferromagnetism due to Weiss may be obtained through an exactly solvable model, where all spins interact among themselves with vanishing size dependent interaction[14,15]. Its solution reveals that the mathematical mechanism responsible for the phase transition occurring in the model is the same as in the more palatable two-dimensional Ising model, i.e., asymptotic degeneracy of the largest eigenvalue of the transfer matrix associated with the partition function of the system. It is then possible to formulate the mean field theory of ferromagnetism within an aesthetically atractive way as the solution of a long-range ferromagnetic model.

In this same spirit, a long-range model intended to represent the mean field theory of spin glasses was

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introduced by Sherrington and Kirkpatrick[9] on the footsteps of the wide general Edwards and Anderson model[8]. It is defined by the Hamiltonian

$$\mathcal{H} = -\sum_{(i,j)} J_{ij}\sigma_i\sigma_j - H\sum_i \sigma_i \ , \qquad (1)$$

where $\sigma_i = \pm 1$, i = 1, 2, ...N is a set of Ising variables under an external magnetic field H. The set of exchange interactions coupling constants $\{J_{ij}\}$ are independent random variables chosen from the gaussian distribution

$$P(J_{ij}) = \sqrt{\frac{N}{2\pi J^2}} exp\left(-\frac{NJ_{ij}^2}{2J^2}\right) \ , \qquad (2)$$

and the sum is taken over all pairs (i, j) of spins. The scaling of the variance as J^2/N is necessary in order to have a finite free energy per particle as $N \rightarrow \infty$. For a system with frozen-in (quenched) disorder the free energy is giver by

$$f\{J_{ij}\} = \frac{F\{J_{ij}\}}{N} = -\frac{kT}{N}\ell n\ Z\{J_{ij}\} \ , \qquad (3)$$

for a given set of $\{J_{ij}\}$. For a very large system it is expected that $f\{J_{ij}\}$ and others densities of extensive quantities are sample independent, i.e., they are self-averaging. Thus, instead of calculating f for a given set (sample) as in Eq. (3), one may obtain an average free energy where the random variables are eliminated by carrying out the averaging of Eq. (3) over its distribution, namely

$$f = -\frac{kT}{N} < \ell n\ Z\{J_{ij}\} >_J \ , \qquad (4)$$

where $<\ldots>_J$ means this average procedure.

It is reasonable to expect the model (1) to have many ground-states, with a complex phase space, in addition to the presence of many metastable states. For such a disordered model it seems almost miraculous that below its critical temperature $T_c = J/K$ the complex structure and organization of the phase space could have its details worked out.

Historically, two complementary approaches were undertaken in order to calculate the free energy of the model. The first was to work out directly Eq.(4) through the replica method[8,9], and the second to obtain $F\{J_{ij}\}$ in Eq.(3) in terms of the local mean magnetizations of the spins [(16), hereafter TAP].

In the TAP approach $F\{J_{ij}\}$ is given by

$$\begin{aligned} F\{J_{ij}\} &= -\sum_{(ij)} J_{ij}m_im_j - \frac{\beta}{2}\sum_{(ij)} J_{ij}^2(1-m_i^2)(1-m_j^2) + \\ &+ \frac{T}{2}\sum_i[(1+m_i)\ell n(1+m_i)/2 + (1-m_i)\ell n(1-m_i)/2] - \sum_i Hm_i \ , \end{aligned} \qquad (5)$$

where $m_i = <\sigma_i>$ is the mean spin on the $i-th$ site, given by

$$m_i = tgh\left[\beta\sum_i J_{ij}m_j - \beta\sum_j J_{ij}^2(1-m_j^2)m_i + H\beta\right] \ . \qquad (6)$$

There are many degenerate solutions to Eq. (6) with the same free energy density yielding the ground states of the system in addition to a huge number of metastable states. While the degeneracy of the former solutions do not contribute to the entropy density, the latter behaves like $exp[N\omega(f)]$ where $\omega(f) > 0$ for f larger than a given critical free energy density f_c [Refs. 17, 18, 19, 5]. One has then the picture that below the critical temperature the phase space of the SK model has many distinct thermodynamic phases (or *pure* states) separated by infinite free energy barriers. Unlike a ferromagnet whose twofold degenerate ground states are related by time reversal symmetry, there is no obvious symmetry among the pure states of the SK model. Each state may be characterized by the set of local average spin $<\sigma_i>_\ell = m_i^\ell$, denoting a possible equilibrium thermodynamic state. Numerical solution of the TAP equations[20] suggests that below the critical temperature, as the temperature decreases there occurs a continuous bifurcation (or better, a multifurcation) cascade in the number of solutions, a picture suggested earlier for the condensed phase reflecting the critical character of the spin glass phase[21,22,3,5].

The distinct thermodynamic states in which the phase space may be decomposed will have a free energy density f_ℓ given by [22,3]

$$exp\,(-Nf_\ell/kT) = Z_\ell = \sum_{\lambda\epsilon\ell} exp(-E_\lambda/kT)\ , \qquad (7)$$

where the sum is over all microscopic states associated with state ℓ and Z_ℓ is the partition function of this state. The Boltzmann-Gibbs partition function involving all states is

$$Z = \sum_\ell Z_\ell = \sum_\ell exp(-Nf_\ell/kT) \qquad (8)$$

and a given thermodynamic state ℓ has a statistical weight

$$P_\ell = \frac{1}{Z}\, exp(-Nf_\ell/kT)\ . \qquad (9)$$

Although the free energy density is self-averaging as $N\rightarrow\infty$, different states may have distinct weights due to fluctuations in f which are $\mathcal{O}(1/N)$ yielding distinct values to (8). The Boltzmann-Gibbs average of an observable A may thus be written

$$<A>_T = \sum_\ell P_\ell <A>_T^{(\ell)}\ , \qquad (10)$$

where $<A>_T^{(\ell)}$ is the thermal average of A in state ℓ. There are quantities like energy and magnetization which are independent of the state (reproducible) and self-averaging while others like the susceptibility is not. Up to now there is no known analytical way of computing the thermal average of an observable in a pure state. This demands the knowledge of how to project out this state through an ordering field[11] although, it may be shown that certain quantities are both sample independent (self-averaging) and state independent (reproducible)[3,5].

Another approach which has been successful in working out the properties of (1) has been the so called replica method[8,9]. One uses the identity $\ell n\, Z = \lim_{n\rightarrow 0}(Z^n-1)/n$ in Eq. (4), interchanges the limits $N\rightarrow\infty$ and $n\rightarrow 0$, considers n an integer to work out $<Z^n>_J$ and at the end of the calculation takes the limit $n\rightarrow 0$. This procedure yields the following expression for the averaged free energy density f

$$\beta f = -\frac{\beta^2 J^2}{4} - \lim_{n\rightarrow 0}\max\left\{-\frac{\beta^2J^2}{2}\sum_{(\alpha\beta)} q_{\alpha\beta}^2 + \ell n\, Tr\, exp\left[\beta^2 J^2 \sum_{(\alpha\beta)} q_{\alpha\beta}\sigma_\alpha\sigma_\beta + \beta H\sum_\alpha \sigma_\alpha\right]\right\}, \qquad (11)$$

where the parameters $q_{\alpha\beta}$, $\alpha,\beta = 1,2,...$n are to be determined variationally from the conditions $\partial f/\partial q_{\alpha\beta} = 0$, which give

$$q_{\alpha\beta} = <\sigma_\alpha\sigma_\beta> = \frac{Tr\{\sigma_\alpha\sigma_\beta exp(\beta\mathcal{H}_n)\}}{Tr\, exp(\beta\mathcal{H}_n)}\ , \qquad (12)$$

with

$$\mathcal{H}_n = \beta J^2 \sum_{(\alpha\beta)} q_{\alpha\beta}\sigma_\alpha\sigma_\beta + H\sum_\alpha \sigma_\alpha\ , \qquad (13)$$

where the traces in Eq. (11) and Eq. (12) are taken over n replicas at a single site. In their original solution Sherrington and Kirkpatrick[9] considered only the solution with all $q_{\alpha\beta} = q$, i.e., a single, order parameter invariant under permutation of replica labels. This solution, however, gives a negative entropy at low temperatures a wrong result for an Ising model[9]. The study of the fluctuations of Eq. (11) around the replica symmetric solution $q_{\alpha\beta} = 1$ [23, 24] revealed that this is an unstable solution below the critical temperature, this instability persisting even in the presence of a magnetic field or when the interactions have a ferromagnetic component. Thus the correct solution for $T < T_c$ must have broken replica permutation symmetry. It took sorne ingenuity to find out how to break the symmetry among the $q_{\alpha\beta}$. By generalizing Blandin[25] work, Parisi[26,27] was able to exhibit the correct ansatz to solve the model. It amounted to introduce an infinite number of order parameters $q_{\alpha\beta}$ which in the limit $n \to 0$ reduces to an order parameter function $q(x)$, $x\epsilon[0,1]$, with the solution being marginally stable throught the condensed phase[17,28,29]. This marginal character seems to reflect the critica1 aspect of the condensed phase and the continuous multifurcation as the temperature is lowered (see Binder and Young[3] for other possible explanations of these zero modes). Within Parisi's ansatz the free energy density takes the following functional form

$$\beta f[q(x)] = -\frac{\beta^2 J^2}{4}\left[1 - 2q(1) + \int_0^1 q^2(x)dx\right] - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-z^2} G(0, H + z\sqrt{q(0)})dz \ , \quad (14)$$

where

$$\frac{\partial G}{\partial x} = -\frac{J^2}{2}\left(\frac{dq}{dx}\right)\left[\frac{\partial^2 G}{\partial y^2} + x\left(\frac{\partial G}{\partial y}\right)^2\right] , \quad (15)$$

with the boundary condition

$$G(1,y) = ln[2\, cosh(\beta y)] \ . \quad (16)$$

During some time the physical interpretation of $q_{\alpha\beta}$ and the broken replica symmetry solution leading to an order parameter function remained misterious. Parisi[30] showed these to have a clear physical interpretation in terms of the overlaps of the local magnetizations and of the probability distribution $P(q)$ of these overlaps between states:

$$q_{\alpha\beta} = \frac{1}{N}\sum_i m_i^\alpha m_i^\beta$$
$$P(q) = \sum_{\alpha,\beta} P_\alpha P_\beta \delta(q - q_{\alpha\beta}) \ , \quad (17)$$

where now α, β label the possible pure states with weight P_α and P_β. It can be shown that $P(q)$ is not a self-averaging quantity but its average over all realizations of $\{J_{ij}\}$, $\overline{P}(q)$, is related to $q(x)$ by $\overline{P}(q) = dx/dq$. So the inverse function $x = x(q)$ gives the cumulative probability for having an overlap p. Moreover, by considering any three pure states α_1, α_2, α_3 and the probability $P(q_1, q_2, q_3)$ for them to have overlaps $q_1 = q_{\alpha_2\alpha_3}$, $q_2 = q_{\alpha_3\alpha_1}$, $q_3 = q_{\alpha_1\alpha_2}$ it can be shown[31] that the space of the pure states of the SK model is organized in an ultrametric fashion: give any three states, at least two pairs will have the same overlap which will be less than or equal to the third pair (for instance, $q_1 = q_2 \le q_3$).

It is rather dificult to work directy with Eqs. (14)-(17). For $T \le T_c$ one may resort to series expansion of the functional Eq. (11) in terms of the order parameters[26-28]. However, by introducing a Lagrange multiplier function $P(x, y)$, related to the local internal field acting on the spins, a new free energy functional may be introduced allowing solution of the SK model for all values of T and H [32-36]. It is worth pointing out that in his work Temesvári[35] argues that the TAP equations and Parisi's theory may not be equivalent. Although it seems that the solution of the SK model has been fairly worked out some points as the free energy barriers[10] and the ordering field[11] may deserve further investigation as well as its dynamical properties[13,37].

III. Finite dimensional systems: competing unfinished theories and bizarre lattices

Up to now there is no generally accepted theory to describe the properties of finite dimensional spin glasses.

For a uniform system like a ferromagnet, its phenomenology is relatively easy to guess and a detailed calculation of the critical properties may be accomplished through Wilson's renormalization group framework. Well above T_c in the paramagnetic phase the long distant spins are decorrelated while below T_c there occurs long-range correlation among them, typical fluctuations involving clusters (droplets) of correlated spins. However, even for a ferromagnet model the concept of a cluster is somewhat vague[38].

In the spin glass case, despite the heroic effort of many people, it seems that much remain to be done. A sound theory a *la* Wilson's renormalization group[39] does not exist. The main existing theories are the Sherrington-Kirkpatrick model and its complex many states ultrametric space with an Almeida-Thouless transition line in a field for one hand and the domain wall *(droplets)* phenomenological approach on the other side, which in its present form does not yield the same rich structure of the SK model, mainly the many states and transition in a field picture[6,40]. On the experimental side there is room to fit the findings favouring one or the other framework[41-43] the same being true in Monte Carlo simulation in small samples[44-47]. Nevertheless, even the very existence of a phase transition at finite temperature in the 3D Ising short-range case is far from being settled[48].

However it is worth to mention that certain effort has been devoted to investigate exactly solvable SG models with short range interactions in bizarre lattices in attempt to understand some aspects of the problem not present in the infinite-range models like the correlation length, the sensibility of the boundary conditions and finite size effects. One of them, the Bethe lattice, has a finite number of nearest neighbours and so might be expected to be closer in nature to real spin glasses than the SK model. However due to its thin and local treelike structure the Bethe lattice contains itself some pathologies: there are no loops and therefore just one path linking any pair of sites, a characteristic of linear systems, and a finite *surface* to bulk sites ratio in the thermodynamic limit. These leads to very subtle and sensitive properties accordingly to the chosen boundary conditions. For a full discussion of these points see Chayes et *al*[49] and Carlson et *al*[50-52].

Many earlier works were done on random systems on the Bethe lattice by many authors, specially in Japan, that derived recursions relations to find the distribution function of the effective field in the $\pm$J Ising SG (for a review of these works see Katsura[53].) However the first mean field study of spin glasses on the Bethe lattice as an alternative approach to the SK model was carried out by TAP[16] where the lowest order $1/z$ expansion was taken on the Bethe cluster. After that Bowman and Levin[54] discussed the entropy and obtain the solution in the absence of a magnetic field while Thouless[55] examined this model for small magnetic field in the neighborhood of the critica1 point. By analizing the correlation between two replicas he found that a replica-symmetry-breaking transition occuring on the same critical curve in the HT plane as obtained by de Almeida and Thouless[23] for the infinite-range model. Thouless[55] notice that on either side of the critica1 curve the correlation functions fall off exponentially with distance but with one correlation length diverging on the curve. He also notice that the thermodynamic averages (internal energy, magnetization and the nonlinear susceptibility) are smooth on the transition curve except in zero field, a distinct behavior of the cusps found in the infinite-range model. A formal replica method was considered by Mottishaw[56] to study this model showing to be necessary to break the replica symmetry on the Bethe lattice just below T_c to have a stable solution. This suggests the existence of many coexisting thermodynamics states as occuring in the SK model. However this conclusion was in contrast with the Thouless conclusion[55] who found a replica symmetric stable solution for zero field. The controversy was later elucidated by Lai and Goldschmidt[57] pointing out the role of the boundary conditions in the Monte Carlo simulation of this model. They found that the Mottishaw's solution holds for the case of *closed* boundary conditions while the Thouless solution is valid for the open uncorrelated case. This latter case has been extensively studied by [50-52] for the case of the $\pm J$ bimodal distribution. Very recently Goldschmidt[58] using the cavity method (see Mèzard et *al*[5,59] obtained equations for the two real replicas that includes an extra parameter m which describes the exponential distributions of free energies of the distinct thermodynamic states. These equations are more general than those of Thouless which are recovered in $m \rightarrow 0$ limit.

Another line of approach to study SG short range models was developed after the study of the spin-glass behaviour in three dimension carried on by Southern and Young[60] who succeed to show by using the very simple scheme known as Migdal-Kadanoff (MK) aproximation[61,62] that there was no transition for d=2 while it occurs for d=3. Recently this approach has its interest renewed to study chaos exponents in SG[63,65]. In this case chaos means that the effective coupling between two given spins at a distance L undergoes multiple and chaotic sign changes with the temperature. It is found by Ney-Nifle and Hilhorst[64,65] that the scaling theory for symmetric SG is characterized by four independent exponents, the thermal ones (y and y_c) at $T = 0$ and the corresponding chaos exponents (ζ and ζ_c) at the criticality. The ζ_c ($> y_c$) exponent appears in the scaling laws that describe the chaotic temperature dependence of the renormalized couplings around the critical region. The condition $\zeta_c > y_c$ is found to be fulfilled in an interval of dimensions ($d_c = 2.46$, $d_+ = 3.4$) in the MK approximation therefore including d=3. These results are based on the calculation of the so called autocorrelation function which measures the sensitivity of the relative deviations from the mean of the renormalized couplings to small changes in the initial coupling distributions. Although these calculations were carried on within the Gaussian projection approximation, that is, after each RG step the distribution of the renormalized couplings is replaced by a Gaussian of the same mean and width, the results were confirmed by numerical estimates obtained by maintaining numerically the renormalized distributions[65].

Following another line Coutinho *et al*[66] studied within the MK scheme the structure of local EA order parameter of the SG Ising model instead of looking to the distributions of the renormalized couplings. They found an exact recursion relation for the local magnetization of the model defined on the diamond hierarchical lattice. These lattices (hereafter *DHL*) are just the lattices where the MK aproximation is exact for the pure Ising model. They found that around the critical temperature a measure constructed with the normalized local EA order parameter for a n-level *DHL* is a fractal measure. The $f(\alpha)$ function that characterizes how the singularities of the measure are distributed was numerically obtained and compared with one for the pure case[66].

The $f(\alpha)$ function for the SG is non-trivial around the critical point while the one for the pure case is non trivial only at the criticality[66]. Furthermore the former extends over a range of values of the α exponent much larger than the one for the latter. This fact suggests that the structure of the local order parameter inside the condensed phase is much more complex and requires an infinite set of exponents to be properly described. On the other hand dynamical simulations for 3d Ising SG in simple cubic lattices carried on by Bernardi and Campbell[67] shows that the dynamic exponent in the power-law relaxation of the autocorrelation function at the ordering temperature is very sensitive with the distribution chosen supporting that the standard universality rules do not seem to hold for these systems.

The present status of the studies in spin glasses, reflects how hard the problem is and the far reaching consequences of its eventual comprehension.

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