Localization in the Anderson Model with Long Range Hopping

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We give a proof of exponential localization in the Anderson model with long range hopping based on a multiscale analysis.

I. Introduction

We consider the random Hamiltonian

$$H = \Gamma + V \quad \text{on} \quad \ell^2(\mathbf{Z}^d) , \qquad (1.1)$$

where

1. Γ is a translation invariant self-adjoint operator with exponentially decaying matrix elements, i.e., $\Gamma(x,y) = \phi(x-y)$ for some function ϕ on $\mathbf{Z}^d$ with $\phi(-x) = \overline{\phi(x)}$ for which there exist $C < \infty$ and $\gamma > 0$ such that
$$|\Gamma(x,y)| = |\phi(x-y)| \leq C e^{-\gamma\|x-y\|} \qquad (1.2)$$
for all $x, y \in \mathbf{Z}^d$.
2. $V(x)$, $x \in \mathbf{Z}^d$, are independent identically distributed random variables with common probability distribution μ.

In the usual Anderson model[1] $\Gamma = -\Delta$, where $\Delta(x,y) = 1$ if $|x-y| = 1$ and zero otherwise.

In this article we are concerned with localization. We say that the random operator H exhibits localization in an energy interval I if H has only pure point spectrum in I with probability one. We have exponential localization in I if we have localization and all the eigenfunctions corresponding to eigenvalues in I have exponential decay. Localization for $-\Delta + V$ has been extensively studied[2-20].

In this article we extend the von Dreifus-Klein[15] proof of localization to random Hamiltonians of the form given in eq. (1.1). (Our methods also extend the original von Dreifus-Spencer[13,14] proof of decay of Green's functions). If μ satisfy certain regularities conditions, localization for such operators at high disorder or low energy has been proved by Aizenman and Molchanov[18]. The proof we give here, as other proofs based on a multiscale analysis[6,15], do not require regularity of μ, it only uses certain *a priori* probabilistic estimates about Green's functions in finite volumes. This has the advantage of allowing the treatment of potentials with singular probability distributions[11,16]. They can also be used to prove localization inside spectral gaps for small disorder[20].

II. Results

We start with notation and definitions.

If $\Lambda \subset \mathbf{Z}^d$, we denote by H_Λ the operator H restricted to Λ with zero boundary conditions outside Λ, i.e.,

$$H_\Lambda(x,y) = \begin{cases} H(x,y) & \text{if} \quad x, y \in \Lambda , \\ 0 & \text{otherwise} . \end{cases} \qquad (2.1)$$

The corresponding Green's function is $G_\Lambda(z) = (H_\Lambda - z)^{-1}$, defined for $z \notin \sigma(H_\Lambda)$. We will write

$$G_\Lambda(z;x,y) = (H_\Lambda - z)^{-1}(x,y) \quad \text{for} \quad x,y \in \Lambda . \qquad (2.2)$$

If $\Lambda = Z^d$ we simply write $G(z;x,y)$. Notice that we omit the dependence of H_Λ and G_Λ on the potential V.

We will use $\mathbf{P}$ to denote the probability measure in the underlying probability space for the random variables $V(x)$, $x \in Z^d$. We will also take $C = 1$ in *(1.2)* without loss of generality.

For $x \in \mathbf{Z}^d$, $x = (x_1, \ldots, x_d)$, we set $\|x\| = \|x\|_\infty \equiv \max\{|x_1|, \ldots, |x_d|\}$. Distances in Z^d will always be taken with respect to this norm.

If $L > 0$, $x \in \mathbf{Z}^d$, we will denote by $\Lambda_L(x)$ the cube centered at x with sides of length L, i.e.,

$$\Lambda_L(x) = \{y \in \mathbf{Z}^d;\ \|y - x\| \leq \tfrac{L}{2}\}. \tag{2.3}$$

We will also use

$$\hat{\Lambda}_L(x) = \{y \in \Lambda_L(x);\ \|y - x\| > \tfrac{L}{4}\}. \tag{2.4}$$

We will say that $\psi \in \ell^2(\mathbf{Z}^d)$ decays exponentially fast with mass $m > 0$ if

$$\limsup_{\|x\| \to \infty} \frac{\log |\psi(x)|}{\|x\|} \leq -m \tag{2.5}$$

The following definition contains the key modification we make in the von Dreifus-Klein proof. We fix β, $0 < \beta < 1$.

DEFINITION *Let $m > 0$, $E \in \mathbf{R}$. A cube $\Lambda_L(x)$ is (m,E)-regular (for a fixed potential and given β) if*

$$d(E, \sigma(H_{\Lambda_L(x)})) \geq \frac{e^{-L^\beta}}{2} \tag{2.6}$$

and

$$|G_{\Lambda_L(x)}(E;x,y)| \leq e^{-m\,\|x-y\|} \tag{2.7}$$

for all $y \in \hat{\Lambda}_L(x)$. Otherwise we say that $\Lambda_L(x)$ is (m,E)-singular.

We can now state our main result.

THEOREM 2.1. Let $E_0 \in \mathbf{R}$ and $L_0 > 0$. Suppose we have:

(P1)

$$\mathbf{P}\{\Lambda_{L_0}(0) \text{ is } (m_0, E_0) - \text{regular}\} \geq 1 - \frac{1}{L_0^p} \tag{2.8}$$

for some $p > 2d$ and m_0 with $0 < m_0 \leq \frac{\gamma}{2}$.

(P2)

$$\mathbf{P}\{d(E, \sigma(H_{\Lambda_L(0)})) < e^{-L^\beta}\} \leq \frac{1}{L^q} \tag{2.9}$$

for some q with $q > 4p + 6d$, all E with $|E - E_0| \leq \eta$ for some $\eta > 0$, and all $L \geq L_0$.

Then, given m, with $0 < m < m_0$, there exists $B = B(p,d,\beta,q,\gamma,m_0,m) < \infty$, such that if $L_0 > B$, we can find $\delta = \delta(L_0,m_0,m,\beta) > 0$ so, with probability one, the spectrum of H in $(E_0 - \delta, E_0 + \delta)$ is pure point and the eigenfunctions corresponding to eigenvalues in $(E_0 - \delta, E_0 + \delta)$ decay exponentially fast at infinity with mass m.

The validity of (P1) and *(P2)* are discussed in ref. [15]. Notice that B and δ do not depend on E_0. Notice also that Theorem *2.1* is still valid if we weaken the requirements on p and m to $p > d$ (as in ref. [15]; notice that if $p > 2d$ we have $J = 3$ in ref. [15]) and $0 < m < \gamma$.

As in ref. [15], Theorem *2.1* will follow from Theorems *2.2* and *2.3*, which we will now state.

THEOREM 2.2. Let $I \subset \mathbf{R}$ be an interval and $L_0 > 0$. Suppose we have:

(K1)

$$\mathbf{P}\{\text{for any } E \in I \text{ either } \Lambda_{L_0}(x) \text{ or } \Lambda_{L_0}(y) \text{ is } (m_0, E) - \text{regular}\} \geq 1 - \frac{1}{L_0^{2p}} \tag{2.10}$$

for some $p > 2d$, m_0 with $0 < m_0 \leq \frac{\gamma}{2}$, and any $x, y \in \mathbf{Z}^d$ with $\|x - y\| > L_0$.

(K2)

$$\mathbf{P}\{d(E,\sigma(H_{\Lambda_L(0)})) < e^{-L^\beta}\} \leq \frac{1}{L^q} \tag{2.11}$$

for some q with $q > 4p + 6d$, all E with $d(E, I) \leq \frac{1}{2}e^{-L^\beta}$, and all $L \geq L_0$.

Then if we fix α, $1 < \alpha < \frac{2p}{p+2d}$, set $L_{k+1} = L_k^\alpha$, $k = 0,1,2,\ldots$, and pick m with $0 < m < m_0$, we can find $Q = Q(p,d,\beta,q,\gamma,m_0,\alpha,m) < \infty$, such that if $L_0 > Q$, we have that for any $k = 0,1,2,\ldots$,

$$\mathbf{P}\{\text{for any } E \in I \text{ either } \Lambda_{L_k}(x) \text{ or } \Lambda_{L_k}(y) \text{ is } (m,E)-\text{regular}\} \geq 1 - \frac{1}{L_k^{2p}} \tag{2.12}$$

for any $x, y \in \mathbf{Z}^d$ with $\|x - y\| > L_k$.

THEOREM 2.3. *Let $\mathbf{I} \subset \mathbf{R}$ be an interval, and let p, L_0, α, m be such that $p > d$, $L_0 > 0$, $1 < \alpha < \frac{2p}{d}$, $0 < m \leq \frac{\gamma}{2}$. Set $L_{k+1} = L_k^\alpha$, $k = 0,1,2,\ldots$. Suppose that we have (2.12) for all $k = 0,1,2,\ldots$ and any $x, y \in \mathbf{Z}^d$ with $\|x - y\| > L_k$. Then, with probability one, the spectrum of H in I is pure point and the eigenfunctions corresponding to eigenvalues in I decay exponentially fast at infinity with mass m.*

Theorems 2.1-2.3 are essentially the the same as in ref. ([15]), except that we have a differentdefinition for when a a cube $\Lambda_L(x)$ is (m,E)-regular, and we require that $m_0 < \gamma$ ($m_0 \leq \frac{\gamma}{2}$ is just to simplify the proofs). The probabilistic part of the proofs are not changed, we modify only the deterministic part ofthe proofs.

III. Proof of theorem 2.2

LEMMA 3.4. *For $1 < \ell < L$ and $0 < m \leq \frac{\gamma}{2}$, let $A = \Lambda_L(x_0)$ for some $x_0 \in \mathbf{Z}^d$, and let $x, y \in A$ be such that $\Lambda_\ell(x) \subset A$ is (m,E)-regular and $y \notin \Lambda_\ell(x)$. Then for any $E \in \mathbf{R}$ we have*

$$|G_\Lambda(E;x,y)| \leq e^{-M\|u-x\|}|G_\Lambda(E;u,y)| \tag{3.1}$$

for some $u \in A \setminus \Lambda_\ell(x)$, where

$$M = m - \frac{4}{\ell^{1-\beta}} - \frac{2\log\{(\ell+1)^d(L+1)^d\}}{\ell}. \tag{3.2}$$

Proof: *It follows from the resolvent identity that*

$$G_\Lambda(E;x,y) = -\sum_{s,t} G_{\Lambda_\ell(x)}(E;x,s)\,\Gamma(s,t)\,G_\Lambda(E;t,y), \tag{3.3}$$

where we sum over all $s \in \Lambda_\ell(x)$ and $t \in A \setminus \Lambda_\ell(x)$.

Thus there exist $s' \in \Lambda_\ell(x)$ and $t' \in A \setminus \Lambda_\ell(x)$ such that

$$|G_\Lambda(E;x,y)| \leq (\ell+1)^d(L+1)^d|G_{\Lambda_\ell(x)}(E;x,s')|\,e^{-\gamma\|t'-s'\|}\,|G_\Lambda(E;t',y)|, \tag{3.4}$$

where we used (1.2).

There are two possible situations:

(i) $s' \in \hat{\Lambda}_\ell(x)$

In this case it follows from eq. (2.7) that

$$|G_{\Lambda_\ell(x)}(E;x,s')|\,e^{-\gamma\|t'-s'\|} \leq e^{-m\|s'-x\|}\,e^{-\gamma\|t'-s'\|} \leq e^{-m\|t'-x\|}, \tag{3.5}$$

by the triangular inequality, since $m < \gamma$.

(ii) $s' \in \Lambda_\ell(x) \setminus \hat{\Lambda}_\ell(x)$

Now we use (2.6) to get

$$\begin{aligned} |G_{\Lambda_\ell(x)}(E;x,s')|\, e^{-\gamma\|t'-s'\|} &< 2e^{\ell^\beta}\, e^{-\gamma\|t'-s'\|} \le e^{-\gamma\|t'-x\|+\gamma(\ell/4)+\ell^\beta+\log 2} \\ &< e^{-(\frac{\gamma}{2}-\frac{4}{\ell^{1-\beta}})\|t'-x\|} \le e^{-(m-\frac{4}{\ell^{1-\beta}})\|t'-x\|}, \end{aligned} \tag{3.6}$$

where we used the reverse triangular inequality, $\|s'-x\| \le (\ell/4)$, $\|t'-x\| > \frac{\ell}{2}$ and m $\le \frac{\gamma}{2}$.

(3.1) now follows immediately from (3.4)-(3.6) and $\|t'-x\| > \frac{\ell}{2}$. ○

Lemma 3.1 replaces (4.2) in ref. [15]. We now need a definition.

DEFINITION *A cube $\Lambda_L(x)$ is non-resonant at the energy* E *if $d(E,\sigma(H_{\Lambda_L(x)})) \ge \frac{1}{2}e^{-L^\beta}$, i.e., if and only if $\|G_{\Lambda_L(x)}(E)\| \le 2e^{L^\beta}$. In this case we will say that $\Lambda_L(x)$ is* E $-$ *N*R.

The following lemma gives the deterministic part of the induction step in the proof of Theorem 2.2. It replaces Lemma 4.2 in [15].

LEMMA 3.5. Let $L = \ell^\alpha$ wih $1 < \alpha < 2$, E $\in$ R, *and m_ℓ with*

$$\left(\gamma\frac{36}{\ell^{\alpha-1}} + \frac{31}{\ell^{1-\beta}}\right) \le m_\ell \le \frac{\gamma}{2}. \tag{3.7}$$

Suppose

(i) *$\Lambda_L(x)$ is* E $-$ *NR*.

(ii) *$\Lambda_{j\ell}(y)$ is* E $-$ *NR for $j = 2,5,8$ and all $y \in \Lambda_L(x)$ with $\Lambda_{j\ell}(y) \subset \Lambda_L(x)$.*

(iii) *There exist at most* **3** *non-overlapping cubes of side ℓ contained in $\Lambda_L(x)$ that are (m_ℓ, E)-singular.*

There exists $Q' = Q'(d,\gamma,\beta,\alpha) < \infty$ such that if $\ell \ge Q'$, then $\Lambda_L(x)$ is (m_L, E)-regular with

$$m_L = m_\ell - \left(\gamma\frac{35}{\ell^{\alpha-1}} + \frac{30}{\ell^{1-\beta}}\right) \ge \left(\gamma\frac{36}{L^{\alpha-1}} + \frac{31}{L^{1-\beta}}\right). \tag{3.8}$$

Proof: Let m $= m_\ell$. By (iii) we have at most **3** non-overlapping cubes of side ℓ contained in $\Lambda_L(x)$ that are (m,E) singular. It follows that we can find u_i E $\Lambda_L(x)$, i $= 1,\dots,$ r, where $r \le 3$, such that if u E $\Lambda_L(x) \setminus \bigcup_{i=1}^r \Lambda_{2\ell}(u_i)$ with $d(u, \partial\Lambda_L(x)) \ge \frac{\ell}{2}$, then $\Lambda_\ell(u)$ is $(m,$ E)-regular.

A geometric argument now shows that we can find cubes $\Lambda_{\ell_i} \subset \Lambda_L(x)$ with side $\ell_i \in \{j\ell,\ j = 2,5,8\}$, i $=$ $1,\dots,$t, where $t \le r$, such that

$$d(\Lambda_{\ell_i}, \Lambda_{\ell_j}) \ge \ell \quad \text{if} \quad i \ne j, \tag{3.9}$$

$$\Xi \equiv \bigcup_{i=1}^t \Lambda_{\ell_i} \supset \bigcup_{i=1}^r \Lambda_{2\ell}(u_i), \tag{3.10}$$

and

$$\sum_{i=1}^t \ell_i \le 8\ell. \tag{3.11}$$

It follows that if u $\in \Lambda_L(x) \setminus \Xi$ and $d(u, \partial\Lambda_L(x)) \ge \frac{\ell}{2}$, we have that $\Lambda_\ell(u)$ is (m, E)-regular.

For $u \in \Lambda_L(x)$ we set $\Lambda'_\ell(u) = \Lambda_{\ell_i}$ if $u \in \Lambda_{\ell_i}$ and $\Lambda'_\ell(u) = \Lambda_\ell(u)$ if u $\notin \Xi$.

SUBLEMMA **3.6.** *Suppose* u *E E, with $d(\Lambda'_\ell(u), \partial\Lambda_L(x)) > \ell$, and* y E $\Lambda_L(x) \setminus \Lambda'_\ell(u)$. *There exist $\ell_1 = \ell_1(d,\beta) < \infty$, such thot for $\ell \ge \ell_1$* we *have*

$$|G_{\Lambda_L(x)}(E;u,y)| \le e^{-(m-\frac{25}{\ell^{1-\beta}})\|t_1-s_1\|}\, |G_{\Lambda_L(x)}(E;t_1,y)|. \tag{3.12}$$

for some $s_1 \in \Lambda'_\ell(u)$ and $t_1 \in \Lambda_L(x) \setminus \Lambda'_\ell(u)$.

Proof: We use the resolvent identity as in (3.3) and (3.4) to get

$$|G_{\Lambda_L(x)}(E;u,y)| \le (\ell_{i'}+1)^d (L+1)^d |G_{\Lambda_{\ell_{i'}}}(E;u,s)| e^{-\gamma\|t-s\|} |G_{\Lambda_L(x)}(E;t,y)|, \tag{3.13}$$

for some $s \in \Lambda'_\ell(u)$ and $t \in \Lambda_L(x) \setminus \Lambda'_\ell(u)$. Since $\Lambda'_\ell(u)$ is E-NR by (ii), we have

$$\begin{aligned} |G_{\Lambda_L(x)}(E;u,y)| &\le (\ell_{i'}+1)^d (L+1)^d 2e^{\ell_{i'}{}^\beta} e^{-\gamma\|t-s\|} |G_{\Lambda_L(x)}(E;t,y)| \\ &\le (8\ell+1)^d (L+1)^d 2e^{(8\ell)^\beta} e^{-\gamma\|t-s\|} |G_{\Lambda_L(x)}(E;t,y)|. \end{aligned} \tag{3.14}$$

If $\|t-s\| \ge \frac{\ell}{2}$, the sublemma follows immediately from (3.14) and (3.7). If not, $\Lambda_\ell(t)$ must be (m, E)-regular, so we can use Lemma 3.1 to estimate $|G_{\Lambda_L(x)}(E;t,y)|$ in (3.14). We get

$$|G_{\Lambda_L(x)}(E;u,y)| \le (8\ell+1)^d (L+1)^d 2e^{(8\ell)^\beta} e^{-\gamma\|t-s\|} e^{-m'\|t'-t\|} |G_{\Lambda_L(x)}(E;t',y)| \tag{3.15}$$
$$\le e^{-\ell^\beta} e^{-\gamma\|t-s\|} e^{-m''\|t'-t\|} |G_{\Lambda_L(x)}(E;t',y)|$$
$$\le e^{-\ell^\beta} e^{-m''\|t'-s\|} |G_{\Lambda_L(x)}(E;t',y)| \tag{3.16}$$

for some $t' \in \Lambda_L(x) \setminus \Lambda_\ell(t)$, where m' is given by the right hand side of (3.2) and

$$m'' = m' - \frac{20}{\ell^{1-\beta}} > m - \frac{25}{\ell^{1-\beta}}, \tag{3.17}$$

where (3.15) and (3.17) are valid for $\ell \ge \ell_1$, for some $\ell_1 = \ell_1(d,\beta) < \infty$, since $\|t'-t\| > \frac{\ell}{2}$.

If $t' \notin \Lambda_{\ell_{i'}}$, the sublemma is proved. If not, it follows from (3.16) that

$$|G_{\Lambda_L(x)}(E;u,y)| \le e^{-\ell^\beta} |G_{\Lambda_L(x)}(E;t',y)| \tag{3.18}$$

since $m'' > 0$ by (3.17) and (3.7). In this case we apply the above argument to $|G_{\Lambda_L(x)}(E;t',y)|$, either obtaining (3.12) for it (and hence for $|G_{\Lambda_L(x)}(E;u,y)|$ by (3.18)), or getting (again using (3.18))

$$|G_{\Lambda_L(x)}(E;u,y)| \le e^{-2\ell^\beta} |G_{\Lambda_L(x)}(E;t'',y)| \tag{3.19}$$

for some $t'' \in \Lambda'_\ell(t') = \Lambda'_\ell(u)$. Since we can keep on repeating this argument, we eventually get (3.12), unless $|G_{\Lambda_L(x)}(E;u,y)| = 0$, in which case there is nothing to prove. □

We can now finish the proof of Lemma 3.2. Let $\ell \ge \ell_1$; for $y \in \hat{\Lambda}_L(x)$ we estimate $|G_{\Lambda_L(x)}(E;x,y)|$ by using either Lemma 3.1 or Sublemma 3.2, as appropriate, starting from the center x of the box $\Lambda_L(x)$. Setting $M' = m - \frac{25}{\ell^{1-\beta}}$, we get, after n steps,

$$|G_{\Lambda_L(x)}(E;x,y)| \le e^{-M'(\|u_1-\tilde{x}\|+\|u_2-\tilde{u}_1\|+\cdots+\|u_n-\tilde{u}_{n-1}\|)} |G_{\Lambda_L(x)}(E;u_n,y)|, \tag{3.20}$$

with $u_1,\dots,u_n \in \Lambda_L(x)$, $\tilde{u} = u$ if $u \notin \Xi$ and otherwise $\tilde{u} \in \Lambda'_\ell(u)$, as long as we obtained $u_i \notin \Lambda'_\ell(y)$ with $d(\Lambda'_\ell(u_i), \partial\Lambda_L(x)) > 1$ for $i = 1,\dots,n-1$. The $(n+1)$th step can now be performed, if we have

$$u_n \notin \Lambda'_\ell(y) \quad \text{and} \quad d(\Lambda'_\ell(u_n), \partial\Lambda_L(x)) > \ell \tag{3.21}$$

By throwing away terms in (3.20), we can assume that each Λ_{ℓ_i} is visited only once, i. e., $u_i, u_j \in \Xi$, $i \ne j$, imply $\Lambda'_\ell(u_i) \ne \Lambda'_\ell(u_j)$. Thus, using (3.11), we have

$$\|u_1-\tilde{x}\| + \|u_2-\tilde{u}_1\| + \cdots + \|u_n-\tilde{u}_{n-1}\| \ge \|u_1-x\| + \|u_2-u_1\| + \cdots + \|u_n-u_{n-1}\| - 8\ell. \tag{3.22}$$

The procedure must be stopped the first time (3.21) is violated; in which case we have

$$\|u_1 - x\| + \|u_2 - u_1\| + \cdots + \|u_n - u_{n-1}\| \geq \|u_n - x\| \geq \|y - x\| - (9\ell + 1)\,. \tag{3.23}$$

It now follows from (3.20), (3.22), (3.23) and (i) that

$$|G_{\Lambda_L(x)}(E;x,y)| \leq 2\mathrm{e}^{L^\beta}\,\mathrm{e}^{-M'(\|x-y\|-(17\ell+1))} < \mathrm{e}^{-M''\|x-y\|}\,, \tag{3.24}$$

with

$$M'' = (m - \frac{25}{\ell^{1-\beta}})(1 - \frac{69}{\ell^{\alpha-1}}) - \frac{5}{L^{1-\beta}} \tag{3.25}$$

since $\|x - y\| > \frac{L}{4}$. It follows that there exists $Q' = Q'(d,\gamma,\beta,\mathrm{a}) < \infty$, such that if $\ell \geq Q'$ we have, using (3.7),

$$\begin{aligned} M'' &\geq \mathrm{m} - \left(m\frac{69}{\ell^{\alpha-1}} + \frac{25}{\ell^{1-\beta}} + \frac{5}{\ell^{\alpha(1-\beta)}}\right) \geq m - \left(\gamma\frac{35}{\ell^{\alpha-1}} + \frac{30}{\ell^{1-\beta}}\right) \\ &\geq \left(\gamma\frac{1}{\ell^{\alpha-1}} + \frac{1}{\ell^{1-\beta}}\right) \geq \left(\gamma\frac{36}{L^{\alpha-1}} + \frac{31}{L^{1-\beta}}\right)\,. \end{aligned} \tag{3.26}$$

The lemma is proved. □

Theorem 2.2 is now proven as in ref. [15]. The probabilistic part of the proof is the same, the deterministic induction step is given by Lemma 3.2.

IV. Proof of theorem 2.3

We recall that E is a generalized eigenvalue for H as in (1.1), if there exists a nonzero polynomially bounded function ψ on $\mathbf{Z}^d$ such that

$$\sum_{y\in\mathbf{Z}^d} H(x,y)\psi(y) = E\psi(y) \qquad \text{for all } x \in \mathbf{Z}^d\,. \tag{4.1}$$

In this case ψ is called a generalized eigenfunction.

We use the following basic result[21,22,23]; notice that in $\ell^2(\mathbf{Z}^d)$ the proof does not require $\Gamma = -\mathbf{A}$, it suffices for Γ to be as in (1.1):

With respect to the spectral measure of H, *almost every energy is a generalized eigenvalue.*

Thus, Theorem 2.3 follows from the following lemma, as in ref. [15].

LEMMA 4.7. *Under the hypothesis of Theorem* **2.3,** *with probability one the generalized eigenfunctions of* H, *corresponding to generalized eigenvalues in I, decay exponentially fast at infinity with mass* m.

Lemma 4.1 is proved in the same way as Lemma 3.1 in ref. [15], the necessary modifications will be given below as lemmas. H will always be as in (1.1) and β, m, a, L_k as in Theorem 2.3. Notice the lemmas are stated for a fixed potential V .

LEMMA 4.8. *Let* **E** *be a generalized eigenvalue for* H, *with corresponding generalized eigenfunction ψ. Suppose $\Lambda_\ell(x)$ is a* (m, *E*)*-regular box, then*

$$|\psi(x)| \leq \sum_{y\notin\Lambda_\ell(x)} \mathrm{e}^{-M\|y-x\|}\,|\psi(y)|\,, \tag{4.2}$$

with

$$M = m - \frac{4}{\ell^{1-\beta}} - \frac{2\log\{(\ell+1)^d\}}{\ell}\,. \tag{4.3}$$

Proof: Since $\Lambda_\ell(x)$ is a (m, E)-regular, E $\notin \sigma(H_{\Lambda_\ell(x)})$, so it follows from (4.1) that

$$\psi(x) = -\sum_{\substack{u\in\Lambda_\ell(x)\\ y\notin\Lambda_\ell(x)}} G_{\Lambda_\ell(x)}(E;u,y)\,\Gamma(u,y)\,\psi(y)\,, \tag{4.4}$$

hence

$$|\psi(x)| \leq \sum_{\substack{u\in\Lambda_\ell(x)\\ y\notin\Lambda_\ell(x)}} |G_{\Lambda_\ell(x)}(E;u,y)|\,\mathrm{e}^{-\gamma\|y-u\|}\,\psi(y)\,. \tag{4.5}$$

The lemma now follows by the same argument as in the proof of Lemma 3.1. □

LEMMA 4.9. *Let E be a generalited eágenvalue for* H, *with corresponding generalized eigenfunction* ψ. *Suppose* $x_0 \in \mathbf{Z}^d$ *is such that* $\psi(x_0) \neq 0$. *Then there exists* $k_1 = k_1(V, E, x_0) < \infty$, *such that* $\Lambda_{L_k}(x_0)$ *is* (m, E)-*singular for all* $\mathrm{k} \geq k_1$.

Proof: Suppose the lemma is false, then there exists a sequence $k_n \to \infty$ such that $\Lambda_{L_{k_n}}(x_0)$ is (m, E)-regular for all n. But then it follows from Lemma 4.2 that

$$|\psi(x_0)| \leq \lim_{n\to\infty} \sum_{y \notin \Lambda_{L_{k_n}}(x_0)} \mathrm{e}^{-\frac{m}{2}\|y-x_0\|} |\psi(y)| = 0 , \tag{4.6}$$

since ψ is polynomially bounded. This is a contradiction. □

We set

$$A^{(b)}_{k+1}(x_0) = \Lambda_{2bL_{k+1}}(x_0) \setminus \Lambda_{2L_k}(x_0) , \tag{4.7}$$

where b is a positive integer.

LEMMA 4.10. *Let E be a generalited eigenvalue for* H, *with corresponding generalized eigenfunction* ψ, *and* $x_0 \in \mathbf{Z}^d$ *is such that* $\psi(x_0) \neq 0$. *Suppose that for all* b *there exists* $\bar{k}_b < \infty$, *such that for* $\mathrm{k} \geq \bar{k}_b$ *we have* $\Lambda_{L_k}(x)$ *(m,E)-regular for all* $\mathrm{x} \in A^{(b)}_{k+1}(x_0)$. *Then* ψ *decays exponentially fast at infinity with mass* m.

Proof: Since ψ is polynomially bounded, there exists $t > 0$ such that

$$|\psi(x)| \leq (1 + \|x - x_0\|)^t \quad \text{for all } \mathrm{x} \in \mathbf{Z}^d , \tag{4.8}$$

if ψ is properly normalized.

Now let p; $0 < \mathrm{p} < 1$, be given, we pick $\mathrm{b} > \frac{1+\rho}{1-\rho}$, and define

$$\tilde{A}_{k+1}(x_0) = \tilde{A}^{(b,\rho)}_{k+1}(x_0) = \Lambda_{\frac{1+\rho}{1+\rho}L_{k+1}}(x_0) \setminus \Lambda_{\frac{2}{1-\rho}L_k}(x_0) . \tag{4.9}$$

Then $\tilde{A}_{k+1}(x_0)$ C $A_{k+1}(x_0) = A^{(b)}_{k+1}(x_0)$, and, if $\mathrm{x} \in \tilde{A}_{k+1}(x_0)$, we have

$$d(x, \partial A_{k+1}(x_0)) \geq \rho\|x - x_0\|. \tag{4.10}$$

Moreover, if $\|x - x_0\| > \frac{L_0}{1-\rho}$, we have $\mathrm{x} \in \tilde{A}_{k+1}(x_0)$ for some k.

So let us fix x E $\tilde{A}_{k+1}(x_0)$, with $\mathrm{k} \geq \bar{k}_b$. It follows that $\Lambda_{L_k}(y)$ is (m, E)-regular for any $\mathrm{y} \in \Lambda_{\rho\|x-x_0\|}(x)$ C $A_{k+1}(x_0)$. We now apply Lemma 4.2 with $\ell = L_k$; it follows from (4.2) that for any $\mathrm{y} \in \Lambda_{\rho\|x-x_0\|}(x)$ with $\delta_y = \rho\|x - x_0\| - \|y - x\| > \frac{L_k}{2}$, there exists $\mathrm{u} \in \Lambda_{2\delta_v}(y) \setminus \Lambda_{L_k}(y)$, such that

$$|\psi(y)| \leq (2\delta_y + 1)^d \mathrm{e}^{-m_1\|u-y\|} |\psi(u)| + \sum_{v \notin \Lambda_{2\delta_y}(y)} \mathrm{e}^{-m_1\|v-y\|} |\psi(v)| , \tag{4.11}$$

with m_1 given by the right hand side of (4.3). We now use (4.8), so

$$\begin{aligned} \sum_{v \notin \Lambda_{2\delta_y}(y)} \mathrm{e}^{-m_1\|v-y\|} |\psi(v)| &\leq \sum_{v \notin \Lambda_{2\delta_y}(y)} \mathrm{e}^{-m_1\|v-y\|} (1 + \|v - x_0\|)^t \\ &\leq \sum_{v \notin \Lambda_{2\delta_y}(y)} \mathrm{e}^{-m_1\|v-y\|} (1 + bL_{k+1} + \|v - y\|)^t \\ &\leq \mathrm{e}^{-m_2\delta_y} , \end{aligned} \tag{4.12}$$

witli $m_2 = \mathrm{m} - \frac{5}{L_k^{1-\beta}}$, in case $L_k \geq \ell_1$ for some $\ell_1 = \ell_1(d,\beta,\mathrm{a},\mathrm{m},\mathrm{b},t) < \infty$. On the other liand,

$$(2\delta_y + 1)^d \, \mathrm{e}^{-m_1\|u-y\|} \leq (2bL_{k+1}+1)\, \mathrm{e}^{-m_1\|u-y\|} \leq \mathrm{e}^{-m_2\|u-y\|} \tag{4.13}$$

if $L_k \geq \ell_2$ for some $\ell_2 = \ell_2(d,\beta,\mathrm{a},b) < \infty$.

Thus, given $y \in \Lambda_{\rho\|x-x_0\|}(x)$ with $\delta_y > \frac{L_k}{2}$, if $L_k \geq \ell_3 = \max(\ell_1,\ell_2)$ there exists u E $\Lambda_{2\delta_y}(y) \setminus \Lambda_{L_k}(y)$, such that

$$|\psi(y)| \leq \mathrm{e}^{-m_2\|u-y\|}\,|\psi(u)| + \mathrm{e}^{-m_2\delta_y} . \tag{4.14}$$

We now start from x and apply (4.14) repeatedly, obtaining, at the nth step,

$$\begin{aligned}
|\psi(x)| &\leq \mathrm{e}^{-m_2\delta_x} + \mathrm{e}^{-m_2\|y_1-x\|}\,|\psi(y_1)| \\
&\leq \mathrm{e}^{-m_2\delta_x} + \mathrm{e}^{-m_2\|y_1-x\|}\,\mathrm{e}^{-m_2\delta_{y_1}} + \mathrm{e}^{-m_2\|y_1-x\|}\,\mathrm{e}^{-m_2\|y_2-y_1\|}\,|\psi(y_2)| \\
&\leq \mathrm{e}^{-m_2\delta_x} + \mathrm{e}^{-m_2\{\|y_1-x\|+\delta_{y_1}\}} + \cdots + \mathrm{e}^{-m_2\{\|y_1-x\|+\|y_2-y_1\|+\cdots\|y_{n-1}-y_{n-2}\|+\delta_{y_{n-1}}\}} \\
&\quad + \mathrm{e}^{-m_2\{\|y_1-x\|+\|y_2-y_1\|+\cdots\|y_n-y_{n-1}\|\}}\,|\psi(y_n)| ,
\end{aligned} \tag{4.15}$$

for some $y_n \in \Lambda_{\rho\|x-x_0\|}(x)$ with $\|y_n - y_{n-1}\| > \frac{L_k}{2}$, in case the first $n-1$ applications of (4.14) gave us $y_1,\ldots,y_{n-1} \in \Lambda_{\rho\|x-x_0\|}(x)$, $\|y_1 - x\|,\ldots,\|y_{n-1}-y_{n-2}\| > \frac{L_k}{2}$, with $\delta_{y_1},\ldots,\delta_{y_{n-1}} > \frac{L_k}{2}$. We have two cases:

1. We obtain (4.15) with n such that

$$\frac{2}{L_k}\rho\|x-x_0\| \leq n < \frac{2}{L_k}\rho\|x-x_0\| + 1 . \tag{4.16}$$

In this case we have

$$\begin{aligned}
|\psi(x)| &\leq n\,\mathrm{e}^{-m_2\rho\|x-x_0\|} + \mathrm{e}^{-m_2 n\frac{L_k}{2}}\,|\psi(y_n)| \\
&\leq \left(\left(\frac{2}{L_k}\rho\|x-x_0\|+1\right) + |\psi(y_n)|\right)\mathrm{e}^{-m_2\rho\|x-x_0\|} \\
&\leq \left(\frac{2bL_{k+1}}{L_k} + 1 + (1+bL_{k+1})^t\right)\mathrm{e}^{-m_2\rho\|x-x_0\|} \\
&\leq \mathrm{e}^{-m_3\rho\|x-x_0\|} ,
\end{aligned} \tag{4.17}$$

if $L_k \geq \ell_4$ for some $\ell_4 = \ell_4(d,\beta,\alpha,b,m) < \infty$, where

$$m_3 = m - \frac{6}{L_k^{1-\beta}} . \tag{4.18}$$

2. We must stop the procedure with n $< \frac{2}{L_k}\rho\|x-x_0\|$. In this case we must have $\delta_{y_n} \leq \frac{L_k}{2}$, so we rnust have $\|y_n - x\| \geq \rho\|x-x_0\| - \frac{L_k}{2}$. It now follows from (4.15) that

$$\begin{aligned}
|\psi(x)| &\leq n\,\mathrm{e}^{-m_2\rho\|x-x_0\|} + \mathrm{e}^{-m_2\|y_n-x\|}|\psi(y_n)| \\
&\leq n\,\mathrm{e}^{-m_2\rho\|x-x_0\|} + \mathrm{e}^{-m_2\left(\rho\|x-x_0\|-\frac{L_k}{2}\right)}\,|\psi(y_n)|
\end{aligned} \tag{4.19}$$

But since y_n E $\Lambda_{\rho\|x-x_0\|}(x)$, we know that $\Lambda_{L_k}(y_n)$ is (m, E)-regular, so it follows from (4.11) and (4.12) with $\frac{L_k}{2}$ substituted for δ_y, that

$$|\psi(y_n)| \leq \mathrm{e}^{-m_2\frac{L_k}{2}} . \tag{4.20}$$

Combining (4.19) and (4.20) we get

$$\begin{aligned} |\psi(x)| &\le n\,e^{-m_2\rho\|x-x_0\|}+e^{-m_2\rho\|x-x_0\|} \\ &\le \left(\frac{2}{L_k}\rho\|x-x_0\|+1\right)e^{-m_2\rho\|x-x_0\|} \\ &\le \left(\frac{2bL_{k+1}}{L_k}+1\right)e^{-m_2\rho\|x-x_0\|} \\ &\le e^{-m_3\rho\|x-x_0\|}, \end{aligned} \tag{4.21}$$

if $L_k \ge \ell_4$.

It follows from (4.17), (4.21) and (4.18) that, given ρ', $0<\rho'<1$, we can find $\hat{k}<\infty$ such tliat, if $k\ge\hat{k}$ we have

$$|\psi(x)| \le e^{-\rho'\rho\|x-x_0\|} \tag{4.22}$$

in case $\|x-x_0\| > \frac{L_k}{1-\rho}$.

We can conclude that ψ decays exponentially, and

$$\limsup_{\|x\|\to\infty}\frac{\log|\psi(x)|}{\|x\|} \le -\rho'\rho m \tag{4.23}$$

for any ρ and $\rho' \in (0,1)$.

Lemma 4.4 is proved. □

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