Path Integral over Velocities for Relativistic Particle Propagator

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A representation for the causal propagator of a relativistic scalar particle by means of a path integral ovèr velocities is presented. In this representation we are integrating over velocities v with arbitrary initial and final conditions, and the matrices which have to be inverted in course of doing Gaussian integrals do not contain any derivatives in time. The integration measure contains a 6-function of $\int v d\tau$, which nevertheless, can be easily incorporated in the usual Gaussian structure. One can define Gaussian and quasi-Gaussian integrals over velocities and rules of handling them. Using such a technique, an explicit expression for the propagator is found in a constant homogeneous electromagnetic field and its combination with a plane wave field.

I. Introduction

Propagators of relativistic particles in external fields (electromagnetic, non-Abelian or gravitational) contain important information about quantum behavior of these particles. Moreover, if such propagators are known in arbitrary external field, one can find exact one-particle Green's functions in the corresponding quantum field theory, taking functional integrals over the external field. It is known that the propagators can be presented by means of path integrals over classical trajectories. Such representations were already discussed in the literature for a long time in different contexts[1-16]. Over recent years this activity got some additional motivation to learn on these simple examples how to quantize by means of path integral more complicated theories, such as string theory, gravity and so on. Path integral representations can be effectively used for calculations of propagators, for example, for calculations of propagators in external electromagnetic or gravitational fields. However, in contrast with the field theory, where path integration rules are enough well defined (at least in the frame of perturbation theory[17,18]), in relativistic and nonrelativistic quantum mechanics there are problenis with uniqueness of definition of path integrals, with boundary conditions, and so on[1,19-23].

In this paper we present spinless relativistic particle propagator by a path integral over velocities. In the representation the integration is going over velocities with arbitrary initial and final conditions, and the matrices which have to be inverted in course of doing Gaussian integrals do not contain any derivatives in time. The integration measure contains a 6-function of $\int v d\tau$, which nevertheless, can be easily incorporated in the usual Gaussian structure. One can define universal Gaussian and quasi-Gaussian integrals over velocities and rules of handling them. This approach is similar the to one used in the field theory (in the frame of perturbation theory[17,18]). We illustrate the approach on calculation of the propagator in a constant homogeneous electromagnetic field and its combination with a plane wave field. For these cases we get closed expression for the propagator. One ought to say that path integral methods were often applied for such kind of calculations. For example, in the works Refs. 2 and 4 the causal propagators for scalar and spinning particles in external electromagnetic field of a plane wave were found by means of path integrations. The same problem was solved in Ref. 5 for the crossed electric and

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magnetic fields. More complicated combination of electromagnetic field, consisting of parallel magnetic and electric field together with a plane wave, propagating along, was considered in Refs. 3 and 12.

II. Representation of scalar particle propagator by means of path integral over velocities

As known, the propagator of a scalar particle in an external electromagnetic field $A_\mu(x)$ is the causal Green's function $D^c(x, y)$ of the Klein-Gordon equation in this field,

$$\left[(i\partial - gA)^2 - m^2 + i\epsilon\right] D^c(x,y) = -\delta^4(x-y) \ , \quad (2.1)$$

where $x = (x^\mu)$, Minkowski tensor $\eta_{\mu\nu} = \text{diag}(1 -1 -1 -1)$, and infinitesimal term $i\epsilon$ selects the causal solution.

Consider the Hamiltonian form of the path integral representation for $D^c(x, y)$. For certainty we will use notations and a definition of the integral by means of discretization procedure presented in Ref. **13**

$$D^c = D^c(x_{out}, x_{in}) = i\int_0^\infty d\lambda_0 \int_{\lambda_0} D\lambda \int D\pi \int_{x_{in}}^{x_{out}} Dx \int Dp \times \exp\left\{i\int d\tau \left[\lambda\left(\mathcal{P}^2 - m^2\right) + p\dot{x} + \pi\dot{\lambda}\right]\right\}, \quad (2.2)$$

where $\mathcal{P}_\mu = -p_\mu - gA_\mu(x)$, and the integration goes over trajectories $x^\mu(\tau)$, $p_\mu(\tau)$, $\lambda(\tau)$, $\pi(\tau)$, parameterized by some parameter $\tau \in [0, 1]$. Boundary conditions hold only for $x(\tau)$ and $\lambda(\tau)$,

$$x(0) = x_{in}, \quad x(1) = x_{out}, \quad \lambda(0) = \lambda_0 \ . \quad (2.3)$$

In (2.2) and in what follows we use the notation

$$\int d\tau = \int_0^1 d\tau \, .$$

Our aim is to transform the integral (2.2) to a form convenient, from our point of view, for calculations. First we shift the momenta,

$$-p_\mu \to p_\mu + \frac{\dot{x}_\mu}{2\lambda} + gA_\mu(x) \ ,$$

make the replacement $e = 2\lambda$ and fulfil the integration over π and λ,

$$D^c = \frac{i}{2}\int_0^\infty \frac{de_0}{e_0^2}\int_{x_{in}}^{x_{out}} Dx \int Dp \times \exp\left\{i\int d\tau \left[-\frac{\dot{x}^2}{2e_0} + \frac{e_0}{2}\left(p^2 - m^2\right) - g\dot{x}A(x)\right]\right\} . \quad (2.4)$$

Then, after the replacement

$$\sqrt{e_0}p \to p \ , \quad \frac{x - x_{in} - \tau\Delta x}{\sqrt{e_0}} \to x \ , \quad \Delta x = x_{out} - x_{in} \ ,$$

taking into account the definition of the integral (2.2) by means of discretization, we get the expression

$$D^c = \frac{i}{2}\int_0^\infty \frac{de_0}{e_0^2}\exp\left[-\frac{i}{2}\left(e_0 m^2 + \frac{\Delta x^2}{e_0}\right)\right]\int_0^0 Dx \int Dp \times \exp\left\{i\int d\tau\left[-\frac{\dot{x}^2}{2} + \frac{p^2}{2} - g(\sqrt{e_0}\dot{x} + \Delta x)A(\sqrt{e_0}x + x_{in} + \tau\Delta x)\right]\right\} , \quad (2.5)$$

where the trajectories $x^\mu(\tau)$ obey already zero boundary conditions,

$$x(0) = x(1) = 0\,. \tag{2.6}$$

Now we replace the integration over the trajectories $x^\mu(\tau)$ by one over velocities $v^\mu(\tau)$,

$$\begin{aligned} x(\tau) &= \int \theta(\tau-\tau')v(\tau')d\tau' = \int_0^\tau v(\tau')d\tau'\,, \\ v(\tau) &= \dot{x}(\tau)\,. \end{aligned} \tag{2.7}$$

The corresponding Jacobian can be written as

$$J = \mathrm{Det}\, \theta(\tau-\tau')$$

and regularized, for example, in the frame of discretization procedure. Note that because of (2.6), the trajectories $v(\tau)$ must obey the conditions

$$\int v(\tau)\,d\tau = 0\,. \tag{2.8}$$

We can take it into account, inserting the corresponding four-dimensional δ-function in the path integral. Thus,

$$\begin{aligned} D^c = \frac{i}{2}\int_0^\infty \frac{de_0}{e_0^2} \exp\left[-\frac{i}{2}\left(e_0 m^2 + \frac{\Delta x^2}{e_0}\right)\right] \\ \int Dv\, J \int Dp\, \delta^4\left(\int v d\tau\right) \exp\left\{i\int d\tau\left[-\frac{v^2}{2}+\frac{p^2}{2}\right.\right. \\ \left.\left. -g\left(\sqrt{e_0}v+\Delta x\right)A\left(\sqrt{e_0}\int_0^\tau v(\tau')d\tau' + x_{in} + \tau\Delta x\right)\right]\right\} \end{aligned} \tag{2.9}$$

One can formally find the Jacobian J, switching off the potential $A_\mu(x)$ in (2.9) and using the expression for the free causal Green function D_0^c,

$$D_0^c = D_0^c(x_{out}, x_{in}) = \frac{1}{2(2\pi)^2}\int_0^\infty \frac{de_0}{e_0^2}\exp\left[-\frac{i}{2}\left(e_0 m^2 + \frac{\Delta x^2}{e_0}\right)\right]\,.$$

So, we get

$$J = i(2\pi)^2\left[\int Dv \int Dp\, \delta^4\left(\int v d\tau\right)\exp\left\{i\int d\tau\left(-\frac{v^2}{2}+\frac{p^2}{2}\right)\right\}\right]^{-1} \tag{2.10}$$

Gathering these results, we may write

$$D^c = \frac{1}{2(2\pi)^2}\int_0^\infty \frac{de_0}{e_0^2}\exp\left[-\frac{i}{2}\left(e_0 m^2 + \frac{\Delta x^2}{e_0}\right)\right]\Delta(e_0)\,, \tag{2.11}$$

$$\begin{aligned} \Delta(e_0) = \int \mathcal{D}v\, \delta^4\left(\int v d\tau\right)\exp\left\{i\int d\tau\left[-\frac{v^2}{2} - g\left(\sqrt{e_0}v+\Delta x\right)\right.\right. \\ \left.\left.\times A\left(\sqrt{e_0}\int_0^\tau v(\tau')d\tau' + x_{in} + \tau\Delta x\right)\right]\right\}\,, \end{aligned} \tag{2.12}$$

where the new measure $\mathcal{D}v$ has the form

$$\mathcal{D}v = Dv\left[\int Dv\ \delta^4\left(\int v d\tau\right)\exp\left\{i\int d\tau\left(-\frac{v^2}{2}\right)\right\}\right]^{-1}\,. \tag{2.13}$$

It is clear that $\Delta(e_0) = 1$ at $A = 0$.

Thus, we got a representation for the propagator of a scalar particle by means of a special kind path integral over velocities, in which the integration over velocities is not subjected to any boundary conditions and no time derivatives appear in integrand, so, e.g. the matrices which have to be inverted in course of doing Gaussian integrals do not contain any time derivatives. One can formulate universal rules of handling such integrals in the frame of perturbation theory, what will be done in the next section.

III. Gaussian and quasi-Gaussian path integrals over velocities

In the previous sections we demonstrated that the propagator of spinless particle can be presented by means of bosonic path integral over velocities of space-time coordinates. This integral have the following structure

$$\int \mathcal{D}v\ \delta^4\left(\int v d\tau\right) F[v], \tag{3.1}$$

where

$$\mathcal{D}v = Dv\left[\int Dv\ \ \delta^4\left(\int v d\tau\right)\exp\{i\int d\tau\left(-\frac{v^2}{2}\right)\}\right]^{-1}, \tag{3.2}$$

with some functional $F[v]$.

Ways of doing path integrals of general form are unknown at present time, only Gaussian path integrals, treated in certain sense, can be taken directly. That is also valid with regards to the integrals in question (3.1). However, if we restrict ourselves with a limited class of functionals $F[v]$, which are called quasi-Gaussian and are defined below, then one can formulate some universal rules of their calculation and handling them. Similar idea has been realized in the field theory[17,18]. The restriction with quasi-Gaussian functionals corresponds, in fact, to a perturbation theory, with Gaussian path integral as a zero order approximation.

Introduce Gaussian functional $F_G[v, I]$,

$$F_G[v,I] = \exp\left\{-\frac{i}{2}\int d\tau d\tau' v^\mu(\tau) L_{\mu\nu}(g,\tau,\tau') v^\nu(\tau') - i\int d\tau I_\mu(\tau) v^\mu(\tau)\right\}, \tag{3.3}$$

where $v^\mu(\tau)$ are the velocities and $I_\mu(\tau)$ are corresponding sources to them. We call a functional $F_{qG}[v,I]$ quasi-Gaussian if

$$F_{qG}[v,I] = F[v] F_G[v,I], \tag{3.4}$$

where $F[v]$ is a functional, which can be expanded in the functional series of v,

$$F[v] = \sum_{n=0}\int d\tau_1 \ldots d\tau_n F_{\mu_1\ldots\mu_n}(\tau_1\ldots\tau_n)\, v^{\mu_1}(\tau_1)\ldots v^{\mu_n}(\tau_n). \tag{3.5}$$

In (3.3) the matrix $L_{\mu\nu}(g,\tau,\tau')$ supposes to have the following form

$$L_{\mu\nu}(q,\tau,\tau') = \eta_{\mu\nu}\delta(\tau-\tau') + gM_{\mu\nu}(\tau,\tau'). \tag{3.6}$$

Define the path integral over velocities v of the Gaussian functional as

$$\int \mathcal{D}v\ \delta^4\left(\int v d\tau\right) F_G[v,I] \tag{3.7}$$
$$= \left[\frac{\text{Det}\ L(g)\ \det\ l(g)}{\text{Det}\ L(0)\ \det\ l(0)}\right]^{-1/2}\exp\left\{\frac{i}{2}\int d\tau d\tau' I(\tau) K(\tau,\tau') I(\tau')\right\},$$

where

$$K(\tau,\tau') = L^{-1}(g,\tau,\tau') - Q^T(\tau) l^{-1}(g) Q(\tau'),$$
$$l(g) = \int d\tau d\tau' L^{-1}(g,\tau,\tau'),\quad Q(\tau) = \int d\tau' L^{-1}(g,\tau',\tau). \tag{3.8}$$

The formula (3.7) can be considered as an infinite dimensional generalization of the straightforward calculations result in the frame of the discretization procedure, connected with the original definition of path integrals for propagators discussed in the previous sections. In course of doing of finite dimensional integrals it is implied a supplementary definition of arisen improper Gaussian integrals by means of the analytical continuation in the matrix elements of the nonsingular matrix L.

To avoid problems with calculations of determinants of matrices with continuous indices we can use some convenient representation. Let us differentiate the well known formula

$$\text{Det } L(g) = \exp\,[Tr \ln\, L(g)]$$

with respect to g. So we get the equation

$$\frac{d}{dg}\text{Det } L(g) = \text{Det } L(g)\ \text{Tr } L^{-1}(g)\frac{dL(g)}{dg} = \text{Det } L(g)\ \text{Tr } L^{-1}(g)M\ ,$$

which can be solved in the form

$$\frac{\text{Det } L(g)}{\text{Det } L(0)} = \exp\left\{\int_0^g dg' \text{Tr } L^{-1}(g')M\right\}\ . \tag{3.9}$$

Taking into account that det $l(0) = -1$, we can rewrite the path integral of the Gaussian functional in the following form

$$\begin{aligned}&\int \mathcal{D}v\ \delta^4\left(\int v d\tau\right) F_G[v,I] \\ &= [-\det\ l(g)]^{-1/2} \exp\left\{\frac{i}{2}\int d\tau d\tau' I(\tau)K(\tau,\tau')I(\tau') - \frac{1}{2}\int_0^g dg' \text{Tr } L^{-1}(g')M\right\}\ .\end{aligned} \tag{3.10}$$

The path integral of the quasi-Gaussian functional we define through one of the Gaussian functional

$$\begin{aligned}&\int \mathcal{D}v\ \delta^4\left(\int v d\tau\right) F_{qG}[v,I] = F\left(i\frac{\delta}{\delta I}\right)\int \mathcal{D}v\ \delta^4\left(\int v d\tau\right) F_G[v,I] \\ &= [-\det\ l(g)]^{-1/2}\, F\left(i\frac{\delta}{\delta I}\right) \exp\left\{\frac{i}{2}\int d\tau d\tau' I(\tau)K(\tau,\tau')I(\tau') - \frac{1}{2}\int_0^g dg' \text{Tr } L^{-1}(g')M\right\}\ .\end{aligned} \tag{3.11}$$

One can formulate rules of handling integrals from quasi-Gaussian functionals, using the formula (3.11). For example, such integrals are invariant under shifts of integration variables,

$$\int \mathcal{D}v\ \delta^4\left(\int (v+u) d\tau\right) F_{qG}[v+u,I] = \int \mathcal{D}v\ \delta^4\left(\int v d\tau\right) F_{qG}[v,I]\ . \tag{3.12}$$

The validity of this assertion for the Gaussian integral can be verified by a direct calculation. Then the general formula (3.12) follows from (3.11). Using the property (3.12), one can derive an useful generalization of the formula (3.11),

$$\int Dv\, \delta^4 \left(\int vd\tau - a\right) F_{qG}[v, I]$$
$$= [-\det l(g)]^{-1/2} F\left(i\frac{\delta}{\delta I}\right) \exp\left\{\frac{i}{2}\int d\tau d\tau' I(\tau)K(\tau,\tau')I(\tau')\right.$$
$$\left. -\frac{i}{2}al^{-1}(g)a - ial^{-1}(g)\int Q(\tau)I(\tau)d\tau - \frac{1}{2}\int_0^g dg' \mathrm{Tr}\, L^{-1}(g')M\right\}, \tag{3.13}$$

where a is a constant vector. The integral of the total functional derivative over $v^\mu(\tau)$ is equal to zero,

$$\int \mathcal{D}v\, \frac{\delta}{\delta v^\mu(\tau)}\, \delta^4\left(\int vd\tau\right) F_{qG}[v, I] = 0\,. \tag{3.14}$$

This property may be obtained as a consequence of the functional integral invariance under the shift of variables, as well as by direct calculations of integral (3.14). Using the latter, one can derive formulas of integration by parts, which we do not present here. If a quasi-Gaussian functional depends on a parameter a , then the derivative with respect to this parameter is commutative with the integral sign,

$$\frac{\partial}{\partial\alpha}\int \mathcal{D}v\, \delta^4\left(\int vd\tau\right) F_{qG}[v, I, \alpha] = \int \mathcal{D}v\, \delta^4\left(\int vd\tau\right) \frac{\partial}{\partial\alpha}F_{qG}[v, I, \alpha]\,. \tag{3.15}$$

Finally, the formula for the change of the variables holds:

$$\int \mathcal{D}v\, \delta^4\left(\int vd\tau\right) F_{qG}[v, I] = \int \mathcal{D}v\, \delta^4\left(\int \phi d\tau\right) F_{qG}[\phi, I]\, \mathrm{Det}\, \frac{\delta\phi_\tau(v)}{\delta v(\tau')}\,, \tag{3.16}$$

where $\phi_\tau(v)$ is a set of analytical functionals in v , parameterized by r . One can prove formulas (3.15, 3.16) in the same manner as it was done in Ref. 18 for the case of the field theory.

Thus, in relativistic quantum mechanics, in the frame of perturbation theory, one can define quasi-Gaussian path integrals over velocities and rules of handling them. This definition is close to one in field theory[17,18], and the analogy is stressed by the circumstance that, as in the field theory, the integrals over velocities do not contain explicitly any boundary condition for trajectories of the integration. After the rules of integration are formulated, one can forget about the origin of the integrals over velocities and fulfil integrations, using the rules only. In the next section we demonstrate this technique on some examples.

IV. Calculation of propagator in external electromagnetic fields

Here we are going to calculate the propagator of a scalar particle in an external electromagnetic field, using representation (2.11) and the rules of integrations presented in the previous sections. We consider a combination of a constant homogeneous field and a plane wave field. The potentials for this field may be taken in the form

$$A,\,(x) = -\frac{1}{2}F_{\mu\nu}x^\nu + f_\mu(nx)\,, \tag{4.1}$$

where $F_{\mu\nu}$ is the field strength tensor of the constant homogeneous field with nonzero invariants

$$\mathcal{F} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \neq 0\,,\ \mathcal{G} = -\frac{1}{4}F^*_{\mu\nu}F^{\mu\nu} \neq 0\,,$$

($F^*_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$, $\epsilon_{\mu\nu\alpha\beta}$ is totally antisymmetric tensor), in terms of which its eigenvalues E and 31 are expressed

$$F_{\mu\nu}n^\nu = -\mathcal{E}n_\mu \ , \ \ F_{\mu\nu}\bar{n}^\nu = \mathcal{E}\bar{n}_\mu \ , \ \ F_{\mu\nu}\ell^\nu = i\mathcal{H}\ell_\mu \ , \ \ F_{\mu\nu}\bar{\ell}^\nu = -i\mathcal{H}\bar{\ell}_\mu \ ,$$
$$\mathcal{E} = \left[(\mathcal{F}^2+\mathcal{G}^2)^{\frac{1}{2}} - \mathcal{F}\right]^{\frac{1}{2}} \ , \ \mathcal{H} = \left[(\mathcal{F}^2+\mathcal{G}^2)^{\frac{1}{2}} + \mathcal{F}\right)]^{\frac{1}{2}} \ . \tag{4.2}$$

The eigenvectors n, $\bar{n}$, ℓ, $\bar{\ell}$ are isotropic and obey the conditions

$$n^2 = \bar{n}^2 = \ell^2 = \bar{\ell}^2 = 0 \ , \ \ n\bar{n} = 2 \ , \ \ell\bar{\ell} = -2 \ , \ n\ell = \bar{n}\ell = n\bar{\ell} = \bar{n}\bar{\ell} = 0 \ . \tag{4.3}$$

The functions f_μ (nx) are arbitrary, except for the fact that they are subject to the conditions

$$f_\mu\,(\mathrm{nx})\,n^\mu = f_\mu\,(\mathrm{nx})\,\bar{n}^\mu = 0 \ . \tag{4.4}$$

The total field strength tensor for the potential (4.1) is

$$F_{\mu\nu}(x) = F_{\mu\nu} + \Psi_{\mu\nu}(nx), \ \Psi_{\mu\nu}(nx) = n_\mu f'_\nu(nx) - n_\nu f'_\mu(nx) \ . \tag{4.5}$$

Since the invariants $\mathcal{F}$, $\mathcal{G}$ of the tensor $F_{\mu\nu}$ are nonzero, there exists a special reference frame where the electric and magnetic fields, corresponding to this tensor, are collinear with respect to one another and to the spatial part n of the four-vector n. In this reference frarne, the total field $F_{\mu\nu}(x)$ corresponds to a constant homogeneous and collinear electric and magnetic fields together with a plane wave propagating along them; $\mathcal{E}$, $\mathcal{H}$, being equal to the strengths of a constant homogeneous electric and magnetic fields, respectively. In terms of the defined eigenvectors the tensor $F_{\mu\nu}$ can be written as

$$F_{\mu\nu} = \frac{\mathcal{E}}{2}(\bar{n}_\mu n_\nu - n_\mu\bar{n}_\nu) + \frac{i\mathcal{H}}{2}\left(\bar{\ell}_\mu\ell_\nu - \ell_\mu\bar{\ell}_\nu\right) \ , \tag{4.6}$$

and the completeness relation holds

$$\eta_{\mu\nu} = \frac{1}{2}\left(\bar{n}_\mu n_\nu + n_\mu\bar{n}_\nu - \bar{\ell}_\mu\ell_\nu - \ell_\mu\bar{\ell}_\nu\right) \ . \tag{4.7}$$

The latter allows one to express any four-vector u in terms of the eigenvectors (4.2),

$$u^\mu = n^\mu u^{(1)} + \bar{n}^\mu u^{(2)} + \ell^\mu u^{(3)} + \bar{\ell}^\mu u^{(4)} \ ,$$
$$u^{(1)} = \frac{1}{2}\bar{n}u \ , \ u^{(2)} = \frac{1}{2}nu \ , \ u^{(3)} = -\frac{1}{2}\bar{\ell}u \ , \ u^{(4)} = -\frac{1}{2}\ell u \ . \tag{4.8}$$

In these concrete calculations it is convenient for us to make a shift of variables in the formula (2.12), to rewrite it in the following form

$$\Delta(e_0) = \exp\left(i\frac{\Delta x^2}{2e_0}\right)\int \mathcal{D}v\,\delta^4\left(\int v d\tau - \frac{\Delta x}{\sqrt{e_0}}\right)$$
$$\times \exp\left\{i\int d\tau\left[-\frac{v^2}{2} - g\sqrt{e_0}vA\left(\sqrt{e_0}\int_0^\tau v(\tau')d\tau' + x_{in}\right)\right]\right\} \ . \tag{4.9}$$

The calculations will be made in two steps: first in a constant homogeneous field only, and then in the total combination (4.1), using some results of the first problem. Thus, on the first step the potentials of the electromagnetic field are

$$A_\mu(x) = -\frac{1}{2}F_{\mu\nu}x^\nu . \qquad (4.10)$$

Substituting the external field *(4.10)* into (4.9), one can find

$$\Delta(e_0) = \exp\left(i\frac{\Delta x^2}{2e_0}\right)\int \mathcal{D}v\ \delta^4\left(\int v d\tau - \frac{\Delta x}{\sqrt{e_0}}\right)$$
$$\times \exp\left\{-\frac{i}{2}\int d\tau d\tau' v(\tau) L(g,\tau,\tau') v(\tau') - i\int \frac{g\sqrt{e_0}}{2} x_{in}F\ v d\tau\right\} , \qquad (4.11)$$

where

$$L_{\mu\nu}(g,\tau,\tau') = \eta_{\mu\nu}\delta(\tau-\tau') - \frac{ge_0}{2}F_{\mu\nu}\epsilon(\tau-\tau') . \qquad (4.12)$$

The path integral *(4.11)* is the Gaussian one (see *(3.13))*. To get an answer, one needs to find the inverse matrix $L^{-1}(g,\tau,\tau')$, which satisfies the equation

$$\int L(g,\tau,\tau'')L^{-1}(g,\tau'',\tau')d\tau'' = \delta(\tau-\tau').$$

One can demonstrate that this equation is equivalent to a differential one,

$$\frac{\partial}{\partial\tau}L^{-1}(g,\tau,\tau') - ge_0F\ L^{-1}(g,\tau,\tau') = \delta'(\tau-\tau') , \qquad (4.13)$$

with initial condition

$$L^{-1}(g,0,\tau') + \frac{ge_0F}{2}\int L^{-1}(g,\tau'',\tau')d\tau'' = \delta(\tau') .$$

Its solution has the form

$$L^{-1}(g,\tau,\tau') = \delta(\tau-\tau') + \frac{ge_0F}{2}\exp\{ge_0(\tau-\tau')F\}\left[\epsilon(\tau-\tau') - \tanh\left(\frac{ge_0F}{2}\right)\right] . \qquad (4.14)$$

Using (4.14), one can find all ingredients of the general formula (3.13), taking into account that $a = -\Delta x/\sqrt{e_0}$, $I(\tau) = g\sqrt{e_0}x_{in}F/2$.
Thus,

$$K(\tau,\tau') = \delta(\tau-\tau') + \frac{ge_0F}{2}\exp\{ge_0(\tau-\tau')F\}\left[\epsilon(\tau-\tau') - \coth\left(\frac{ge_0F}{2}\right)\right] , \qquad (4.15)$$
$$\int d\tau d\tau' K(\tau,\tau') = 0 , \int d\tau Q(\tau) = l(g) , \ l(g) = \frac{\tanh ge_0F/2}{ge_0F/2} ,$$
$$M(\tau,\tau') = -\frac{e_0}{2}F\epsilon(\tau-\tau') , \int_0^g dg' \mathrm{Tr}L^{-1}(g')M = \mathrm{tr}\ln(\cosh ge_0F/2) ,$$

where the symbol “tr” is being taken over four dimensional indices only. Then

$$\Delta(e_0) = \left[-\det\left(\frac{\sinh ge_0F/2}{gF/2}\right)\right]^{-1/2}$$
$$\times \exp\left\{\frac{i}{2}\left[\frac{\Delta x^2}{e_0} + g\ x_{out}Fx_{in} - \frac{1}{2}\Delta x\ gF\coth\left(\frac{ge_0F}{2}\right)\Delta x\right]\right\} \qquad (4.16)$$

Substituting *(4.16)* into (2.11), we get a final expression for the causal propagator of a scalar particle in a constant homogeneous electromagnetic field

$$D^c(x_{out},x_{in})=\frac{1}{2(2\pi)^2}\int_0^\infty de_o\left[-\det\left(\frac{\sinh ge_0F/2}{gF/2}\right)\right]^{-1/2}$$
$$\times\exp\left\{\frac{i}{2}\left[gx_{out}Fx_{in}-e_0m^2-\frac{1}{2}\Delta x\, gF\coth(\frac{ge_0F}{2})\Delta x\right]\right\}. \tag{4.17}$$

This result was first derived by Schwinger, using his proper time method[24].

Now we return to the total electromagnetic field (4.1). Let us substitute the potential (4.1) into (4.9),

$$\Delta(e_0)=\exp\left(i\frac{\Delta x^2}{2e_0}\right)\int\mathcal{D}v\,\delta^4\left(\int v d\tau-\frac{\Delta x}{\sqrt{e_0}}\right) \tag{4.18}$$
$$\times\exp\left\{-\frac{i}{2}\int d\tau d\tau' v(\tau)L(g,\tau,\tau')v(\tau')-i\int\frac{g\sqrt{e_0}}{2}x_{in}F\,vd\tau\right.$$
$$\left.-ig\sqrt{e_0}\int d\tau v(\tau)f\left(nx_{in}+\sqrt{e_0}\int_0^\tau nv(\tau')d\tau'\right)\right\},$$

with $L(g,\tau,\tau')$ defined in (4.12). One can take the integral (4.18) as quasi-Gaussian, in accordance with the formula (3.13). So, one can write

$$\Delta(e_0) = \exp\left\{g\sqrt{e_0}\int d\tau f\left(nx_{in}+i\sqrt{e_0}\int_0^\tau n\frac{\delta}{\delta I(\tau')}d\tau'\right)\frac{\delta}{\delta I(\tau)}\right\}B(I)|_{I=0}, \tag{4.19}$$

where

$$B(I)=\exp\left(i\frac{\Delta x^2}{2e_0}\right)\int\mathcal{D}v\,\delta^4\left(\int vd\tau-\frac{\Delta x}{\sqrt{e_0}}\right) \tag{4.20}$$
$$\times\exp\left\{-\frac{i}{2}\int d\tau d\tau' v(\tau)L(g,\tau,\tau')v(\tau')-i\int\left(\frac{g\sqrt{e_0}}{2}x_{in}F+I(\tau)\right)v(\tau)d\tau\right\}$$

The integral can be found similar to (4.11). As a result we get

$$B(I)=\exp\left\{\frac{i}{2}\int d\tau d\tau' I(\tau)K(\tau,\tau')I(\tau')-i\int I(\tau)a(\tau)d\tau\right\}\Delta(e_0)|_{\Psi=0}, \tag{4.21}$$

where $\Delta(e_0)|_{\Psi=0}$ is the expression given by (4.16), $K(\tau,\tau')$ is defined in (4.15), and

$$a(\tau)=\frac{\Delta x}{2\sqrt{e_0}}\left(1+\coth(ge_0F/2)\right)ge_0F\exp(-ge_0F\tau).$$

To obtain the action of the operator involved in (4.19) on the functional $B(I)$, we decompose the sources $I^\mu(\tau)$ in the eigenvectors (4.2), using (4.8)

$$I^\mu(\tau)=\frac{1}{2}\left(n^\mu\,\bar{n}I(\tau)+\bar{n}^\mu\,nI(\tau)-\ell^\mu\,\bar{\ell}I(\tau)-\bar{\ell}^\mu\,\ell I(\tau)\right).$$

Then, it is possible to write

$$n\frac{\delta}{\delta I(\tau)}=2\frac{\delta}{\delta\bar{n}I(\tau)},\quad f\frac{\delta}{\delta I(\tau)}=\bar{\ell}f\frac{\delta}{\delta\bar{\ell}I(\tau)}+\ell f\frac{\delta}{\delta\ell I(\tau)}.$$

Using this, we get

$$f\left(nx_{in}+i\sqrt{e_0}\int_0^\tau n\frac{\delta}{\delta I(\tau')}d\tau'\right)\frac{\delta}{\delta I(\tau)}$$
$$=\bar{\ell}f\left(nx_{in}+i\sqrt{e_0}\int_0^\tau\frac{\delta}{\delta\bar{n}I(\tau')}d\tau'\right)\frac{\delta}{\delta\bar{\ell}I(\tau)}+\ell f\left(nx_{in}+i\sqrt{e_0}\int_0^\tau\frac{\delta}{\delta\bar{n}I(\tau')}d\tau'\right)\frac{\delta}{\delta\ell I(\tau)},$$

$$\int d\tau d\tau' I(\tau)K(\tau,\tau')I(\tau')$$
$$=\int d\tau d\tau'\left[\bar{n}I(\tau)\,nI(\tau')K(\tau,\tau',\mathcal{E})-\bar{\ell}I(\tau)\,\ell I(\tau')K(\tau,\tau',i\mathcal{H})\right],$$

$$\int I(\tau)a(\tau)d\tau=\frac{1}{2}\int\left[\bar{n}I(\tau)\,na(\tau)+nI(\tau)\,\bar{n}a(\tau)-\bar{\ell}I(\tau)\,\ell a(\tau)-\ell I(\tau)\,\bar{\ell}a(\tau)\right]d\tau,$$

where

$$K(\tau,\tau',\mathcal{E}) = \delta(\tau-\tau') + \frac{ge_0\mathcal{E}}{2}\exp\{ge_0(\tau-\tau')\mathcal{E}\}\left[\epsilon(\tau-\tau') - \coth\left(\frac{ge_0\mathcal{E}}{2}\right)\right],$$
$$K(\tau,\tau',i\mathcal{H}) = \delta(\tau-\tau') + \frac{ige_0\mathcal{H}}{2}\exp\{ige_0(\tau-\tau')\mathcal{H}\}\left[\epsilon(\tau-\tau') + i\cot\left(\frac{ge_0\mathcal{H}}{2}\right)\right].$$

Now the exponent of the functional $B[I]$ is linear in $nI(\tau)$, $\bar{n}I(\tau)$, $\ell I(\tau)$,$\bar{\ell}I(\tau)$. Thus, one can easyly get the result

$$\Delta(e_0) = \exp\left\{\frac{i}{2}g^2e_0\int d\tau d\tau' f(nx_{cl}(\tau))K(\tau,\tau')f(nx_{cl}(\tau')) + ig\sqrt{e}_0\int d\tau a(\tau)f(nx_{cl}(\tau))\right\}\Delta(e_0)|_{\Psi=0}, \tag{4.22}$$
$$nx_{cl}(\tau) = nx_{in} + \frac{1-\exp(ge_0\mathcal{E}\tau)}{1-\exp(ge_0\mathcal{E})}n\Delta x, \quad nx_{cl}(0) = nx_{in}, \quad nx_{cl}(1) = nx_{out},$$

where $x_{cl}(\tau)$ is the solution[12] of the Lorentz equation in the external electromagnetic field *(4.1)*. Gathering *(4.22)* and (4.16), we get

$$A(e_0) = \left[-\det\frac{\sinh(ge_02}{ge_0F/2}\right]^{-1/2}\exp\left\{\frac{i}{2}[gx_{out}Fx_{in} - \frac{1}{2}(\Delta x + l(e_0,1))\, gF\coth(ge_0F/2)\,(Ax + l(e_0,1)) + 2\Phi(e_0) + \Delta x\, gFl(e_0,1) + \frac{\Delta x^2}{e_0}]\right\}, \tag{4.23}$$

where

$$\Phi(e_0) = e_0\int gf(nx_{cl}(\tau))[gf(nx_{cl}(\tau)) + gFl(e_0,\tau)]d\tau, \tag{4.24}$$
$$l(e_0,\tau) = e_0\int_0^\tau \exp\{ge_0(\tau-\tau')F)gf(nx_{cl}(\tau')\}d\tau'.$$

Substituting *(4.23)* into (2.11), we arrive to the final expression for the causal propagator of a scalar particle in the external electromagnetic field *(4.1):*

$$D^c(x_{out},x_{in}) = \frac{1}{2(2\pi)^2}\int_0^\infty de_0\left[-\det\left(\frac{\sinh ge_0F/2}{gF/2}\right)\right]^{-1/2} \tag{4.25}$$
$$\times\exp\left\{\frac{i}{2}[gx_{out}Fx_{in} - e_0m^2 + Ax\, gFl(e_0,1) - \frac{1}{2}(Ax + l(e_0,1))\, gF\coth(ge_0F/2)\,(\mathbf{Ax} + l(e_0,1)) + 2\Phi(e_0)]\right\}.$$

This expression coincides with the one obtained in Ref. *25,* by means of the method of summation over exact solutions of Klein-Gordon equation in the external field *(4.1)*. *A* detailed description of quantum electrodynamical processes in such a field can be found in Refs. *12* and *26*.

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