Instability for a Class of Randomly Kicked Systerns

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A class of randomly kicked systems is studied. It is shown that the kinetic energy is unbounded and its expectation value grows "linearly" with the number of kicks, in spite of the stability of the periodic kicked case. The relation to the dynamical localization theory is briefly discussed.

In this note we study a class of randomly kicked systems; we prove that, almost surely, the kinetic energy is unbounded and that its expectation value grows "linearly" with the number of kicks. In order to underline the physical side of our results we begin with a brief exposition of some well-known results related to dynarnical localization.

One of the most important results obtained in the field of quantum chaos is the dynamical localization phenomenon. It was discovered in Ref. [1] and its significance is the "absence, in some cases, of chaotic features in quantum mechanics." More precisely, in [1] the periodically kicked rotator (KR) was investigated from both classical and quantum points of view. The KR Hamiltonian is given by

$$H(p,x,t) = \frac{p^2}{2} + k \cos x \sum_n \delta(t - 2\pi n T) , \quad x \in \mathbb{T}, \quad (1)$$

where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the unit circle. Integration of the classical Hamiltonian equations over one period $2\pi T$ of the external perturbation yields

$$
\begin{aligned}
x_{n+1} &= x_n + 2\pi T p_{n+1} \quad (\text{mod } 2\pi) \\
p_{n+1} &= p_n + \sin(x_n)
\end{aligned}
\tag{2}
$$

This map is known as standard map, and it has a transition to chaos[1,2]. In the chaotic regime, numerical

computations show that for typical parameter values the average kinetic energy grows linearly with the number of kicks.

When compared to the quantum mechanics of (1), the numerical simulations have shown a surprise: the kinetic energy is bounded; it follows the classical linear growth for a short number of kicks and then stops growing[1,3]. This quantum absence of classical instability is the dynamical localization phenomenon. This phenomenon has also been invoked to explain some features in the realistic case of ionization of Rydberg Hydrogen atoms interacting with a microwave field[4].

The term dynamical localization comes from a formal analogy between the quantum suppression of classical diffusion of energy just described, and the one-dimensional Anderson localization in solid-state physics[5]. In Ref. [5] a map, the so-called Maryland construction, was obtained associating tight-binding models with pseudo-random potentials to kicked systems. For truly random potentials the one-dimensional Anderson localization has been rigorously proved[6] for a large class of distributions, but this is not the case for dynamical localization yet.

The situation is in fact more involved. Recently, a disordered model (random dimer model) has been

proposed that exhibit localization-delocalization transitions even in one dimension[7]. It was rigorously shown in ref. [8] that for a generic choice of periods (although of zero Lebesgue measure) the quantum periodic KR presents unbounded energy (in fact, in ref. [8] it was proved that the KR-Floquet operator has a continuous component in the spectrum; see the Appendix). There are also other cases of interest: quasi-periodic kicks[9], disorder induced by substitution sequences[10], Hamiltonians on quasi-periodic lattices[11,12], etc. These cases constitute active areas of research, where most of the results are related to specific examples, showing the lack of a general theory.

Despite of the numerical simulations indicating stability for the quantum periodic KR, in ref. [13] it was proved that the randomness in the time dependente of the kicks implies quantum instability for that model. That instability would sound at least not intuitive were the dynamical localization for the KR rigorously proved. This fact is accentuated if we recall the Maryland construction and that the Anderson localization has been rigorously proved. Here we exhibit a class of stable periodically kicked systems that become unstable as soon as the kicks are randomly distributed; this is the content of the theorem presented in the next Section. Unfortunately, the physical reason for this behavior is not clear yet; nevertheless, it tells us that a small uncertainty about the value of the period of the external pertiirbation is enough to cease localization and to establish instabilities.

II. Statement of the Results

We study the kicked system represented by eq.(4) that is stable if periodically kicked, and show that it becomes unstable when the kicks are randomly distributed. Th: simplicity of the system we consider permits us to get rather explicit results for quite general potentials and distribution functions, so it is interesting enough to be mentioned. It is known that stable systems can become unstable if the perturbation

is randomly depending on time (see[14,15] for the case of the Schrodinger equation depending on time by some Markov processes), but it seems that so far there were no rigorous results for kicked systerns.

For quantum time-periodic systems a suitable tool for the study of the quantum behavior is the Floquet operator[3,16], i.e. the quantum time evolution operator $U(T)$ between the time 0 and the period T. The main information on the longtime behavior of the system is contained in the spectral properties of the Floquet operator: states in the continuous spectral subspace of $U(T)$ exhibit some diffusive behavior in phase space and states in the point spectral subspace of $U(T)$ correspond to a regular quantum behavior. In the case of random or quasi-periodic time driving forces there is no Floquet operator and the instability in the quantum dynamics is usually related to a growth of the expectation value of the energy. For convenience of the reader we present an Appendix with a discussion on the relation between spectral type of $U(T)$ and kinetic energy growth with time.

As already mentioned, the KR model has been extensively studied; nevertheless, there are few rigorous results[3,8,16] for that model. In Section I of Ref. [16] the kicked system

$$H = -i\partial/\partial x + u(x)\sum_{n\in\mathbb{Z}}\delta(t - 2n\pi T), \quad x \in \mathbb{T}, \quad (3)$$

was considered. For a large class of potentials u it was proved that there exists a set M C ℝ of zero Lebesgue measure (M depends on u) such that, if $0 < T \in \mathbb{R}/M$ the spectrum of the corresponding Floquet operator is pure point, so (3) can be considered a stable system. In this paper we study the system given by the Hamiltonian

$$H = -i\partial/\partial x + u(x)\sum_{n\in\mathbb{Z}}\delta(t - 2n\pi t_n), \quad x \in \mathbb{T}, \quad (4)$$

where the gaps between two consecutive kicks are independent random variables. In the Theorem presented

below it is shown that if the kicks are randomly distributed according to some laws, then there is an "average linear growth" (diffusion in momentum) of the expectation value of the kinetic energy.

If two consecutive kicks occur at $2\pi t_n$ and $2\pi t_{n+1}$, respectively, the unitary operator U_{n+1} that connects the state ψ at $2\pi(t_n + 0)$ to ψ at $2\pi(t_{n+1} + 0)$ is

$$(U_{n+1}\psi) := e^{iu(x)}\psi(x - 2\pi(t_{n+1} - t_n)) \qquad (5)$$

We take (5) as the definition of the time-evolution.

If $2\pi t_n$, n E ℕ, are the instants at which kicks occur, set $\tau_1 = t_1$ and r, $= (t_n - t_{n-1})$, for n > 1, and assume that r, are independent non negative random variables distributed according to a common probability distribution function $F(\tau)$. Denote the characteristic function of the variable r at s by P,:

$$P_s := E(e^{i2\pi\tau s}) := \int_0^\infty e^{i2\pi\tau s}dF(\tau) \qquad (6)$$

Setting $\psi_N(x)$ for the wave function ψ E $L^2(\mathbb{T})$ at the instant immediately after the N-th kick, the average kinetic energy

$$K(\psi_N) = (1/2\pi)\int_0^{2\pi} |id\psi_N/dx|^2 dx \qquad (7)$$

is defined for ψ E $dom(\Delta) \subset L^2(\mathbb{T})$, where A $= (-d^2/dx^2)$ is the kinetic energy operator.

THEOREM **A)** Suppose the kicks in (5) are independent randomly distributed according to the distribution function F. If $|P_s| < 1$ for $0 \neq s \in \mathbb{Z}$ and $\lim sup_{|s|\to\infty}|P_s| < 1$, then for u $\in dom(\Delta)$ with $||u'|| \neq 0$ there exist $\gamma > 0$ and $M > 0$ such that

$$K(\psi_0)+\gamma N-M \leq E(K(\psi_N)) \leq K(\psi_0)+\gamma N+M \quad (8)$$

for N > 2 and any $\psi_0 = \psi$ E $dom(\Delta), ||\psi|| = 1$.

B) Let **F** as above, $u \in dom(\Delta)$ with $u' \neq 0$ a.e. and ψ E $dom(\Delta)$. Then, the sequence $\{K(\psi_N)\}_{N=0}^\infty$ is unbounded with full probability.

COROLLARY: If the probability distribution function **F** is absolutely continuous (with respect to

Lebesgue measure) and $|P_s| < 1$ for $0 \neq s \in \mathbb{Z}$, then the conclusions of the above Theorem hold.

Proof The proof is a simple application of the Riemann-Lebesgue lemma. □

It is clear from the theorem that we have got results for quite general potentials u and distribution functions **F**. Notice the different conditions on u' from parts **A** and B above. The proofs are presented in Section II. In Section III we work out some simple examples.

A natural question is about the behavior of the classical analogue of the random kicked system (4), which is given by the map (in correspondence to (2))

$$\begin{aligned} x_{n+1} &= x_n + 2\pi\tau_n \quad (\text{mod} 2\pi) \\ p_{n+1} &= p_n - u'(x_n) \end{aligned}$$

By iteration we get $p_N = p_0 - \sum_{i=0}^{N-1} u'(x_i)$, and it is not difficult to see that

$$(U_N...U_1)^* p(U_N...U_1) = p_0 - \sum_{i=0}^{N-1} u'(x_i).$$

Hence the classical and quantum energy growth are essentially identical.

III. Proof of the Theorem

Proof of Part A

For $\psi \in dom(\Delta)$,

$$\psi_N(x) = (U_N...U_2U_1)\psi(x) =$$
$$\exp\left\{\sum_{j=1}^N u(x - 2\pi(\tau_{j+1} + ... + \tau_N))\right\}$$
$$\psi(x - 2\pi(\tau_1 + ... + \tau_N)) \qquad (9)$$

with the convention: $(\tau_N+\tau_N) = \tau_N$ and $(\tau_{N+1}+\tau_N) = 0$.

It is possible to assume that $< u >:= \int_0^{2\pi} u(x)d(x/2\pi) = 0$. Indeed, if $< u > \# 0$, we have

$$(U_N \psi_{N-1})(x) = e^{i(u(x))} \psi_{N-1}(x - 2\pi\tau_N)$$

$$= e^{i<u>} e^{i(u(x)-<u>)}$$

$$\psi_{N-1}(x - 2\pi\tau_N) \qquad (10)$$

and the phas: factor $e^{i<u>}$ does not change the average kinetic energy.

Let us first consider vectors of the form $\psi^k(x) = e^{ikx}$, $k \in \mathbb{Z}$. By (9) we have

$$\psi_N^k(x) = \exp\left\{ i\sum_{j=1}^{N} u(x - 2\pi(\tau_{j+1} + ... + \tau_N)) \right.$$

$$\left. -k2\pi(\tau_1 + ... + \tau_N) \right\} \psi^k(x), \qquad (11)$$

and the average kinetic energy is given by

$$K(\psi_N^k) = k^2 + \int_0^{2\pi} d(x/2\pi) \sum_{j,l=1}^{N}$$

$$u'(x - 2\pi(\tau_{j+1} + ... + \tau_N))$$

$$u'(x - 2\pi(\tau_{l+1} + ... + \tau_N)) \qquad (12)$$

To evaluate

$$A_{lj} := \int_0^{2\pi} d(x/2\pi) u'(x - 2\pi(\tau_{l+1} + ... + \tau_N))$$

$$u'(x - 2\pi(\tau_{j+1} + ... + \tau_N)), \qquad (13)$$

we expand u' in Fourier series, $u'(x) = \sum_{s\in\mathbb{Z}} a_s e^{isx}$ ($a_0 = 0$, since $<u> = 0$), and get

$$A_{lj} = \int_0^{2\pi} d(x/2\pi) u'(x - b_{lj}) u'(x), \qquad (14)$$

with $b_{lj} = 2\pi[(\tau_{l+1} + ... + \tau_N) - (\tau_{j+1} + ... + \tau_N)]$.

$$A_{lj} = \int_0^{2\pi} d(x/2\pi) \left(\sum_{s\in\mathbb{Z}} a_s e^{isx} \right) \left(\sum_{r\in\mathbb{Z}} a_{-r} e^{-irx} \right) e^{irb_{lj}} =$$

$$= \sum_{s\in\mathbb{Z}} |a_s|^2 e^{isb_{lj}} + (1/2\pi) \sum_{s\neq r\in\mathbb{Z}} a_s a_{-r} e^{irb_{lj}} \left(e^{i2\pi(s-r)} - 1 \right)/(i(s-r)) =$$

$$= \sum_{s\in\mathbb{Z}} |a_s|^2 e^{isb_{lj}}. \qquad (15)$$

Thus, $K(\psi_N^k) = k^2 + \sum_{s\in\mathbb{Z}}(\sum_{j,l=1}^{N} |a_s|^2 e^{isb_{lj}})$. Since $b_{lj} =: 0$ if $l = j$, we get

$$K(\psi_N^k) = + \sum_{s\in\mathbb{Z}} \left\{ \left(N + \sum_{j\neq l=1}^{N} e^{isb_{lj}} \right) |a_s|^2 \right\} \quad l$$

The expectation value of the kinetic energy is

$$E(K\psi_N^k) = k^2 + \sum_{s\in\mathbb{Z}} \{ (N + E(B_s))|a_s|^2 \}, \qquad (17)$$

where $B_s := \sum_{j\neq l=1}^{N} e^{isb_{lj}}$. $E(B_s)$ is found to be

$$E(B_s) = \sum_{l=2}^{N}\sum_{j=1}^{N} \left(P_s^{j-l+1} + P_{-s}^{j-l+1} \right) = \text{(since } |P_s| < 1 \text{ and } N > 2) =$$

$$= 2(N-1)Re[P_s/(1-P_s)] - 2Re[(P_s^2 + ... + P_s^N)/(1-P_s)] =$$

$$= 2(N-1)Re[P_s/(1-P_s)] - 2Re[(P_s^2/(1-P_s))(1 + ... + P_s^{N-2}) =$$

$$= 2N\, Re[P_s/(1-P_s)] - 2Re[P_s/(1-P_s) + (P_s^2 - P_s^{N+1})/(1-P_s)^2]. \qquad (18)$$

Denote

$$g(s) = 2Re[P_s/(1-P_s)] \quad \text{and} \quad h(s, \text{N}) = -2Re[P_s/(1-P_s) + (P_s^2 - P_s^{N+1})/(1-P_s)^2]. \tag{19}$$

Then

$$E(K\psi_N^k) = k^2 + \sum_{s \in \mathbb{Z}} \{(N(1+g(s)) + h(s,N))|a_s|^2\}. \tag{20}$$

Claim 1 There exists $D > 0$ such that $SUP_{s,N}|h(s,N)| \leq D$.

Claim 2 There exist $Q, q > 0$, such that $Q \geq (1 + g(s)) > q$ for any $s \in \mathbb{Z}\backslash\{0\}$.

Claim 1 as well as the existence of $Q > 0$ such that $Q \geq (1 + g(s))$ in Claim 2 follow easily from the hypotheses of the Theorem. If for some $s \in \mathbb{Z}\backslash\{0\}$, $(1 + g(s)) \leq 0$, we get

$$1 + (r_s + im_s)(1 - (r_s + im_s))^{-1} +$$
$$(r_s - im_s)(1 - (r_s - im_s)^{-1}) \leq 0,$$

where $r_s = Re(P_s)$ and $m_s = Im(P_s)$; the above expression is equivalent to

$$|P_s|^2 = r_s^2 + m_s^2 \geq 1,$$

but $|P_s| < 1$ for $s \in \mathbb{Z}\backslash\{0\}$ and $\lim \sup |P_s| < 1$, which imply the existence of such $q > 0$ This proves Claim 2.

By (20) and Claims 1 and 2 we obtain

$$k^2 + \sum_{s \in \mathbb{Z}} \{(N(1+g(s)) - D)|a_s|^2\} \leq E(K\psi_N^k) \leq$$
$$\leq k^2 + \sum_{s \in \mathbb{Z}} \{(N(1+g(s)) + D)|a_s|^2\} \tag{21}$$

with convergent series.

Thus, Part A of the Theorem is proved for ψ^k with $0 < y := \sum_{s \in \mathbb{Z}} |a_s|^2(1 + g(s))$ and $M := D \sum_{z \in \mathbb{Z}} |a_s|^2$.

Since $\{\psi^k : k \in \mathbb{Z}\}$ is a basis for $L^2(\mathbb{T})$, $\psi \in dom(\Delta)$, $||\psi|| = 1$, can be written $\psi = \sum_{k \in \mathbb{Z}} c_k \psi^k$, and by (21) the expectation of the average kinetic energy $E(K\psi_N) = \sum_{k \in \mathbb{Z}} |c_k|^2 E(K\psi_N^k)$ satisfies

$$\sum_{k \in \mathbb{Z}} |c_k|^2(k^2 + N\gamma - M) \leq E(K\psi_N)$$
$$\leq \sum_{k \in \mathbb{Z}} |c_k|^2(k^2 + N\gamma + M),$$

or

$$(K\psi) + N\gamma - M \leq E(K\psi_N) \leq (K\psi) + N\gamma + M. \quad \square$$

Proof of Part B The proof of Part B is reduced to proving Claim **3.** Since we shall follow Ref. [13] closely, we shall not go into some details.

Claim 3 Let $\psi \in L^2(\mathbb{T})$. There exists a subsequence $\{\psi_{N_k}\}$ of $\{\psi_N\}$ which converges weakly to zero with full probability.

For $\psi \in dom(\Delta)$, $K\psi_N = \sum_{k \in \mathbb{Z}} k^2| < \psi^k|\psi_N > |^2$ is finite. Suppose, per absurdum, that the sequence $\{K\psi_N\}$ is bounded, so it belongs to a compact set in $L^2(\mathbb{T})$ and, with full probability, a convergent subsequence $\{\psi_j\}$ (of {$,v,)) exists. By Claim 3 $\{\psi_j\}$ converges to the null vector, which contradicts the unitarity of U_N. Therefore, $\{K\psi_N\}$ is unbounded with full probability.

Now we turn to the proof of Claim 3. Let $\psi, \phi \in L^2(\mathbb{T})$, $\phi = \sum_{k \in \mathbb{Z}} r_k \psi^k$, $\psi = \sum_{k \in \mathbb{Z}} q_k \psi^K$ and $e^{iu} = \sum_{k \in \mathbb{Z}} c_k \psi^k$. After a little algebra we get

$$| < \phi|\psi_N > |^2 = \sum_{n,n',k_1,k_1',\ldots,k_N} \left\{ r_n r_{n'}^* c_{n-k_1}(c_{n'-k_1'})^* \ldots c_{n-k_N}(c_{n'-k_N'})^* q_{k_1} q_{k_1'}^* \right.$$
$$\left. \times \ e^{-i(k_1-k_1')\tau_1} e^{-i(k_2-k_2')\tau_2} \ldots e^{-i(k_N-k_N')\tau_N} \right\} \tag{22}$$

and

$$E(|<\phi|\psi_N>|^2) = \sum_{n,n',k_1,k_1',\ldots,k_N,k_N'} \left\{ r_n r_{n'}^* c_{n-k_1}(c_{n'-k_1'})^* \ldots c_{n-k_N}(c_{n'-k_N'})^* q_{k_1} q_{k_1'}^* \right.$$

$$\left. \times \quad P_{(k_1'-k_1)} P_{(k_2'-k_2)} \ldots P^{(k_N'-k_N)} \right\} \tag{23}$$

with all indices running from $-\infty$ to $+\infty$.

By using of Fourier series the vectors ψ, ϕ can be represented by elements of the space H of square summable sequences, $\phi \Longrightarrow R = \{r_k\}$ and $\psi \Longrightarrow Q = \{q_k\}$; thus

$$E(|<\phi|\psi_N>|^2) = [(R^* \otimes R), (VC)^N (Q^* \otimes Q)] \tag{24}$$

where the inner product $[.,.]$ is in $H^2 := H \otimes H$, i.e., the space of double sequences $a = \{\alpha_{ij}\}$ such that $\sum_{ij} |\alpha_{ij}|^2 < \infty$. The unitary operator V and the contraction C (i.e., $||C\alpha|| \leq ||\alpha||$) are defined by

$$(V\alpha)_{km} = \sum_{ij}' c_{k-i} c_{m-j}^* \alpha_{ij} \quad \text{and} \quad (C\alpha)_{km} = P_{(k-m)} \alpha_{km} \tag{25}$$

In order that $\beta \in H^2$ be such that $||VC\beta|| = ||\beta||$ it is necessary that $\beta \in \Omega = \{\alpha \in H^2 : \alpha_{km} = 0 \text{ if } k \neq m\}$. Let $\beta \in \Omega$; although $(C\beta) \in R$, $(Vb) \in R$. Indeed under the identification of H^2 with $L^2(\mathbb{T}) \otimes L^2(\mathbb{T})$, an element $\beta \in R$ can be represented by a function of the form $f(x - y) \in L^2(\mathbb{T}) \otimes L^2(\mathbb{T})$ and, from the definition of V, $(V\beta) \in \Omega$ can be represented by $g(x - y)$, i.e.

$$e^{iu(x)} e^{-iu(y)} f(x - y) = g(x - y) \tag{26}$$

Therefore, for any $z \in \mathbb{R}$

$$e^{iu(x+y)} e^{-iu(y+z)} f(x - y) = g(x - y) \tag{27}$$

Taking into account that $u'(\cdot) \neq 0$ a.e., from (26) and (27) we see that $f(\cdot) = 0$ a.e. Thus VC acts as a unitary operator only on the trivial subspace and we can conclude that the minimal unitary dilation of VC has absolutely continuous spectrum. Therefore,

$$\lim_{N \to \infty} E(|<\phi|\psi_N>|^2) =$$

$$\lim_{N \to \infty} [(\phi^* \otimes \phi), \ (VC)^N (\psi^* \otimes \psi)] = 0 \tag{28}$$

for any ψ, $\phi \in L^2(\mathbb{T})$.

Hence there exists a subsequence $\{\psi_{N_k}\}$ of $\{\psi_N\}$ such that $<\phi|\psi_{N_k}>$ converges to zero with full probability. An argument about nested subsequences concludes Claim **3**. $\square$

IV. Some particular systems

Now we briefly discuss some examples.

a) Even though in the case of uniform distribution over $(0,1)$ we can apply the Corollary of the Theorem, it is worth working it out. In this case for $0 \neq s \in \mathbb{Z}$ $P_s = \int_0^1 e^{i2\pi\tau s} dt = 0$. Then from (20) we get $E(K\psi_N^k) = k^2 + N \sum_{s \in \mathbb{Z}} |a_s|^2$ and, for $\psi \in dom(\Delta)$,

$$E(K\psi_N) = (K\psi) + N \sum_{s \in \mathbb{Z}} |a_s|^2 = (K\psi) + N ||u'||^2. \tag{29}$$

b) Let us consider $u(x) = C \cos x$, $0 \neq C \in \mathbb{R}$. We notice that this is the potential of the KR model studied in Ref. [13]. Following the proof of the Theorem of Section I we get

$$E(K\psi_N) =$$

$$(K\psi) + NC^2/2 + C^2/2 \sum_{l=2}^{N} \sum_{j=1}^{N} (P_1^{j-l+1} + P_{-1}^{j-l+1}) \tag{30}$$

If $|P_1| \neq 1$, we can apply the Theorem and the expectation value of the energy "grows linearly" with the number of kicks N. If $|P_1| = 1$ a direct calculation shows

that $E(K\psi_N)$ grows quadratically with N. This example has straightforward generalization to potentials $u(x) = \sum_{s=-n}^{n} b_s e^{isx}$.

c) In this last example the consider the case in which the gaps between two consecutive kicks are symmetric distributed over a positive integer

n and o $\notin \mathbb{Q}$; we take the potential u be such that $\sum_{0 \neq s \in \mathbb{Z}} |a_s|^2/(\sin^2(\pi sx)) < \infty$, where $u'(x) = \sum_{0 \neq s \in \mathbb{Z}} a_s e^{isx}$ (we can suppose a general diophantine-like condition on o, namely, $\sum_{0 \neq s \in \mathbb{Z}} sin^{-2}(\pi sx) < \infty$). In this case we can not apply the Theorem presented in Section I, since we have

$$F(\tau) = (1/2)(\delta_{x,\tau} + \delta_{n,\tau}), \quad P_s = e^{i\pi sx} \cos(\pi sx) \tag{31}$$

and *lim sup*$_{|s| \to \infty} |P_s| = 1$.

For $0 \neq s \in \mathbb{Z}$ we have $g(s) = g(x, \text{n}; s) = 0$ and

$$h(s, \text{N}) = h(x, \text{n}; s, \text{N}) = 2[\cos^2(\text{wso}) - \cos^{(N+1)}(\pi sx)\cos(\pi s(N-1)x)]/\sin^2(\pi sx).$$

From (20) we obtain

$$E(K\psi_N) = (K\psi) + N||u'||^2 + 2\sum_{s \in \mathbb{Z}} |a_s|^2 \tan^2(\pi sx)[1 - \cos^{(N-1)}(\pi sx)\cos(\pi s(N-1)x)] \tag{32}$$

Since the last term in the r.h.s. of (31) is positive (and finite), we are assumed that (31) grows at least "linearly" with N.

Acknowledgement s

Appendix

In this appendix we present a short discussion on the relation between spectral type and energy growth behavior. For simplicity we shall not be concerned with technical questions involving domains of unbounded operators. The main role is played by the point *(associated* to *stable* vectors) and continuous (unstable vectors) spectra.

Let $U(T) : L^2(T) \to L^2(T)$ be the Floquet operator associated to some time-periodic Hamiltonian. If the state vector φ belongs to the subspace spanned by the eigenvectors of $U(T)$ (the so-called point subspace $\mathcal{H}_p$), $U(T)\varphi_n = e^{i\lambda_n}\varphi_n$, we have

$$\varphi = \sum_j a_j \varphi_j \tag{A1}$$

and at $t = \text{NT}$, $\text{N} \in \mathbb{Z}$,

$$U(T)^N \varphi = \sum_j a_j e^{iN\lambda_j} \varphi_j \tag{A2}$$

From (A1) and (A2) one obtains

$$< U(T)^N \varphi |p^2 U(T)^N \varphi > | = \sum_{j,k} a_k a_j^* e^{iN(\lambda_k - \lambda_j)} < \varphi_j |p^2 \varphi_k > \tag{A3}$$

which is a bounded quasi-periodic function of N. More details can be found, e.g. in ref. [17].

Now pick $\psi \neq 0$ in the continuous subspace of $U(T)$, i.e. ψ is orthogonal to H. Bellow it is argued that $< U(T)^n \psi | p^2 U(T)^N \psi >$ is an unbounded function of time N [16]. Let $\psi^k(x) = e^{iLx}$, $k \in \mathbb{Z}$, be the eigenvectors of the momentum operator p. Since (ψ^k) is a basis of $L^2(\mathbb{T})$, we can write

$$\psi_N = U(T)^N \psi = \sum_j b_j(N)\psi^j \qquad (A4)$$

We shall make use of the following corollary of the RAGE theorem[18,16]. Set $m(N, b_j) = \frac{1}{N}\sum_{s=0}^{N-1} |b_j(s)|^2$. Then

$$\lim_{N \to \infty} m(N, b_j) = 0 \quad \forall j \in \mathbb{Z} \qquad (A5)$$

By using of (A4) we get $< \psi_N | p^2 \psi_N > = \sum_j j^2 |b_j(N)|^2$. Let $r \in \mathbb{N}$; by (A5) we can take N large enough so that $m(N, b_j)$ is very small for $j = 0, \pm1, \pm2, ..., \pm(r-1)$, thus $\sum_{|j| \geq r} m(N, b_j) \cong ||\psi||^2$.

Therefore

$$\frac{1}{N}\sum_{s=0}^{N-1} < \psi_s | p^2 \psi_s > \cong \sum_{|j| \geq r} j^2 m(N, b_j) \geq$$
$$r^2 \sum_{|j| \geq r} m(N, b_j) \cong r^2 ||\psi||^2 (N \text{ large}).$$

Since r s arbitrary the average value of $< \psi_N | p^2 \psi_N >$ is unbounded, so is $< \psi_N | p^2 \psi_N >$ as well. It is left to the interested reader the translation of the above argument to a rigorous one, which can be found in ref. [16].

To concludt:, let us mention that the above discussion is not restricted to the operator p^2, but it holds for any unbounded positive operator with discrete spectrum, for exaniple, the absolute value of momentum, which plays th: role of unperturbed energy in model (4).

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