Quantum Noise Reduction in Coupled Oscillators

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We obtain exact solutions for the time evolution of a quantum linear harmonic oscillator coupled to a second one with quadratic self-interacting terrns. As known in the literature the second oscillator exhibits the squeezing effect in its quadratures while the first one doesn't exhibit it in a coherent state, i. e., the eigenfunctions of the annihilation operator. We show that due to the coupling the squeezing effect contagiate the first linear oscillator. We observe also interesting features in these solutions, as the so called collapse and revival phenomena, independent of the existence of anharmonicity in the second oscillator, as recently suggested in the literature.

Squeezed states for harmonic oscillators (HO) and for electromagnetic fields have been investigated in the recent years for a variety of physical situations[1]. They have been observed for light in many experiments[2] and, more recently, the observation of squeezing for oscillators has been proposed in the literature[3]. These states do not admit a positive non singular diagonal representations in the coherent basis, thus becoming examples of non classical states. Squeezing occurs whenever one of the variances of quadrature phase amplitudes, here defined as $\hat{x} = \sqrt{\hbar/2}(\hat{a}+\hat{a}^{+}), \hat{p} = \sqrt{\hbar/2}(\hat{a}-\hat{a}^{+})/i$, is below the shot noise level of $\sqrt{\hbar/2}$. The symbols $\hat{a}^{+}(\hat{a})$ stand for the raising (lowering) operator for the HO, or the creation (annihilation) operator of photons.

Here we examine the possibility of obtaining the squeezing effect in a linear oscillator (HO) coupled to another "nonlinear" oscillator (QO). The term "nonlinear" being used here on account for the fact that the QO Hamiltonian have quadratic self-interacting terms in the lowering operator $\hat{a}$ and the raising operator $\hat{a}^{+}$, as distinguished from the HO, which do not possesses these terrns. As is known[1], this "quadratic" Hamiltonian is a candidate to exhibit the squeezing effect and we will verify that the squeezing generated in the QO may contagiate one of the variances of the HO, thus becoming also squeezed. In the present paper we obtain an exact closed solution for the time evolution of the $\hat{x}(t)$ and $\hat{p}(t)$ operators for both oscillators, allowing us to compute exactly their variances as a function of time. This time evolution shows interesting features such as the so called collapse and revival phenomena, which appears to be independent of the existence of anharmonicity terrns, as suggested in the work of Agarwal and Puri.[5]

Recently, Agarwal and Gupta[4] investigated the combined system consisting of an atomic oscillator interacting with a squeezed light field, both damped by a heat bath. They found (exact) steady state solutions for the (non-Hamiltonian) dynamical equations of the combined system. From these solution they were able to analyze relaxation of the atomic oscillator and also effects of the squeezed radiation on the vacuum-field Rabi splitting. Due to the presence of the heat bath these authors have employed the (necessary) density state formalism, through the Wigner function. In our case, however, there is no heat bath, hence our approach is different: our system has a Hamiltonian dynamics, which allow us to employ the pure state formalism. In this way, we found (exact) undamped solutions for the Hamiltonian dynamical equations, of our combined system. These solutions allow us to investigate the possibility of occurrence of squeezing effect in the HO interacting with the QO.

Following Gordon, Walker and Louisell[6] we take the model Hamiltonian for the whole system as

$$\hat{H} = \hat{H}_{HO}(\hat{A},\hat{A}^{+}) + \hat{H}_{QO}(\hat{a},\hat{a}^{+}) + \hat{V}(\hat{a},\hat{a}^{+},\hat{A},\hat{A}^{+}) , \tag{1.1}$$

where

$$\hat{H}_{HO} = \hbar g(\hat{A}^{+}\hat{A} + \frac{1}{2}) , \tag{1.2}$$

is the Hamiltonian describing the (free) HO,

$$\hat{H}_{QO} = \hbar f_1(\hat{a}^{+}\hat{a} + \frac{1}{2}) + \hbar f_2(\hat{a}^{+^2} + \hat{a}^2) , \tag{1.3}$$

is the Hamiltonian describing the (free) QO and

$$\hat{V} = -\hbar\gamma(\hat{a}\hat{A}^{+} + \hat{a}^{+}\hat{A}) , \tag{1.4}$$

is their interaction. We also have the commutation relations

$$[\hat{a},\hat{a}^{+}] = \left[\hat{A},\hat{A}^{+}\right] = 1 ,$$

$$\left[\hat{a},\hat{A}\right] = \left[\hat{a},\hat{A}^{+}\right] = \left[\hat{a}^{+},\hat{A}\right] = \left[\hat{a}^{+},\hat{A}^{+}\right] = 0 .$$

Next, we set the canonical transformation[7]

$$\hat{a} = \frac{1}{\sqrt{2\hbar}}(\hat{x} + ip) \tag{1.5}$$

and

$$\hat{A} = \frac{1}{\sqrt{2\hbar}}(\hat{X} + i\hat{P}) , \tag{1.6}$$

and their corresponding a Hermitin conjugate operators, to obtain the Hamiltonian (1.1) in the form

$$\mathrm{H} = \hat{H}_{HO}(\hat{X},\mathrm{P}) + \hat{H}_{QO}(\hat{x},\hat{p}) + \hat{V}(\hat{x},\hat{X},\hat{p},\hat{P}) , \tag{1.7}$$

where

$$\hat{H}_{HO} = \frac{1}{2}g\hat{P}^2 + \frac{1}{2}g\hat{X}^2 , \tag{1.8}$$

$$\hat{H}_{QO} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}k\hat{x}^2 , \tag{1.9}$$

and

$$\hat{V} = -\gamma(\hat{x}\hat{X} + \hat{p}\hat{P}) , \tag{1.10}$$

with

$$m = (f_1 - 2f_2)^{-1} , \tag{1.11}$$

and

$$k = (f_1 + 2f_2) . \tag{1.12}$$

From Eqs. (1.7) - (1.10) we obtain the Heisenberg equations of motion given by

$$\dot{\hat{X}} = g\hat{P} - \gamma\hat{p} , \tag{1.13}$$

$$\dot{\hat{P}} = -g\hat{X} + \gamma\hat{x} , \tag{1.14}$$

$$\dot{\hat{x}} = \frac{\hat{p}}{m} - \gamma\hat{P} , \tag{1.15}$$

$$\dot{\hat{p}} = -k\hat{x} + \gamma\hat{X} . \tag{1.16}$$

If $\gamma = 0$ the above system of equations decouples and we easily find the solutions

$$\hat{X}(t) = \hat{X}(0)\cos gt + \hat{P}(0)\sin gt , \tag{1.17}$$

and

$$\hat{P}(t) = -\hat{X}(0)\sin gt + \hat{P}(0)\cos gt , \tag{1.18}$$

for 'the HO and

$$\hat{x}(t) = \hat{x}(0)\cos\omega_0 t + \frac{\hat{p}(0)}{\sqrt{mk}}\sin\omega_0 t , \tag{1.19}$$

$$\hat{p}(t) = -\sqrt{mk}\hat{x}(0)\sin\omega_0 t + \hat{p}(0)\cos\omega_0 t , \tag{1.20}$$

for the QO, where $\omega_0 = \sqrt{k/m}$.

In this case, we find the variances for an initial coherent state

$$\Delta X(t) = \Delta P(t) = \sqrt{\frac{\hbar}{2}} \tag{1.21}$$

with $\Delta X(t).\Delta P(t) = \hbar/2$ and

$$\Delta x(t) = \sqrt{\frac{\hbar}{2}}\left[1 - \left(1 - \frac{1}{mk}\right)\sin^2\omega_0 t\right]^{1/2} , \tag{1.22}$$

and

$$\Delta p(t) = \sqrt{\frac{\hbar}{2}}\left[1 - (1 - mk)\sin^2\omega_0 t\right]^{1/2} . \tag{1.23}$$

Here we are using the definitions, for a general operator Ô (Ô =: $\hat{x}, \hat{p}, \hat{X}$ or P),

$$\Delta O(t) \equiv \sqrt{\left\langle \hat{O}(t)^2 - \left\langle \hat{O}(t) \right\rangle^2 \right\rangle} \, , \tag{1.24}$$

where the brackets stand for the expectation value with respect to the eigenfunctions (at time $t = 0$) of the corresponding lowering operator, that is, â in the case of $\hat{x}$, $\hat{p}$ and Â in the case of $\hat{X}$ and P.

Hence, if y = 0 the variances of $\hat{X}(t)$ and $\hat{P}(t)$ of the HO remains coherent for all t ≥ 0 whereas either $\hat{x}(t)$ or $\hat{p}(t)$ of the QO exhibit squeezing for t > 0, as expected.

If y $\neq 0$ the Eqs. (1.13) - (1.16) can be decoupled using the transformation

$$\begin{pmatrix} \hat{x}' \\ \hat{X}' \\ \hat{p}' \\ \hat{P}' \end{pmatrix} = \begin{pmatrix} e^{\frac{\lambda}{2}} \cos\frac{\phi}{2} & -e^{\frac{-\lambda}{2}} \sin\frac{\phi}{2} & 0 & 0 \\ e^{\frac{\lambda}{2}} \sin\frac{\phi}{2} & e^{\frac{-\lambda}{2}} \cos\frac{\phi}{2} & 0 & 0 \\ 0 & 0 & e^{\frac{-\lambda}{2}} \cos\frac{\phi}{2} & -e^{\frac{\lambda}{2}} \sin\frac{\phi}{2} \\ 0 & 0 & e^{\frac{-\lambda}{2}} \sin\frac{\phi}{2} & e^{\frac{\lambda}{2}} \cos\frac{\phi}{2} \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{X} \\ \hat{p} \\ \hat{P} \end{pmatrix} , \tag{1.25}$$

where

$$\lambda = \frac{1}{2} \ln \left[\frac{m(k+g)}{1+mg} \right] \, , \tag{1.26}$$

and

$$\varphi = \arctan \left[\frac{2\gamma}{ke^{-\lambda} - ge^{\lambda}} \right] \, . \tag{1.27}$$

The application of the above transformatiun (which corresponds to a rotation plus a dilation[8]) to the coupled Hamiltonian given in eq. (1.7) gives

$$\hat{H}' = \left\{ \frac{1}{2}\alpha_1 \hat{p}'^2 + \frac{1}{2}\beta_1 \hat{x}'^2 \right\} + \left\{ \frac{1}{2}\alpha_2 \hat{P}'^2 + \frac{1}{2}\beta_2 \hat{X}'^2 \right\} \, , \tag{1.28}$$

where

$$\alpha_1 = ge^{-\lambda} \sin^2\frac{\phi}{2} + \frac{1}{m} e^{\lambda} \cos^2\frac{\phi}{2} + \gamma \sin\phi \, , \tag{1.29}$$

$$\beta_1 = ge^{\lambda} \sin^2\frac{\phi}{2} + ke^{-\lambda} \cos^2\frac{\phi}{2} + y \sin\phi \, , \tag{1.30}$$

$$\alpha_2 = ge^{-\lambda} \cos^2\frac{\phi}{2} + \frac{1}{m} e^{*} \sin^2\frac{\phi}{2} - \gamma \sin\phi \, , \tag{1.31}$$

$$\beta_2 = ge^{\lambda} \cos^2\frac{\phi}{2} + ke^{-\lambda} \sin^2\frac{\phi}{2} - \gamma \sin\phi \, . \tag{1.32}$$

Hence, the Heisenberg equations of motion. for the new variables assume the form of eqs. (1.13) - (1.16) for the case $\gamma = 0$ simply by replacing g $\rightarrow \alpha_2, g \rightarrow \beta_2, 1/m \rightarrow \alpha_1, k \rightarrow \beta_1$ and therefore their solutions have similar forms as those in eqs. (1.17) - (1.20). We find the following results

$$\hat{X}'(t) = \hat{X}'(0) \cos\omega_2 t + \sqrt{\frac{\alpha_2}{\beta_2}} \hat{P}'(0) \sin\omega_2 t \, , \tag{1.33}$$

$$\hat{P}'(t) = -\sqrt{\frac{\beta_2}{\alpha_2}} \hat{X}'(0) \sin\omega_2 t + \hat{P}'(0) \cos\omega_2 t \, , \tag{1.34}$$

$$\hat{x}'(t) = \hat{x}'(0) \cos\omega_1 t + \sqrt{\frac{\alpha_1}{\beta_1}} \hat{p}'(0) \sin\omega_1 t \, , \tag{1.35}$$

$$\hat{p}'(t) = -\sqrt{\frac{\beta_1}{\alpha_1}} \hat{x}'(0) \sin\omega_1 t + \hat{p}'(0) \cos\omega_1 t \, , \tag{1.36}$$

where $\omega_1 = \sqrt{\alpha_1\beta_1}, \omega_2 = \sqrt{\alpha_2\beta_2}$ and whenever $\alpha_1\beta_1 > 0$ and $\alpha_2\beta_2 > 0$, otherwise hyperbolic solutions occur, as is the case for intermediate values of the parameter y. For small or large values of y , how small or how large depending on the values of the other parameters m, k and g , we have always real oscillatory solutions as shown in eqs. (1.33) - (1.36).

Next the application of the inverse transformation of eq. (1.25) to the solutions given by eqs. (1.33) - (1.36) results in the solution for our original coupled system as given by

$$\hat{X}(t) = -\left[ad\sqrt{\frac{\alpha_1}{\beta_1}}\sin\omega_1 t - bc\sqrt{\frac{\alpha_2}{\beta_2}}\sin\omega_2 t\right]\hat{p}(0) + \left[d^2\sqrt{\frac{\alpha_1}{\beta_1}}\sin\omega_1 t + c^2\sqrt{\frac{\alpha_2}{\beta_2}}\sin\omega_2 t\right]\hat{P}(0)\ ,$$

and

$$- cd\,[\cos\omega_1 t - \cos\omega_2 t]\,\hat{x}(0) + [bd\cos\omega_1 t + ac\cos\omega_2 t]\,\hat{X}(0)\ , \tag{1.37}$$

$$P(t) = \left[bc\sqrt{\frac{\beta_1}{\alpha_1}}\sin\omega_1 t - ad\sqrt{\frac{\beta_2}{\alpha_2}}\sin\omega_2 t\right]\hat{x}(0) - \left[b^2\sqrt{\frac{\beta_1}{\alpha_1}}\sin\omega_1 t + a^2\sqrt{\frac{\beta_2}{\alpha_2}}\sin\omega_2 t\right]\hat{X}(0)\ ,$$

$$- ab\,[\cos\omega_1 t - \cos\omega_2 t]\,\hat{p}(0) + [bd\cos\omega_1 t + ac\cos\omega_2 t]\,\hat{P}(0), \tag{1.38}$$

for the HO , and

$$\hat{x}(t) = \left[a^2\sqrt{\frac{\alpha_1}{\beta_1}}\sin\omega_1 t + b^2\sqrt{\frac{\alpha_2}{\beta_2}}\sin\omega_2 t\right]\hat{p}(0) - \left[ad\sqrt{\frac{\alpha_1}{\beta_1}}\sin\omega_1 t - bc\sqrt{\frac{\alpha_2}{\beta_2}}\sin\omega_2 t\right]\hat{P}(0)\ ,$$

and

$$+ [ac\cos\omega_1 t + bd\cos\omega_2 t]\,\hat{x}(0) - ab[\cos\omega_1 t - \cos\omega_2 t]\,\hat{X}(0)\ , \tag{1.39}$$

$$\hat{p}(t) = -\left[c^2\sqrt{\frac{\beta_1}{\alpha_1}}\sin\omega_1 t + d^2\sqrt{\frac{\beta_2}{\alpha_2}}\sin\omega_2 t\right]\hat{x}(0) + \left[bc\sqrt{\frac{\beta_1}{\alpha_1}}\sin\omega_1 t - ad\sqrt{\frac{\beta_2}{\alpha_2}}\sin\omega_2 t\right]\hat{X}(0)$$

$$+ [ac\cos\omega_1 t + bd\cos\omega_2 t]\,\hat{p}(0) - cd\,[\cos\omega_1 t - \cos\omega_2 t]\,\hat{P}(0)\ , \tag{1.40}$$

for the QO, where

$$a = e^{\frac{-\lambda}{2}}\cos\frac{\phi}{2}\ , \tag{1.41}$$

$$b = e^{\frac{-\lambda}{2}}\sin\frac{\phi}{2}\ , \tag{1.42}$$

$$c = e^{\frac{\lambda}{2}}\cos\frac{\phi}{2}\ , \tag{1.43}$$

and

$$d = e^{\frac{\lambda}{2}}\sin\frac{\phi}{2}\ , \tag{1.44}$$

with λ, ϕ given by eqs. (1.26) and (1.27).

By applying our definition (eq. (1.24)) to the above solutions, we find, after some algebra, the variances

$$\begin{aligned}\Delta X(t) &= \sqrt{\frac{\hbar}{2}}\,[d^2(b^2+c^2)\cos^2\omega_1 t + c^2(a^2+d^2)\cos^2\omega_2 t + d^2\frac{\alpha_1}{\beta_1}(a^2+d^2)\sin^2\omega_1 t \\ &+ c^2\frac{\alpha_2}{\beta_2}(b^2+c^2)\sin^2\omega_2 t + 2cd(ab-cd)\cos\omega_1 t\cos\omega_2 t \\ &+ 2cd\sqrt{\frac{\alpha_1\alpha_2}{\beta_1\beta_2}}(cd-ab)\sin\omega_1 t\sin\omega_2 t\Big]^{\frac{1}{2}}\ ,\end{aligned} \tag{1.45}$$

$$\Delta P(t) = \sqrt{\frac{\hbar}{2}}\Big[b^2(a^2+d^2)\cos^2\omega_1 t + a^2(b^2+c^2)\cos^2\omega_2 t + b^2\frac{\beta_1}{\alpha_1}(b^2+c^2)\sin^2\omega_1 t$$

$$c^2\frac{\beta_2}{\alpha_2}(a^2+d^2)\sin^2\omega_2 t+2ab(cd-ab)\cos\omega_1 t\cos\omega_2 t$$

$$+2ab\sqrt{\frac{\beta_1\beta_2}{\alpha_1\alpha_2}}(ab-cd)\sin\omega_1 t\sin\omega_2 t\Bigg]^{\frac{1}{2}}, \tag{1.46}$$

$$\Delta x(t)=\sqrt{\frac{\hbar}{2}}\,[a^2(b^2+c^2)\cos^2\omega_1 t+b^2(a^2+d^2)\cos^2\omega_2 t+a^2\frac{\alpha_1}{\mathrm{Pl}}(a^2+d^2)\sin^2\omega_1 t$$

$$+b^2\frac{\alpha_2}{\beta_2}(b^2+c^2)\sin^2\omega_2 t+2ab(cd-ab)\cos\omega_1 t\cos\omega_2 t$$

$$+2ab\sqrt{\frac{\alpha_1\alpha_2}{\mathrm{Pl}\beta_2}}(ab-cd)\sin\omega_1 t\sin\omega_2 t\Bigg]^{\frac{1}{2}}, \tag{1.47}$$

$$\Delta p(t)=\sqrt{\frac{\hbar}{2}}\,[c^2(a^2+d^2)\cos^2\omega_1 t+d^2(b^2+c^2)\cos^2\omega_2 t+c^2\frac{\beta_1}{\alpha_1}(b^2+c^2)\sin^2\omega_1 t$$

$$d^2\frac{\beta_2}{\alpha_2}(a^2+d^2)\sin^2\omega_2 t+2cd(ab-cd)\cos\omega_1 t\cos\omega_2 t$$

$$+2cd\sqrt{\frac{\beta_1\beta_2}{\alpha_1\alpha_2}}(cd-aà)\sin\omega_1 t\sin\omega_2 t\Bigg]^{\frac{1}{2}} \tag{1.48}$$

The results given in eqs. (1.45) - (1.48) are plotted for some values of the parameters m, k, g and y for the HO and for the QO. These are shown in Figures 1, 2, 5, 6, for the HO and Figures 3, 4, 7, 8 for the QO. Solid lines are for variances A X and Ax, while dashed lines are for variances A P and Ap. The vertical scales are in units of $\sqrt{\hbar/2}$. From the behavior of the variantes shown in these plots we point out the following interesting features:

i - if $\zeta = mk = 1$ then $f_2 = 0$ (eqs. (1.11) and (1.12)) and there is no squeezing for the QO. As a consequence no squeezing is "transferred" to the HO, with both oscillators remaining in a coherent state if their initial states are coherent.

ii- if $\zeta = mk > 1$ then $f_2 \neq 0$, and the QO exhibits squeezing in one quadrature for small value of the coupling constant γ (Figure 3), but the squeezing is alternating for both quadratures for large values of y (Figure 4). In this case the squeezing is transferred to the HO and contrary to the corresponding QO, the squeezing in the HO is alternating for both quadratures for small values of y (Figure 1) while existing only in one of the two quadratures for large values of y (Figure 2). Also we notice that whereas variance A P of the HO starts squeezing it is the variance Ax of the QO that starts squeezing. Hence the variance A P (AX) of the HO has a behavior similar to the variance Ax (Δp) of the QO when $\zeta = mk > 1$.

iii- if $\zeta = mk < 1$ we have again $f_2 \neq 0$. Figures 5, 6, 7, 8 stand for this case and they show a similar behavior as in Figures 1, 2, 3, 4 the difference being a change in the roles played by the variances $\Delta X \leftrightarrow$ A P and Ax $\leftrightarrow$ Ap when we pass from $\zeta > 1$ to $\zeta < 1$. Compare Figure 5 with Figure 1 (Figure 7 with Figure 3) and Figure 6 with Figure 2 (Figure 8 and Figure 4). Again there is here also a change of behavior when we pass from a small value of y (y = 0.2) to a large value of y (y = 2.0), as commented in item ii.

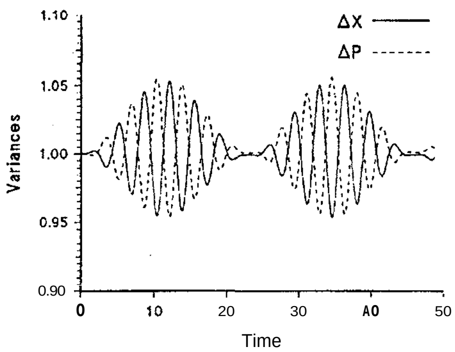

Figure 1: Variances of the HO as function of. time, for parameters values $\mathbf{m} = 1.5, k = 1, g = 1$, and $\gamma = 0.2; \zeta = 1.5$.

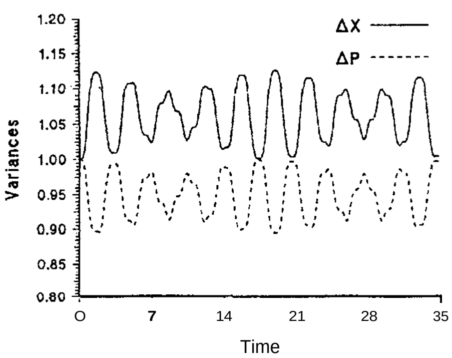

Figure 2: Variances of the HO as function of time, for parameters values $\mathbf{m} = 1.5, k = 1, g = 1$, and $\mathbf{y} = 2.0; \zeta = 1.5$.

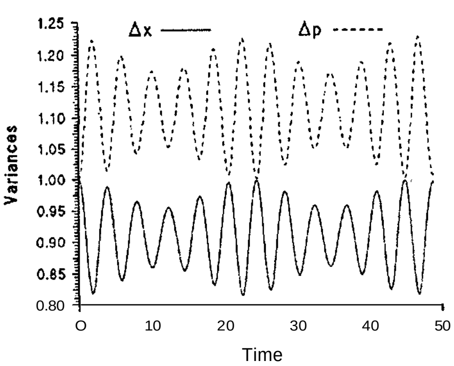

Figure 3: Same quantities as in Figure 1, for the QO.

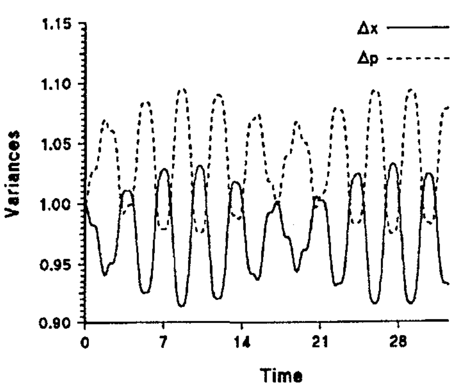

Figure 4: The same as in Figure 2, for the QO.

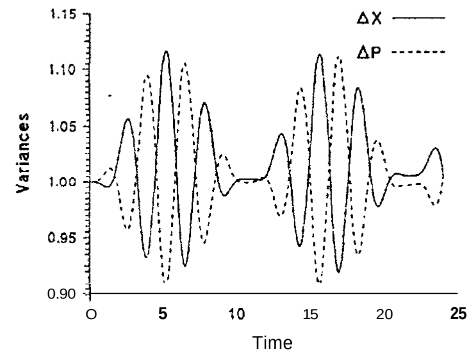

Figure 5: Variances of the HO as function of time, for parametersvalues $m = 0.5, k = 1, g = 1$, and $y = 0.2; \zeta = 0.5$.

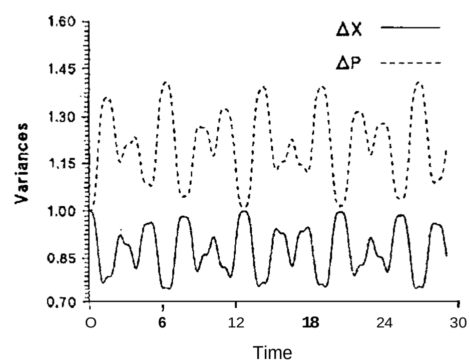

Figure 6: Variances for the HO as function of time for parameters values $m = 0.5, k = 1, g = 1$ and $y = 2.0, \zeta = 0.5$.

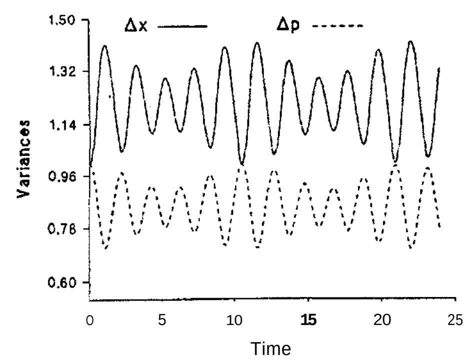

Figure 7: Same quantities as in Figure 5, for the QO.

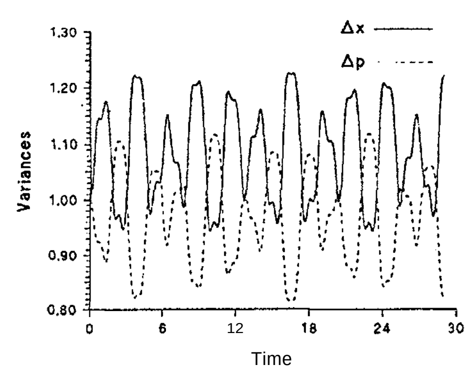

Figure 8: Same quantities as in Figure 6, for the QO.

Other interesting features displayed by this system, such as collapses and revivals of oscillations in the variances (see. Figures 1, 5), blow-up in variantes for asymptotic times and intermediate values of y (y $\sim$ 0.9), statistics of excitations, etc. are being investigated and will be discussed in details elsewhere.

As final remarks we mention that our HO acted upon by the QO displays features that are somewhat similar to those displayed by a HO acted upon by a tirne-dependent magnetic field[9]. Also, we should mention that a two-level atom interacting with a radiation field originates squeezing in the field, under certain conditions, which emerges from the squeezing generated in the atomic dipole operators[10].

Acknowledgements

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