Controlling the Effect of Griffiths' Singularities in Random Ferromagnets: Smoothness of the Magnetization

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The magnetization of random ferromagnets with site or bond disorder is shown to be a differentiable function of the externa1 magnetic field at sufficiently high temperatures. This is shown to happen even in a region of the parameters where this function is not analytic as a consequence of the so called Griffiths' singularities. The result is proven through the use of correlation inequalities specific of ferromagnetic systems and with the weakest possible assumptions on the probability distribution of the random couplings. With the same methods one can actually prove infinite differentiability in the same region of parameters. We also discuss the analogous problem for ferromagnets in (d + 1) dimensions with d-dimensional disorder. In this case translation invariance in the extra direction produces slower decay rates in the presence of Griffiths' singularities. The results for the latter systems are of importance in the analysis of quantum disordered ferromagnetic models.

I. Introduction

In the Statistical Mechanics of Disordered Systems a special role is played by the family of random ferromagnets. **A** typical representative of this class is the Ising model in $\mathbf{Z}^d$, with configurations a $= \{a, = \pm 1, x \in \mathbf{Z}^d\}$ and with energy function for the system in a finite volume h $\subset \mathbf{Z}^d$ given by:

$$H_\Lambda(\sigma, \mathbf{J}) = -\sum_{\langle xy \rangle \in \Lambda} J_{xy}\sigma_x\sigma_y + h\sum_{x\in\Lambda}\sigma_x \ , \qquad (1.1)$$

where $\mathbf{J} = \{J_{xy}, (xy) \in \mathbf{Z}^d\}$ is a family of identically distributed random variables satisfying: $J_{xy} \geq 0$, where the notation (xy) means that $|x - y| = 1$, i.e. the summation is taken only over pairs of nearest neighbor sites in $\mathbf{Z}^d$. The model will be called bond disordered if the random variables J $= \{J_{xy}, \langle xy \rangle \in \mathbf{Z}^d\}$ are taken as independent, whereas if $J_{xy} = \xi_x\xi_y$J where $\xi = \{\xi_x \geq 0, x$ E $\mathbf{Z}^d\}$ are independent random variables the model will be called site disordered. For a given realization of the random parameters J $= \{J_{xy}, (xy)$ E $\mathbf{Z}^d\}$ the thermal expectations at inverse temperature β of an observable A**(a)** are defined in the usual way:

$$(A),(\mathbf{J}) = \frac{1}{Z_\Lambda(\mathbf{J})}\sum_\sigma \exp\left[-\beta H_\Lambda(\sigma, \mathbf{J})\right] \ . \qquad (1.2)$$

The so called quenched expectations are defined as averages over the random parameters J and will be denoted by $\overline{(A)},$. We will be primarily interested in these quantities considered in the thermodynarnical limit:

$$(\mathbf{A})(\mathbf{J}) = \lim_{\mathrm{A} - \mathbf{Z}^d} \langle A \rangle_\Lambda(\mathbf{J}) \quad \text{and} \quad (\mathrm{A}) = \lim_{\Lambda \to \mathbf{Z}^d} \overline{\langle A \rangle_\Lambda} \ .$$

We shall be interested mainly in the case where the external magnetic field h is zero, but in many cases we will have to take the limit h $\to$ 0 only after taking the thermodynamic limit A $\to \mathbf{Z}^d$.

In 1969 R. Griffiths[1] considered the site diliite model, with

$$\xi_x = \begin{cases} 1 \text{ with probability} p \\ 0 \text{ with probability } (1-p) \end{cases}, \qquad (1.3)$$

and called the attention to the fact that the quenched magnetization

$$m(z) = \overline{\langle \sigma_0 \rangle}$$

viewed as a function of $z = e^{-\beta h}$ displayed a non analytic behavior at $z = 0$ even at values of the inverse temperature β for which the system has neither spontaneous magnetization nor long range order, provided only $\beta J > K_c(d)$, the critical value for a homogeneous d-dimensional deterministic system with coupling $J_{xy} = J$ for all bonds (xy). This phenomenon goes nowadays under the name of Griffiths' singularities; its existence has been rigorously proved[2] and it is now recogriized as a regular feature in the Statistical Mechanics of disordered systems[3]. The physical origin of this behavior may be better understood in the following situation which dramatizes the phenomenon. Consider the site dilute model given by (1.3) and suppose p $< p_c(d)$, where $p_c(d)$ is the critical value for the occupation probability of a site in the site percolation problem in $\mathbb{Z}^d$. We are therefore in a situation where, with probability one, only finite clusters of sites which are coupled (i e. $J_{xy} = J$) with their nearest neighbors, so that the system decomposes into a collection of finite independent subsystems. In this case we may conclude that with probability one, there is no spontaneous magnetization nor long-range order for all values of the temperature. However, as a consequence of the law of large numbers, also with probabilty one, there are arbitrarily large d-dimensional boxes inside which, sites are coupled with their nearest neighbors with strength J. Now, if $\beta J > K_c(d)$ there will be *arbitrarily* large (but finite!) boxes inside which the system is strongly correlated, i.e. below the critical temperature, thus generating the singular behavior.

The above simplified situation suggests the general mechanisni for the phenomenon. Define a region $S \subset \mathbb{Z}^d$ to be singular when $\beta J_{ij} > K_c(d)$ for all bonds (ij) in S. Then, even if the inverse temperature is such that the system is not ordered (so that singular regions are mostly finite or even small) as a whole, there exist with probability one, arbitrarily large singular regions, i.e. finite regions inside which the system is strongly correlated.

Another remarkable consequence of these singularities is that for those models, with either site or bond disorder, where the couplings J_{xy} may assume arbitrarily large values (even if with very small but non-zero probability) the usual high temperature expansions do not converge for any values of β. An example of this situation is the bond disordered model where J_{xy} has a probability p distribution like a one-sided gaussian:

$$\rho(J) = \begin{cases} 0 \text{ for } J < 0 \\ \frac{2}{\sqrt{2\pi\sigma^2}} \exp - \frac{x^2}{2\sigma^2} \end{cases}$$

with arbitrary variance σ^2, with the property that $\rho(J) > 0$ for all J.

The next question to be asked concerns the nature of this singularity. The first rigorous result controlling the effect of Griffiths' singularities was obtained by Olivieri, Perez and Goulart Rosa[4] who discussed the bond disordered ferromagnetic model and showed exponential decay of correlation functions in the presence of Griffiths' singularities. Their results implied also, although not explicitly stated in the paper, infinite differentiability of the quenched magnetization (see discussion below). Their techniques however, could only be applied to the specific situation of bond disorder and for Ising systems, i.e. $\sigma_x = \pm 1$. It required also finite average of the coupling J_{xy}.

More general results, concerning exponential decay of truncated correlation functions for not necessarily ferromagnetic models, where obtained by Berretti[5] with strong restrictions on the probability distribution of the random parameters and subsequently by Fröhlich and Imbrie[6] through an intricate resummation of high-temperature or low-activity expansions. More recently Dreifus, Klein and Perez[7] produced a very general and simple proof of infinite differentiability of the magnetization at sufficiently high temperatures, with no assumption on the probabiblity distribution of the random parameters.

The purpose of this note is to show how the use of correlation inequalities specific of purely ferromagnetic systems drastically simplifies the analysis. We are going to show exponential decay of the two point functions and differentiability of the quenched magnetization as a function of the external magnetic field, in the presence of Griffiths' singularities at sufficiently high-temperatures. Actually our methods can be sharpened, along the lines discussed in ref. [7] to prove exponential decay of all truncated correlation functions and infinite differentiability of the qnenched magnetization, but this would go beyond the scope of this contribution. Our assumptions on the probability distribution of the random variables $\{J_{xy}\}$, apart from ferromagnetism are the weakest possible. In the case of bounded spins ($|\sigma_x| \leq$ c, for some positive constant c) J_{xy} may even take the value $+\infty$ if this happens with sufficiently small prohability. This is to be compared with the methods of ref.[4], which crucially requires a, $= \pm 1$ and $\overline{J_{xy}} < \infty$. With some restrictions on the probability distribution of the coupling J_{xy} we can also eliminate the restriction on the values of the spin variable 'Jx.

Another interesting class of random ferromagnetic spin sytems are the so called $(d+1)$-dimensional systeni with d-dimensional disorder. A typical representative of these models is the Ising model in $\mathbf{Z}^{d+1}$with energy function given by

$$H_{\Lambda\times T} = -\sum_{<xy>\in\Lambda}\sum_{t\in T} J(x,y)\,\sigma(x,t)\,\sigma(y,t) - \sum_{x\in\Lambda}\sum_{<s,t>\in T} K(x)\,\sigma(x,s)\,\sigma(x,t) \ , \qquad (1.4)$$

where $J = \{J(x,y), < xy > \in \mathbf{Z}^d\}$ and $K = \{K(x), x \in \mathbf{Z}\}$ are two families of independent, and within each family identically distributed random variables with J (x,y) $\geq$ 0 and K (x) $\geq$ O. The spin variables a $(x,t)$are taken to be ± 1, the symbol < uv >denotes that u and v are nearest neighbor points in $\mathbf{Z}^d$ or Z i.e.$|u-v| = 1$; $\Lambda \subset \mathbf{Z}^d$ and T $\subset$ Z are finite subsets. The characteristic feature of these systems is the fact that both "horizontal" J (x,y) and "vertical" $K(x)$ couplings do not depend on the "time" variable t E Z.

Apart from its own interest (see [8]), these models when realized in $\mathbf{Z}^d$ x $\left(\frac{1}{n}\mathbf{Z}\right)$, that is with lattice spacing $\left(\frac{1}{n}\right)$in the "time" direction and with the replacements

$$\begin{array}{c} J(x,y) \rightarrow \frac{1}{n} J(x,y) \\ K(x) \rightarrow -\frac{1}{2}\ln\tanh\frac{1}{n}h(x) \end{array} \ ,$$

produce in the limit n $\rightarrow \infty$ the path space of a quantum disordered system: the Ising model with a random transverse field in $\mathbf{Z}^d$ [9], [10] with Hamiltonian formally given by:

$$H = -\sum_{\langle xy\rangle} J(x,y)\,\sigma_3(x)\,\sigma_3(y) + \sum_x h(x)\,\sigma_1(x) \ ,$$

where $\sigma_i(x)$, i = 1, 2, 3 are usual Pauli spin operators. These models appear also in the study of contact processes in random environments[11,10].

In this paper we briefly revisit the problem of tlie phase diagram of such models and provide some insight into the nature of the effects of Griffiths' singularities for these systerns. It is intuitively clear that their effect should be even more serious than those for the standard random ferromagnets in $\mathbf{Z}^{d+1}$: as a consequence of translation invariance in the "time" direction the singular regions (defined as those regions inside which the system is strongly correlated) are now infinitely extended tubes of the form S x Z where S $\subset \mathbf{Z}^d$

is a finite set. For simplicity, we shall consider only a simplified model where one can explicitly compute how one looses exponential decay of the two-point function in the "time" direction. Our technique is considerably simpler, though similar in many respects to those used in refs. [8], [9], [11] and many of the results there may be obtained with the techniques described here. Our main ingredients are ferromagnetic correlation inequalities which in the general case, not considered here will have to be coupled to a multiscale analysis of the type used in ref. [13] and will be discussed elsewhere.

This paper is organized as follows. In section II we discuss the models given by (1.1) showing exponential decay of the two-point function and differentiability of the quenched magnetization. In section III we discuss the models in (d + 1)-dimensions with d-dimensional disorder given by (1.4).

II. Exponential decay of correlation functions

Let us consider the model given by eq. 1.1. For a given configuration $J = \{J_{xy}\}$, the two point function of the system can be estimated by

$$\langle \sigma_x \sigma_y \rangle_\Lambda \leq \sum_{\omega: x \to y} \prod_{\langle ij \rangle \in \omega} \tanh \beta J_{<ij>} , \qquad (2.1)$$

where the summation is taken over all self-avoiding paths $w : x \to y$ in $\mathbf{Z}^d$ connecting x to y, that is $w = \{\langle i_1 j_1 \rangle, \langle i_2 j_2 \rangle, \dots \langle i_n j_n \rangle\}$ with $i_1 = x$, $i_{k+1} = j_k$ for $k = 1, \dots, n-1$ and $j_n = y$; $i_k \neq i_l$ for $k \neq l$. We shall denote by $|\omega|$ the number n of steps in the walk w. This bound, derived by Fisher[12], is an immediate consequence of ferromagnetism ($J_{xy} \geq 0$) and of the spins being ± 1. Suppose now we are in the bond disordered model, so that $\{J_{xy}\}$ are independent and identically distributed random variables and let

$$\zeta = \overline{\tanh \beta J_{ij}} ,$$

where the bar indicates the average over the random parameters. In this case we may compute the average over the randomness of the above inequality, to obtain:

$$\overline{\langle \sigma_x \sigma_y \rangle_\Lambda} \leq \sum_{\omega: x \to y} \prod_{\langle ij \rangle \in \omega} \overline{\tanh \beta J_{ij}} = \sum_{\omega: x \to y} \prod_{\langle ij \rangle \in \omega} \zeta = \sum_{\omega: x \to y} \zeta^{|\omega|}$$

so that, if $2d\zeta < 1$

$$\overline{\langle \sigma_x \sigma_y \rangle_\Lambda} \leq \frac{1}{1 - 2d\zeta} (2d\zeta)^{|x-y|} = C e^{-m|x-y|}$$

where $m = -\ln(2d\zeta)$ and $C = (1 - 2dJ)^{-1}$. Since this estimate is uniform in the volume Λ, we can conclude that the quenched two-point function, in the infinite volume limit also decays exponentially, provided $2d\zeta < 1$:

$$\overline{\langle \sigma_x \sigma_y \rangle} \leq C e^{-m|x-y|} .$$

It is now simple to convert this bound on the averaged correlation function $\overline{\langle \sigma_x \sigma_y \rangle}$ into a bound implying exponential decay of the two point function $\langle \sigma_x \sigma_y \rangle$ (J) that is obtained with probability one in the realization of the random paramenters $J = \{J_{xy}\}$. This is through the use of Chebyschev's inequality:

$$\mathbf{P}\{|f| \geq \gamma\} \leq \frac{1}{\gamma} \overline{f} ,$$

where $P\{A\}$ denotes the probability of the event A. If we take

$$f = \sum_{\gamma} e^{\mu|x-y|} \langle \sigma_x \sigma_y \rangle ,$$

where $0 < \mu < m$, we obtain

$$\mathbf{P}\left\{ \sum_y e^{\mu|x-y|} \langle \sigma_x \sigma_y \rangle \geq \gamma \right\} \leq \frac{1}{\gamma} \sum_y e^{\mu|x-y|} \overline{\langle \sigma_x \sigma_y \rangle} < \infty ,$$

and this together with the Borel-Cantelli lemma implies that there exists $n_0(x) < \infty$, such that for $|x - y| \geq$

$n_0(x)$ with probability one

$$\langle\sigma_x\sigma_y\rangle \le \gamma e^{-\mu|x-y|} ,$$

or alternatively, for every $0 < \mu < m$ with probability one there exists $C_x(\mathbf{J},\mu) < \infty$, such that

$$\langle\sigma_x\sigma_y\rangle \le C_x(\mathbf{J},\mu)\, e^{-\mu|x-y|} .$$

It should be noticed that the condition $\zeta < 1$, may be satisfied even if $J_{xy} = \infty$ with non zero (but small) probability. If, for example,

$$J_{ij} = \begin{cases} \mathrm{J} & \text{with probability } (1-\mathrm{p}) \\ \infty & \text{with probability p} \end{cases}$$

then

$$\zeta = p + (1-p)\tanh\beta J ,$$

so that $(2d\zeta) < 1$ if p and βJ are sufficiently small.

In the region of parameters implying exponential decay of the correlation functions we can prove differentiability of the quenched magnetization at zero external field from the formula:

$$\frac{dm}{dh}(h=0) = \sum_x \overline{\langle\sigma_0\sigma_x\rangle} .$$

The proof of infinite differentiability of m(h) at h = 0 requires the analysis of the decay properties of the n-point correlation functions.

III. Discussion of a simple model for d-dimensional disorder in (d+ 1) dimensions

We begin our discussion with an extremely simple model which incoporates the basic features of the more general situation. It is given by the following conditions:

$$J(x,y) = \begin{cases} \infty, & \text{with probability p} \\ 0 & \text{with probability } 1-\mathrm{p} \end{cases} \qquad (3.1)$$
$$K(x) = K$$

For a given realization J of the random parameters we say that two sites $x, y \in \mathbf{Z}^d$ are in the same connected cluster if there is a path $w: x \to y$ such that $J(i,j) = \infty$ for $(i,j) \in w$. In this model, for any two sites $(x,t),(y,t) \in \mathbf{Z}^d \times Z$ in the same horizontal layer the corresponding spin variables are equal if the two sites $x, y \in \mathbf{Z}^d$ lie in the same connected cluster or else they are statistically independent. Therefore, for a given realization J, the only nontrivial correlation functions are those between the spin variables in sites lying on the same vertical line $(x,s),(x,t)$ and they are given by the correlation functions of a one dimensional Ising model with coupling $N(x)K$ where $N(x)$ is the number of sites in the connected cluster of sites containing x. We can therefore write down the correlation functions for this model:

$$\langle\sigma(x,t)\,\sigma(y,s)\rangle = \begin{cases} (\tanh KN(x))^{|s-t|} & \text{if x and y are in the same cluster} \\ 0 \text{ otherwise} \end{cases} \qquad (3.2)$$

We should first remark that if $p < p_c(d)$, where $p_c(d) > \frac{1}{2d}$ is the critical value for the bond percolation problem in $\mathbf{Z}^d$ the random variables $N(x)$ are finite with probability one, since

$$\sum_{n=1}^{k} \mathbf{P}\{N(x) = n\} < \infty.$$

Therefore with probability one we have exponential decay of correlation functions in the time-direction:

$$\langle\sigma(x,t)\,\sigma(x,s)\rangle \le \exp -m\,|t-s| \qquad (3.3)$$

where $m = e^{-2N(x)K}$. This is a trivial remark since, although finite with probability one, $N(x)$ takes arbitrarily large values, also witli probability one. This phenomenon is the trade mark of the Griffiths' singularities. The quenched two-point function is then given by:

$$\overline{\langle\sigma(x,t)\sigma(x,s)\rangle} = \sum_{n=1}^{\infty} P(n)(\tanh Kn)^{|s-t|} \qquad (3.4)$$

where $P(n) = \mathbf{P}\{N(x) = n\}$, and $\overline{f} \equiv \mathbf{E}(f)$ denotes the expectation with respect to the random parameters of the random variable f. Using the obvious bounds $P(n) \leq (2d)^{n-1}p^n$ (actually for d = 1 we have $P(n) = (1-p)^2 np^n$) and $\tanh x \leq (1-\exp-(2x)) \leq \exp-(\exp-(2x))$ it follows from (3.4) that

$$\overline{\langle\sigma(x,t)\sigma(x,s)\rangle} \leq \sum_{n=1}^{\infty}(2dp)^{n-1}\, e^{-|t-s|\exp-(2nK)}$$

and so, if $e^{-l} \equiv 2dp < 1$,

$$\overline{\langle\sigma(x,t)\sigma(x,0)\rangle} \leq e^{l}\int_0^{\infty} e^{-nl}e^{-|t|\exp-(2nK)}dn =$$

$$= \frac{1}{|t|^{\frac{l}{2K}}}\frac{e^l}{2K}\int_0^t u^{\frac{l}{2K}}e^{-u}\frac{du}{u} \leq \frac{1}{|t|^{\frac{l}{2K}}}\frac{e^l}{2K}\int_0^{\infty} u^{\frac{l}{2K}}e^{-u}\frac{du}{u} \qquad (3.5)$$

that is.

$$\overline{\langle\sigma(x,t)\sigma(x,s)\rangle} \leq \frac{c(K,l)}{|t-s|^{\frac{l}{2K}}} \qquad (3.6)$$

where

$$c(K,l) = \frac{e^l}{2K}\int_0^{\infty} u^{\frac{l}{2K}}e^{-u}\frac{du}{u}\,.$$

It should be remarked that for d = 1 with little extra work, the above inequalities can be reversed to yield the lower bound

$$\overline{\langle\sigma(x,t)\sigma(x,s)\rangle} \geq \frac{C(K,l)}{|t-s|^{\frac{l}{2K}}}\,, \qquad (3.7)$$

for some constant $C(K,1) > 0$.

For $x \neq y$ we have only the clusters with $n > |x-y|$ give a non trivial contribution:

$$\overline{\langle\sigma(x,t)\sigma(y,s)\rangle} \leq \sum_{n>|x-y|}^{\infty} e^{-l(n-1)}e^{-|t-s|\exp-(2nK)}$$

so that

$$\overline{\langle\sigma(x,t)\sigma(y,s)\rangle} \leq e^{-l|x-y|}\sum_{n=1}^{\infty}(2dp)^{n-1}\, e^{-|t-s|\exp-(2|x-y|K)\exp-(2nK)}$$

This shows that

$$\overline{\langle\sigma(x,t)\sigma(y,s)\rangle} \leq e^{-l|x-y|}\frac{c(K,l)}{[|t-s|\exp-(2|x-y|K)]^{\frac{l}{2K}}} \qquad (3.8)$$

or

$$\overline{\langle\sigma(x,t)\sigma(y,s)\rangle} \leq c(K,l)\frac{e^{-(l-2K)|x-y|}}{|t-s|^{\frac{l}{2K}}} \qquad (3.9)$$

so that if $l > 2K$ we have exponential decay in the space direction and polynomial decay in the time direction.

The upper bound (3.6) can be used to prove differentiability of the quenched magnetization provided $\frac{l}{2K}$ is sufficiently large so that

$$\sum_{x\in\mathbf{Z}^d}\sum_{t\in\mathbf{Z}}\overline{\langle\sigma(x,t)\sigma(0,0)\rangle} < \infty\,,$$

whereas the lower bound precludes exponential decay of the quenched correlation functions. The lower bound is in this case the visible sign of the Griffiths' singularity.

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