Crossover of Thermodynamic Properties of Ising Model on Sierpinski-Type Fractals

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We use an exact renormalization group method to study the Ising model on a family of Sierpinski gasket type of fractals. For general values of b, where b is an integer ($b \geq 2$) characterizing the fractal, we derive exact recursion relations between partition functions at successive generations of the fractals. We calculate the temperature dependence of the thermodynamic functions for the ferromagnetic and antiferromagnetic models with $b \leq 10$, arid the antiferromagnetic residual entropies up to $b = 11$. We find that these functions show a smooth crossover to the corresponding quantities of the same models defined on a regular triangular lattice.

I. Introduction

Considerable attention has been devoted in the last quarter of this century to the investigation of physical models related with fractal[1,2] and hierarchical[3–5] lattices. When such lattices exhibit self-similarity and finite order of ramification, some physical models on these lattices can be solved exactly through a renormalization transfcrmation. Several aspects of these models have been stuc ied in different contexts, among which we may mention those related with percolation[6,7], thermodynamic behavior of frustrated Ising models[8–10], superconductivity in disordered media[11], magnon dynamics on diluted Heisenberg magnets[12], crossover between fractal and Euclidean lattices[13], fractal interpolating dimensions between Euclidean dimensions[14,15], and a great diversity of other problems.

In this paper, the Ising model with either ferromagnetic or antiferromagnetic couplings on a family of fractals which are generalizations of the Sierpinski Gasket is studied using an exact renormalization group technique[16]. In the absence of externa1 magnetic field we calculate the thermodynamic potentials for arbitrary temperature and we get, as a subproduct, the residual entropy of the family. The method presented here is based on an adaptation of the Real Space Renormalization Group via Decimation, where we impose a well specific invariance of the partition function during each renormalization step, and on the ideas of the scale theory of finite size systems.

Several authors have considered the calculation of thermodynamic potentials of spin systems on the *Sierpinski* Gasket and some of its generalizations either in the two-dimensional Euclidean space[17–25] or in spaces of higher Euclidean dimensions[21]. However, most of these works concentrate themselves on the ferromagnetic region of the coupling parameter and in the zero temperature limit. We stress here three exceptions; the first is the calculations of the thermodynamic potentials of the Sierpinski Gasket for arbitrary temperature in the ferromagnetic region[22]; the second is an exact calculation of the residual entropy of fully frustrated *generalized* Sierpinski *Gaskets*[23,24], with results corroborating with those obtained here within a precision of more than fifteen decimals; and the third is the calculus of the residual entropy of the Sierpinski Gasket family in the zero-temperature limit[25].

II. Model

The model we consider for illustration of the method introduced here consists of a family of fractals embedded in a two-dimensional Euclidean space and represents one of the possible generalizations of the *Sierpinski* Gasket. Each member of this farnily, designated by $SG_b^{(n)}$, where b is an integer greater or equal two identifying the member of the family, is constituted by a set of

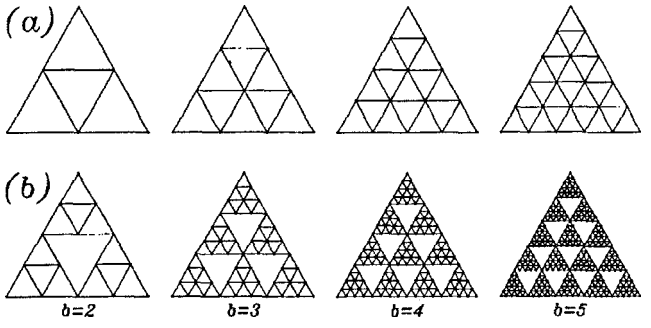

Figure 1. (a) The generators $SG_b^{(1)}$ of the two-dimensional Sierpinski type of fractals, for b = 2, 3, 4 and 5. (b) The fractal ($SG_b^{(2)}$ structures) at the $n = 2$ stage of the construction.

self-similar structures identified by the integer n greater or equal zero. The onset of the process of construction of these structures is an upward triangle of unity sides, denominated root of the member and designated by $SG_b^{(0)}$. In the first stage of the construction of each member of the family, we divide the root in b^2 equilateral triangles and drop out the $b(b-1)/2$ downward triangles. In this way, we get $M_{b,1} = b(b+1)/2$ upward triangles of sides equal $1/b$ physically present in the root, spanning an infinity set of generators of the family designated by $SG_b^{(1)}$. This process is iterated, with the length scale of the members of the family decreasing by a factor b in each step of the construction, and it stops at some suitable microscopic length b^{-n} corresponding to the structures $SG_b^{(n)}$ with $M_{b,n} = [b(b+1)/2]^n$ upward triangles of sides b^{-n} physically present. The fractal lattices SG_b are, by definition, the $n \to \infty$ limit of this process. The Hausdorff dimension of each member of the family is expressed by $D_b = \ln(M_{b,1})/\ln(b)$, and the number of vertexes at the nth step is given by $N_{b,n} = 3 + [(b+4)/(b+2)][M_{b,n} - 1]$. Fig. 1 depicts the construction procedure.

We construct an Ising model on the $SG_b^{(n)}$ attaching a spin variable $\sigma_i = \pm 1$ to each vertex of this structure, with nearest-neighbor adimensional interactions and in the absence of external fields. The reduced Hamiltonian is

$$H_{b,n} = -K_n \sum_{<ij>} (\sigma_i \sigma_j), \tag{1}$$

where $< ij >$ indicates the sum over all nearest-neighbor spins variables. In Eq. (1) $K_n = \beta J_n$ is the coupling parameter, with $\beta = 1/k_B T$ and J_n is the exchange interaction for nearest neighbor pair of spins, k_B is the Boltzmann constant, and T is an arbitrary temperature. The partition function is given by

$$Z_b(K_n) = \sum_{\{\sigma_i\}} \exp[K_n \sum_{<ij>} (\sigma_i \sigma_j)], \tag{2}$$

where the composed symbol $\{\sigma_i\}$ represents all possible configurations that the spin variables of the structure can assume.

III. Formalism

III.1. Partial partition functions

The partial partition functions of the $SG_b^{(n)}$ are those partition functions obtained when we keep fixed a determined configuration of the external spins, and accomplish the trace over the internal spins of such structures. These functions play a crucial role in the evaluation of the thermodynamic potentials of the fractals.

Let us work out the process of determining the partial partition functions of the second order structure of the member *Sierpinski Gasket* $SG_2^{(2)}$ of the family. According to the scheme of Fig. 2, we can factorize the internal clusters of the $SG_2^{(2)}$ by tracing over the internal spin variables indexed by **1**, **2** and **3** separately. Then the partial partition functions may be written as:

$$Z_2(K_2; \{\sigma_i\}) = F(\sigma_1, \sigma_2, \sigma_3) F(\sigma_2, \sigma_4, \sigma_5) F(\sigma_3, \sigma_5, \sigma_6), \tag{3}$$

where

$$F(\sigma_i, \sigma_j, \sigma_k) = x_2^9 + 3x_2 + 4x_2^{-3} \tag{4}$$

for $\sigma_i = \sigma_j = \sigma_k$, and

$$F(\sigma_i, \sigma_j, \sigma_k) = x_2^5 + 4x_2 + 3x_2^{-3} \tag{5}$$

otherwise. From now on we use the notation $x_n = \exp(K_n)$ for the Boltzmann weights.

Denoting these two expressions by $A_2(x_2)$ and $B_2(x_2)$, respectively, and inspecting the possible combinations of their arguments in Eq. (3), we find that the $SG_2^{(2)}$ has only four distinct partial partition functions which may be written in a compact manner as

$$Z_2(x_2; l) = A_2^{3-l}(x_2) B_2^l(x_2), \tag{6}$$

where $0 \le l \le 3$ represents the energy levels $\epsilon_{2,l}(K_1) = -(3M_{2,1} - 4l)K_1$ of the $SG_2^{(1)}$ related with the external

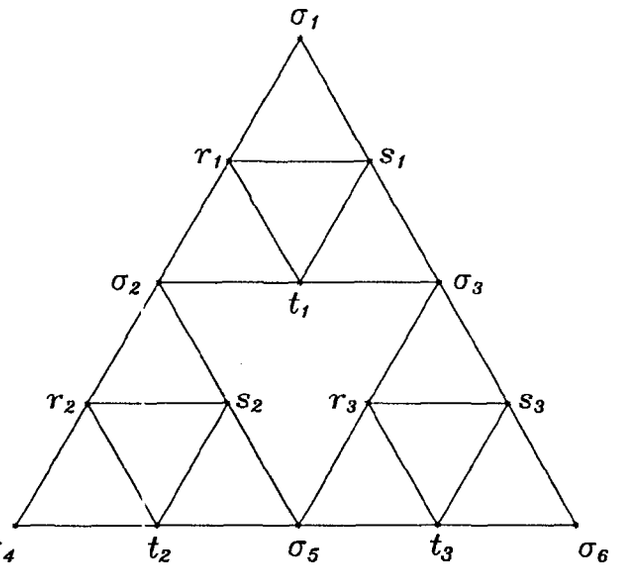

Figure 2. The sets of external and interna1 spins of this structure are designated by $\{a; 1 \leq i \leq 6\}$ and $\{r_i, s_i, t_i; 1 \leq i \leq 3\}$, respectively.

spin configurations of the structure $SG_2^{(2)}$. Note that we change the partial partition functions notation from a representation in terms of the external spin configurations $\{\sigma_i\}$ of the structure $SG_2^{(2)}$ to a representation in terms of energy levels $\{l\}$ of the structure $SG_2^{(1)}$. As there are $2^6 = 64$ possible configurations of the external spins, a given partial partition function is degenerated, and its degeneracy degree is equal to the degeneracy of the energy level associated with the configuration of the external spins.

Extending the idea of factorizing of the internal clusters to higher orders and to other members of the family, we get for any $SG_b^{(n)}$

$$Z_b(x_n; l) = A_b^{M_{b,n-1}-l}(x_n) B_b^l(x_n), \tag{7}$$

where $0 \leq 1 \leq M_{b,n-1}$, $M_{b,n-1}$ is the number of physical triangles of the structure $SG_b^{(n-1)}$, and the structure functions $A_b(x_n)$ and $B_b(x_n)$ are determined from the sum over the internal spins of the generator $SG_b^{(1)}$ of the member b of the family. We see from Eq. (7) that these two functions are the only relevant ones to write down the partial partition functions for any $SG_b^{(n)}$.

111.2. Partition functions

From the root of the members of the family, an upward triangle, is easy to see that the $SG_b^{(0)}$ has eight possible physical states, and two energy levels. The first level corresponds to the ground state of a ferromagnetic system with energy $-3K_0$ and has degeneracy two, the other, with energy K_0 has degeneracy six. The partition function in terms of Boltzmann weights is then

$$Z_b(x_0) = 2x_0^3 + 6x_0^{-1}. \tag{8}$$

Each of these levels generates a partial partition function for the structures of immediate higher order with the same degeneracy of the own level, and with the first level of the $SG_b^{(0)}$ corresponding to the function $A_b(x_1)$ and the other level to $B_b(x_1)$. Accordingly we may express the partition function for the set of generators $SG_b^{(1)}$ as

$$Z_b(x_1) = 2A_b(x_1) + 6B_b(x_1). \tag{9}$$

The structure functions $A_b(x_1)$ and $B_b(x_1)$ are calculated by a slightly different method from that suggested by Fernandes and Brady Moreira[26]. In the Appendix we give the functions for $b \leq 8$ (for larger values of b see Ref. 27). By substituting these functions into Eq. (9), we can write the partition functions of the generators of the members of the family in terms of the Boltzmann weights.

Proceeding to higher orders, we observe that the structure of order n of member b of family has $M_{b,n}+1$ energy levels, with energies $\epsilon_{b,l}(K_n) = -(3M_{b,n} - 4l)K_n$. Here $M_{b,n} = [b(b+1)/2]^n$ is the number of upward triangles in the structure, and $l = 0, 1, \ldots, M_{b,n-1}$. Each of these levels generates a partial partition function for the structure of order superior, with the same degeneracy of the level, and then we can generalize the foregoing reasoning and express the partition function of $SG_b^{(n)}$ as

$$Z_b(x_n) = \sum_{l=0}^{M_{b,n-1}} C_{l,n-1} Z_b(x_n; l), \tag{10}$$

where $Z_b(x_n; l)$ is given by Eq. (7), and the $C_{l,n-1}$ are the same factors of the partition function $Z_b(x_{n-1})$ when expresses in terms of Boltzmann weights, that is

$$Z_b(x_{n-1}) = \sum_{l=0}^{M_{b,n-1}} C_{l,n-1} \exp[(3M_{b,n-1} - 4l)K_{n-1}]. \tag{11}$$

The last two expressions, linked by the coefficients $C_{l,n-1}$, formalize the physical idea suggested in this section. They may be thought of as a recursion relation

which allows us to write down the partition function of $SG_b^{(n)}$ at any arbitrary order.

111.3. Renormalization

The recursion relations for the coupling parameter K_n and for the free energy $F_b(x_n)$ of the structure $SG_b^{(n)}$ can be suitably derived from an adaptation of a decimation transformation. By assuming the invariance of the Boltzmann weights of a given energy level of $SG_b^{(n-1)}$, which are related with the partial partition function originated by the trace over the internal spins of the structures $SG_b^{(n)}$, that is,

$$Z_b(x_n;l) = \exp[G_b(x_n)]\exp[(3M_{b,n-1} - 4l)K_{n-1}], \tag{12}$$

where $l = 0, 1, ..., M_{b,n-1}$ represents the energy levels of structure $SG_b^{(n-1)}$ with $M_{b,n-1}$ physical triangles. In the above equation $G_b(x_n) = N_{b,n}g_b(x_n)$ is an auxiliary function of the free energy distribution during the decimation, $N_{b,n}$ is the number of spin variables (vertexes) of $SG_b^{(n)}$, and $g_b(x_n)$ is a specific function to be determined in the renormalization process.

Starting from Eq. (12), multiplying both members by the degeneracies $C_{l,n-1}$ of the energy levels of the $SG_b^{(n-1)}$, performing the sum over the l levels, and substituting the result by Eqs. (10) and (11), we find

$$Z_b(x_n) = \exp[G_b(x_n)]Z_b(x_{n-1}). \tag{13}$$

Therefore, the partition functions of $SG_b^{(n)}$ and $SG_b^{(n-1)}$ are related by the same factor as the partial partition functions of $SG_b^{(n)}$ and the corresponding Boltzmann weights of $SG_b^{(n-1)}$. This implies that the invariance of the partial partition functions led us to the invariance of the partition function under the renormalization.

The idea of imposing the invariance of the partial partition functions of clusters of decimated spins, in relation to the Boltzmann weights of the remaining spins, and thence to reach the invariance of the partition functions under the renormalization was previously presented in determining the thermodynamic properties of Ising spins on the triangular lattice[28]. In fact, the Eq. (13) represents the invariance requirement for the partition function of the original and renormalized systems under a scale transformation[29].

Eq. (12) represents a system with $M_{b,n-1} + 1$ equations, nevertheless we only have two quantities to be determined, K_{n-1} and $g_b(x_n)$, and then only two of those equations are needed. In fact, for any pair of equations indexed by 1 and m, with $l \neq$ m referring to distinct levels of energy, we obtain:

$$K_{n-1} = \frac{1}{4}\ln\left[\frac{A_b(x_n)}{B_b(x_n)}\right] \tag{14}$$

$$g_b(x_n) = \frac{M_{b,n-1}}{4N_{b,n}}\ln[A_b(x_n)B_b^3(x_n)], \tag{15}$$

which are independent of l and m. The irrelevance of the choice of the energy levels is then established.

111.4. Thermodynamic functions

We now turn to the recursion relation for the free energy. Defining the dimensionless free energy per spin of the $SG_b^{(n)}$ as

$$F_b(x_n) = \frac{\ln[Z_b(x_n)]}{N_{b,n}} \tag{16}$$

and using (13), we can write

$$F_b(x_n) = \frac{N_{b,n-1}}{N_{b,n}}F_b(x_{n-1}) + g_b(x_n)\ . \tag{17}$$

Iterating Eq. (17) we obtain

$$F_b(x_n) = \frac{N_{b,0}}{N_{b,n}}F_b(x_0) + \sum_{i=0}^{n-1}\left[\frac{N_{b,n-i}}{N_{b,n}}\right]g_b(x_{n-1}), \tag{18}$$

where $F_b(x_0) = \frac{1}{3}\ln(2x_0^3 + 6x_0^{-1})$ is the free energy per spin of the root $SG_b^{(0)}$. Thus, Eqs. (14-18) enable us to determine the free energy of any $SG_b^{(n)}$ for general values of the coupling parameter K_n. The exact expression for $F_b(x_n)$, Eq. (18), was first derived in Ref. 16.

The internal energy $E_b(x_n) = dF_b(x_n)/dK_n$ is derived from Eq. (17). After (n − 1) iterations we get:

$$E_b(x_n) = \frac{N_{b,0}}{N_{b,n}}\frac{dK_0}{dK_n}E_b(x_0) + \sum_{i=1}^{n}\left[\frac{N_{b,i}}{N_{b,n}}\frac{dK_i}{dK_n}\frac{dg_b(x_i)}{dK_i}\right], \tag{19}$$

where $E_b(x_0) = [x_0^4 - 1]/[x_0^4 + 3]$ is the internal energy per spin of $SG_b^{(0)}$, and $dK_i/dK_n = \Pi_{l=i}^{n-1}(dK_l/dK_{l+1})$. The entropy can then be calculated from the thermodynamic relation

$$S_b(x_n) = F_b(x_n) - K_nE_b(x_n), \tag{20}$$

and the specific heat can be written as

$$C_b(x_n) = \left(\frac{N_{b,0}}{N_{b,n}}\right)\left(\frac{K_n dK_0}{K_0 dK_n}\right)^2 C_b(x_0) + K_n^2 \sum_{i=1}^{n}\left(\frac{dK_{n+1-i}}{dK_n}\right)^2 \times \left[\left(\frac{N_{b,i}}{N_{b,n}}\right)\left(\frac{d^2K_{n-i}}{dK_{n+1-i}^2}\right) E_b(x_{n-i}) + \left(\frac{N_{b,n+1-i}}{N_{b,n}}\right)\left(\frac{d^2 g_b(x_{n+1-i}}{dK_{n+1-i}^2}\right)^2\right], \quad (21)$$

where $C_b(x_0) = 16K_0^2 x_0^2/(x_0^3 + 3x_0^{-1})$ represents the specific heat of the $SG_b^{(0)}$. The results of the numerical calculation are presented in the next Section.

IV. Results

We first present our results for the thermodynamic potentials calculated at successive stages of construction of the Sierpinski gasket type of fractals, described in Sec. I, and then we examine the question of how the thermodynamic functions of these fractals converge to the correspcnding functions of the regular triangular lattice.

The free energies $F_b(x_n)$ present the high-temperature limit of ln 2 in $K = 0$, and increase with $|K|$ with a well defined concavity as expected for free energy functioris. For the case b = 2, the calculated free energies for the first ten stages of construction saturate numerically within a specified precision. For example, for the structure $SG_2^{(8)}$ made up from 9,843 particles, the thermodynamic lirnit (n $\rightarrow \infty$) is achieved within a precision of 10^{-5}. This means that the surface effects due to the finite size of the systerns become negligible for those structures labeled by $b = 2$ and n $\geq$ 8, and, therefore, the thermodynamic properties of the *Sierpinski Gasket* is, above this order, dominated predominantly by the characteristics of the bulk. Of course, any desired precision may be reached by iterating the renormalizatioii at a compatible order.

Figure 3(a) presents the internal energies per spin of the $SG_2^{(n)}$, for $1 \leq$ n ≤ 10. They must be understood negative in entire K space. The two plateaux indicate that the low-temperature regime is quickly reached, and that the chosen interval of values of K is therefore quite suitable :o represent the thermodynamic potentials. We note that the internal energy curves saturate after just a few renormalization steps.

The entropies of the $SG_2^{(n)}$, $1 \leq n \leq 10$, are shown in Fig. 3(b). There is a maximum in the origin, corresponding to T $\rightarrow \infty$, equals to ln 2. In the ferromagnetic side the curves tend towards the asymptotic value zero, while in the antiferromagnetic one the curves tend towards the asymptotic residual value 0.493006107. This residual entropy for the *Sierpinski Gasket* is higher than the value 0.32066 calculated exactly by Wannier[30] for the Ising antiferromagnet on the triangular lattice. Such unexpected result is contrary to the idea that the presence of holes in the lattice can reduce the frustration. Indeed, it was found that the *random* removal of bonds (sites) reduces the frustration in randomly diluted disordered systems[31].

In Fig. 3(c) we show the specific heat per spin of the $SG_2^{(n)}$, for $1 \leq n \leq 10$. Each curve tends towards zero in both limits of T $\rightarrow$ 0 and T $\rightarrow \infty$, and presents two finite maxima in the pseudocritical temperatures. For antiferromagnetic interaction ($K <$ O), the rounded peaks saturate at C $\simeq$ 0.1184, with the pseudocritical temperatures being located in the neighborhood of $K = -0.58$. For $K >$ 0 the peaks also become higher as n increases, saturating at C $\simeq$ 0.7453 and, except for the first two orders, their positions remain fixed around $K = 0.52$. Furthermore, the curves intersect each other in the region where the systems reach the low-temperature regime, that is $K \simeq 0.67$.

Thermodynamic functions of the higher members (b $\geq$ 3) of the family present, in general terms, the same dependence with temperature as the appropriate curves of the *Sierpinski Gasket*. However, we have noted for all thermodynamic potentials that the rate of convergence towards the appropriate asymptotic curves (n $\rightarrow \infty$ limit) become more pronounced for larger b.

Table I presents the results for the antiferromagnetic residual (zero-temperature) entropies of the first six members of the family $SG_b^{(n)}$. The data were listed for a chosen accuracy of $O(10^{-7})$ and, for fixed b, we included only those entries needed to achieve the desired precision. In this way we might establish a stop criterion (avoiding unnecessary calculations) to calculate, within a specified precision, the thermodynamic functions of a given fractal lattice through a sequence of renormalizations.

Once accomplished the sequences of renormalizations which allow us to determine the order n of the

Table 1 - The antiferromagnetic residual entropy per spin of the $SG_b^{(n)}$ structures. For a given b, only those entries needed to achieve an accuracy of $O(10^{-7})$ were quoted.

n	b=2	b=3	b=4	b=5	b=6	b=7
1	0.5430161	0.5075174	0.4822785	0.4633318	0.448456	0.4366635
2	0.5130202	0.4789589	0.4588857	0.4443779	0.4330644	0.4238393
3	0.5001540	0.4733224	0.4562160	0.4430026	0.4322783	0.4233576
4	0.4954469	0.4723541	0.4559495	0.4429103	0.4322407	0.4233404
5	0.4938263	0.4721918	0.4559228	0.4429042	0.4322388	0.4233398
6	0.4932839	0.4721647	0.4559202	0.4429038		
7	0.4930759	0.4721602	0.4559199			
8	0.4930366	0.4721594				
9	0.4930163	0.4721593				
10	0.4930095					
11	0.4930072					
12	0.4930064					
13	0.4930062					

generation needed to reach the thermodynamic limit for each member b of the family, we proceed with the calculation of the potentials for increasing values of b. The difficult of going very far away with this purpose resides in obtaining the exact expressions for the partial partition functions since, as b increases, the degree of degeneracy are represented by very big numbers. (Actually, for the case b = 12 not included in the present calculation, the degeneracies cannot be accommodated in an integer word of the computer.)

We have calculated the temperature dependence of the thermodynamic potentials on the bgaskets with $2 \leq b \leq 10$ and for both ferromagnetic (F) and antiferromagnetic (AF) interactions. The internal energy, the entropy, and the specific heat per spin are shown in Fig. 4. In Fig. 4(c) we also included the specific heat of the regular triangular lattice. All curves were obtained with n = 10. By comparing these results with those shown in Fig. 3, one can see that the convergence of any thermodynamic potential towards an asymptotic curve is faster when we increase the number of spins in a lattice with a fixed fractal dimension than when, in the limit of large n, we increase the fractal dimension.

As b increases, the specific heat curves present round peaks with increasing amplitudes in both ferromagnetic and antiferromagnetic regions, but while in the AF case they spread out and are located at lower temperatures, in the F region the round peaks become narrower and occur at higher temperatures. Such a general trend of crossing over from fractal to Euclidean behavior has also been observed for the other thermodynamic functions shown in Fig. 4.

The rate of convergence of the potentials with increasing b is better illustrated through the b-dependence of the ground-state energy and of the zero-temperature entropy. In this limit, for both the AF and F systems, we can obtain a relation between the internal energy per spin and the parameter b, that is

$$E_0 = \epsilon_0(b+2)/(b+4), \tag{22}$$

where $\epsilon_0 = -1(-3)$ is the antiferromagnetic (ferromagnetic) ground-state energy per site on the regular triangular lattice. As b increases, the internal energy in the limit $T \rightarrow 0$ increases systematically in modulus and, for a fixed b, its absolute value in the ferromagnetic fractal is three times larger than the antiferromagnetic one, as in the Euclidean triangular lattice. Besides, the crossover behavior of E_0, when $b \rightarrow \infty$, is given by $E_0 = (1 - 2/b)\epsilon_0$.

The entropy of all ferromagnetic systems vanishes in the limit $T \rightarrow 0$. For antiferromagnetic interactions the zero-temperature entropy reaches its maximum value for b = 2, and then decreases as b increases approaching smoothly the entropy of the Euclidean triangular lattice. In Table II we show the residual entropy for the antiferromagnetic fractal lattices with $b = 8, 9, 10, 11$. The seven-digit values for $S_b^A(T \rightarrow 0)$ with $2 \leq b \leq 7$ can be read from the last entries of Table I.

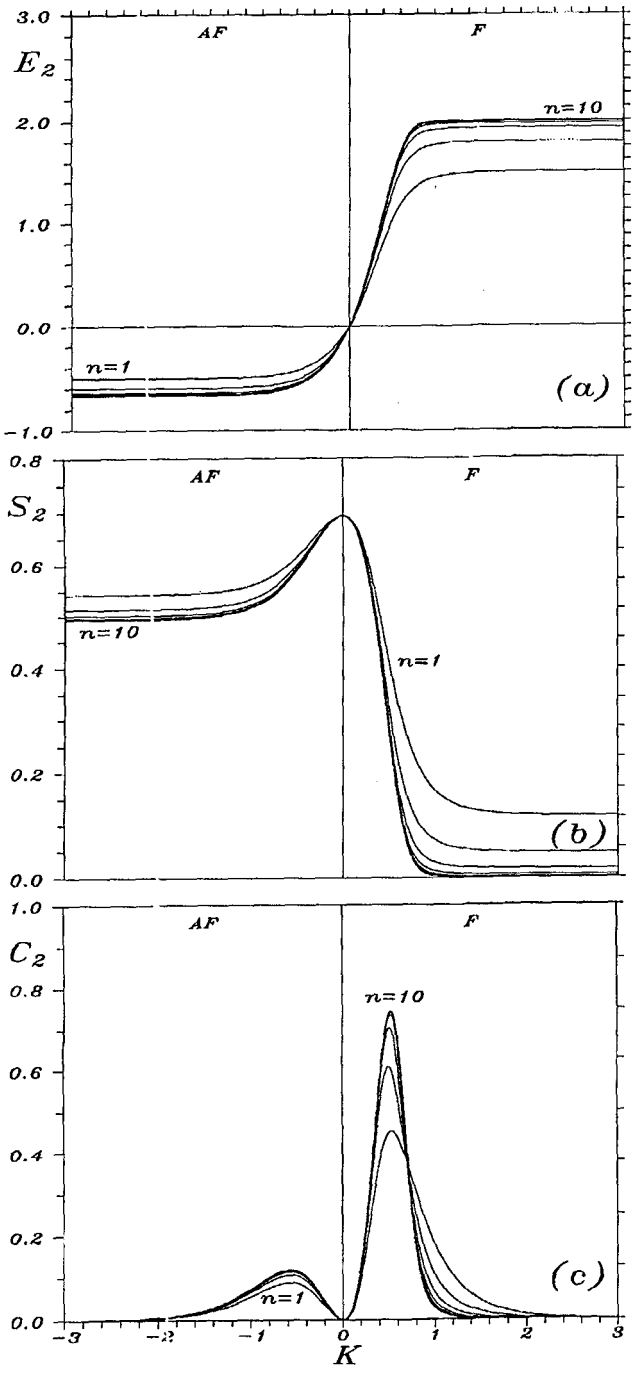

Figure 3. Temperature dependence of (a) the internal energy, (b) the entropy, and (c) the specific heat per spin of Ising spins with ferromagnetic and antiferromagnetic interactions at the nth stage of construction of the $b = 2$ Sierpinski gasket fractal.

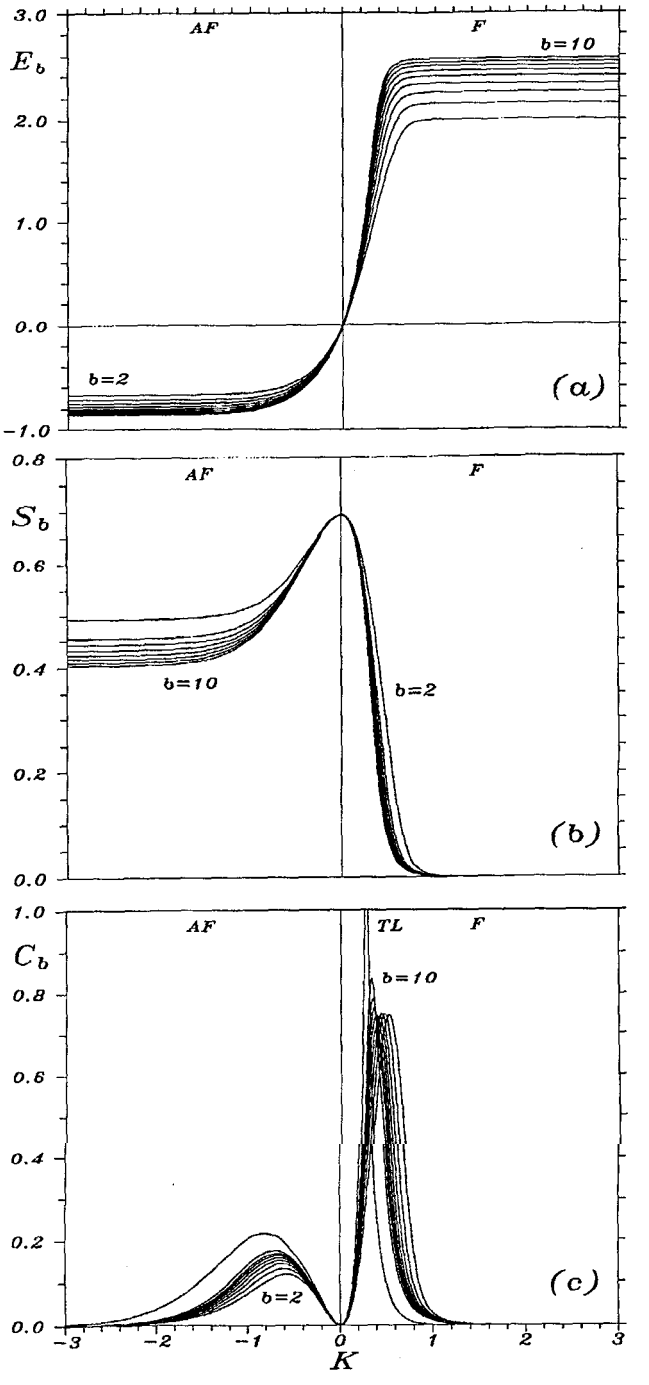

Figure 4. Temperature dependence of **(a)** the internal energy, (b) the entropy, and (c) the specific heat per spin for both ferromagnetic and antiferromagnetic models on the b-gaskets, for $2 \leq b \leq 10$. The specific heat curve (TL) for the regular triangular lattice is included for comparison.

Table II - Antiferromagnetic residual entropies $S_b^A(T \rightarrow 0)$ for the fractal lattices with $b = 8, 9, 10$ and 11. The values of n stand for the number of iterations needed to achieve the precision shown. The corresponding values for the first six members ($2 \leq b \leq 7$) of the family can be read froin the last entries of Table I.

b	n	$S_b^A(T \rightarrow 0)$
8	4	0.4158001
9	4	0.4093292
10	4	0.4037138
11	4	0.3987939

The results obtained from several rescaling factors can be used to form a. basis for the application of a scale theory to determine a law for the values of the residual entropy in any higher dimension. In Fig. 5 we have plotted the antiferromagnetic residual entrcpy as a. function of the number of spins in the generator of each member of the family. This figure shows $S_b^A(T \rightarrow 0)$, for $4 \leq b \leq 11$, in the thermodynamic limit (large n). A last-square fitting to the data results in a power-law expression given by

$$S_b^A(T \rightarrow 0) = 0.323094 + 0.384133(N^{-1/2})^{0.746} \quad (23)$$

where $N = (b + 1)(b + 2)/2$ is the number of spins in the generator. The intersection with the vertical axis ($N = \infty$) corresponds to the residual entropy $S_\infty^A(T \rightarrow 0) = 0.323094$ of tlie fully frustrated triangular lattice. The exponent $w_S = 0.746$ found for the family of fractals studied in this work, should be compared to the exponent $w_S = 1$ for their generators which was first derived from an exact enumeration on finite triangular clusters[26].

V. Concluding remarks

We have studied tlie thermodynamic behavior of both the ferromagnetic and antiferromagnetic Ising models on a family of fractal lattices. This work extends the results of Ref. 16 to higher fractal dimensionalities. Our exact calculations of the thermodynamic potentials presented in the above Sections may be viewed as an exact study of the Ising model on a generalized family of fractals, or as an investigation of the behavior from fractal to Euclidean crossover.

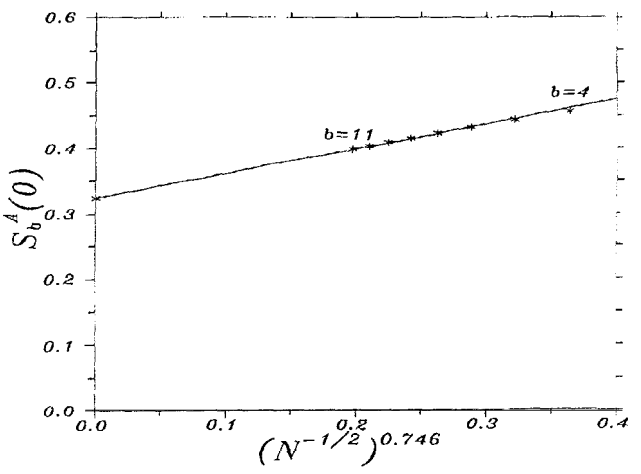

Figure 5. The antiferromagnetic zero-temperature entropy per spin, in the tliermodynamic limit of members with $4 \leq b \leq 11$, as a function of tlie number of spins in the generators.

The results obtained for the $SG_b^{(n)}$ lead us to conclude that the method developed here is perfectly feasible for determining the thermodynamic potentials of non-homogeneous and self-similar fractal structures. The non-homogeneity allow us to decompose a given system of particles in several subsystems with internal structures being connected by a finite number of links. In the case of the $SG_b^{(n)}$, this property becomes manifest through the possibility of factorizing the clusters of interna1 spins; cf. Eq. (7). On the other hand, the self-similarity is also necessary to obtain the recursion relation for the partition functions of two successive structures; cf. Eqs. (10) and (11).

The family of fractals considered in this work is uniform, nevertheless this property is not necessary. The application of the present formalism to non-uniform self-similar lattices may render more defiances, but it is also possible, in principle. We choose this generalization of the *Sierpinski* ***Gasket*** because we were interested in the thermodynamic behavior of cooperative systems where frustration plays an important role in their ground-state properties. We could eventually be interested in other quantities, such as the magnetization and susceptibility. The inclusion of an externa1 magnetic field on our formalism, spite of being much more complicated, is perfectly feasible.

In the case of the antiferromagnetic model, the application of the present formalism to calculate the residual entropy is an cxact study of the frustration effects.

We found that these fractals present degeneracies of their respective ground states which are even larger than that of the corresponding Ising model on the regular triangular lattice. The higher ground-state degeneracy found in the b-gaskets, where the bonds (sites) are removed in some organized fashion, should be contrast with Monte Carlo simulations of randomly diluted Ising models on the triangular lattice. In these latter systems, the random removal of bonds (sites) leads to less frustration and lower ground-state degeneracy.

We have also found from the results on the fractals with $4 \leq b \leq 11$, that the residual entropy $S_b^A(T \to 0)$ shows a power law behavior given by Eq. (23). Whether a crossover expression of the same type can be found in the case of other fractals presenting a finite order of ramification is a subject of further work. The question of finding universal behavior on fractals is still open, and our calculations may suggest the universal character of the exponent $\omega_S = 0.75$.

To summarize, we have developed a renormalization group approach to calculate exactly the thermodynamic functions of Ising models on non-homogeneous and self-similar fractal lattices. The method was applied to both ferromagnetic and antiferromagnetic Ising models on a family of Sierpinski gasket type of fractals. We concluded that all thermodynamic functions calculated on the fractals approach the corresponding functions of the Euclidean triangular lattice in a smooth way.

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Appendix

Partial partition functions of the generators $SG_b^{(1)}$
$[x = \exp(K_1)]$

$$A_2(x) = x^9 + 3x + 4x^{-3}$$
$$B_2(x) = x^5 + 4x + 3x^{-3}$$

$$A_3(x) = x^{18} + 6x^{10} + 8x^6 + 45x^2 + 48x^{-2} + 20x^{-6}$$
$$B_3(x) = x^{14} + 4x^{10} + 18x^6 + 32x^2 + 53x^{-2} + 20x^{-6}$$

$$A_4(x) = x^{30} + 9x^{22} + 13x^{18} + 75x^{14} + 213x^{10} + 595x^6 + 975x^2 + 1200x^{-2} + 847x^{-6} + 168x^{-10}$$
$$B_4(x) = x^{26} + 4x^{22} + 21x^{18} + 81x^{14} + 223x^{10} + 545x^6 + 991x^2 + 1243x^{-2} + 812x^{-6} + 175x^{-10}$$

$$A_5(x) = x^{45} + 12x^{37} + 19x^{33} + 114x^{29} + 339x^{25} + 1301x^{21} + 3846x^{17} + 10341x^{13} + 22414x^9 + 40566x^5 + 58359x + 63112x^{-3} + 43455x^{-7} + 16137x^{-11} + 2128x^{-15}$$
$$B_5(x) = x^{41} + 4x^{37} + 24x^{33} + 99x^{29} + 402x^{25} + 1303x^{21} + 3905x^{17} + 10038x^{13} + 22405x^9 + 40794x^5 + 58906x + 62151x^{-3} + 43848x^{-7} + 16171x^{-11} + 2093x^{-15}$$

$$A_6(x) = x^{63} + 15x^{55} + 26x^{51} + 162x^{47} + 519x^{43} + 1992x^{39} + 7119x^{35} + 23730x^{31} + 73659x^{27} + 206574x^{23} + 521583x^{19} + 1172052x^{15} + 2325213x^{11} + 3992784x^7 + 5790653x^3 + 6823029x^{-1} + 6240489x^{-5} + 4148331x^{-9} + 1782315x^{-13} + 408546x^{-17} + 35640x^{-21}$$
$$B_6(x) = x^{59} + 4x^{55} + 27x^{51} + 118x^{47} + 514x^{43} + 2047x^{39} + 7284x^{35} + 24271x^{31} + 73662x^{27} + 205395x^{23} + 519202x^{19} + 1172135x^{15} + 2330608x^{11} + 3998845x^7 + 5780212x^3 + 6814685x^{-1} + 6253657x^{-5} + 4149229x^{-9} + 1776475x^{-13} + 410487x^{-17} + 35574x^{-21}$$

$$A_7(x) = x^{84} + 18x^{76} + 34x^{72} + 219x^{68} + 762x^{64} + 2944x^{60} + 11190x^{56} + 39930x^{52} + 141756x^{48} + 469728x^{44} + 1490328x^{40} + 4407910x^{36} + 12212220x^{32} + 31362744x^{28} + 74480732x^{24} + 162216357x^{20} + 321512928x^{16} + 573270666x^{12} + 907434594x^8 + 1254525855x^4 + 1485220986 + 1467077592x^{-4} + 1167837678x^{-8} + 712749872x^{-12} + 311063028x^{-16} + 87876468x^{-20} + 13692132x^{-24} + 835920x^{-28}$$
$$B_7(x) = x^{80} + 4x^{76} + 30x^{72} + 138x^{68} + 639x^{64} + 2710x^{60} + 10800x^{56} + 40454x^{52} + 142854x^{48} + 476092x^{44} + 1493564x^{40} + 4409944x^{36} + 12181630x^{32} + 31328724x^{28} + 74456504x^{24} + 162333468x^{20} + 321654781x^{16} + 573249668x^{12} + 907065294x^8 + 1254426090x^4 + 1485547763 + 1467481990x^{-4} + 1167325992x^{-8} + 712667550x^{-12} + 311285516x^{-16} + 87840972x^{-20} + 13670856x^{-24} + 840564x^{-28}$$

$$A_8(x) = x^{108} + 21x^{100} + 43x^{96} + 285x^{92} + 1077x^{88} + 4208x^{84} + 17037x^{80} + 62025x^{76} + 234271x^{72} + 832131x^{68} + 2914695x^{64} + 9778350x^{60} + 31602186x^{56} + 97640163x^{52} + 287800246x^{48} + 806524818x^{44} + 2142496410x^{40} + 5377217634x^{36} + 12701606340x^{32} + 28111800417x^{28} + 57989098873x^{24} + 110813359398x^{20} + 194798696457x^{16} + 312495041721x^{12} + 453188767851x^8 + 587545995579x^4 + 671785931011 + 666128088924x^{-4} + 560880884472x^{-8} + 390386838615x^{-12} + 216882235620x^{-16} + 91757795259x^{-20} + 27723241260x^{-24} + 5464697595x^{-28} + 607727703x^{-32} + 27578408x^{-36}$$
$$B_8(x) = x^{104} + 4x^{100} + 33x^{96} + 159x^{92} + 777x^{88} + 3493x^{84} + 14608x^{80} + 59149x^{76} + 225897x^{72} + 830931x^{68} + 919915x^{64} + 9835195x^{60} + 31720542x^{56} + 97802902x^{52} + 287828071x^{48} + 806028482x^{44} + 2141286838x^{40} + 5375925466x^{36} + 12702570558x^{32} + 28116664404x^{28} 57995016389x^{24} + 110810954881x^{20} + 194784162734x^{16} + 312481904929x^{12} + 453201945537x^8 + 587576120623x^4 + 71787003339 + 666092292775x^{-4} + 560869547716x^{-8} + 390416524852x^{-12} + 216887411563x^{-16} + 91743925376x^{-20} + 27724178735x^{-24} + 5466868016x^{-28} + 607422299x^{-32} + 27513915x^{-36}$$