The Christ - Lee Model in the Approach of the Symplectic Projector Method

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We apply tlie syinplectic projector method to obtain tlie pliysical variables aiid the pliysical Hamiltonian ill the Christ-Lce model.

One of tlie possibilities in treating constrained systems, as pointed out by Fradkin and Vilkovisky[1], is to reduce tlie phase-space ill order to work only witli tliose variables which are gauge independents ("physical" or "true" variables). Tliere are, however, great difficulties in finding a procedure to display tliose variables iii a systeinatical way, which limits drastically their applicabilities[2].

Tlie projector technique was developed to deal witli tlie constrained systems of particles[3], and after extension to tlie symplectic phase space it leads to tlie quantization of non - regular systems[4]. Recently tlie symplectic projector inetliod was used in order to obtain the physical variables and tlie pliysical Hamiltonian ill gauge theories aiid two examples have been treated[5]: (i) tlie electrodynamics in tlie Couloinb Gauge and, (ii) the two dimensional Bosonic Schwinger Model. We remark that iii case (ii), tlie new field $B = \Pi_{\Phi} - A_1$ arise in a natural way, instead of tlie procedure in [G].

In this letter we apply tlie symplectic projector inetliod in a simple, but sufficieiitly illuminating, non-relativist, gauge-iiivariant model, proposed by Clirist aiid Lee[7]. Tliis model, which lias been analyzed by several authors,[8-10] lias been used by Costa aiid Girotti[8] as a check of tlie so-called Dirac Bracket Quantization Procedure (DBQP) when, although in an intuitive way, they have succeded ill obtaining tlie pliysical variables.

We show in this letter liow to obtain tlie physical variables in a systematic way.

Let us star witli the following Lagrangian:

$$L = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) - (x_1\dot{x}_2 - \dot{x}_1 x_2)x_3 + \frac{1}{2}x_3^2(x_1^2 + x_2^2) - V(x_1^2 + x_2^2). \quad (1)$$

In order to implement tlie syinplectic projector inetliod we need a local vector space generated by tlie second class constraints of the theory, which are

$$\phi_1 = p_3 = 0, \quad (2.a)$$
$$\phi_2 = p_2 - ep_3 = 0, \quad (2.1))$$
$$\phi_3 = x_2 - ex_1 = 0, \quad (2.c)$$
$$\phi_4 = x_3 = 0, \quad (2.d)$$

where $e = \tan\, b/c$, and b, c are nonzero constants[8].

Tlie projector on tlie manifold provided by tlie constraints has components in free coordinates given by

$$P^{\mu}_{\eta} = \delta^{\mu}_{\eta} - g_{ij}\epsilon^{\mu\beta}\partial_{\beta}\phi^i\partial_{\eta}\phi^j, \quad (3)$$

wliere $\epsilon^{\mu\eta}$ is tlie global symplectic metric aiid g_{ij} is tlie inverse of tlie syinplectic local inetric defined by $g^{ij} = \{\phi^i, \phi^j\}$.

When we look for projected coordinates considering that

$$d\chi^{*\mu} = p^{\mu}_{\eta} d\chi^{\eta}, \quad (4)$$

we find

$$\chi_1^* = (1+e^2)^{-1}\chi_4 + e(1+e^2)^{-1}\chi_5, \quad (5.a)$$
$$\chi_2^* = e\chi_1^*, \quad (5.b)$$
$$\chi_3^* = 0, \quad (5.c)$$
$$\chi_4^* = -(1+e^2)^{-1}\chi_1 - e(1+e^2)^{-1}\chi_2, \quad (5.d)$$
$$\chi_5^* = e\chi_4^*, \quad (5.e)$$
$$\chi_6^* = 0, \quad (5.f)$$

where χ^{μ} is a symplectic coordinate aiid the correspondence witli the canonical coordinates is specified by

$$(x_1, x_2, x_3, p_1, p_2, p_3) \leftrightarrow (\chi_1, \chi_2, \chi_3, \chi_4, \chi_5, \chi_6). \quad (6)$$

Equations (5.a) to (5.f) sliow tliat tlie manifold allowed by the constraints is one-dimensional and the that motion is driven by a Hamiltonian whose form is derived from the canonical one

$$\begin{aligned} H &= \frac{1}{2}\, p_1^2 + \frac{1}{2}\, p_2^2 + V(x_1^2 + x_2^2) \\ &= \frac{1}{2}\, \chi_4^2 + \frac{1}{2}\, \chi_5^2 + V(\chi_1^2 + \chi_2^2). \end{aligned} \quad (0.7)$$

After projection, tlie Hamiltonian assumes tlie form

$$\begin{aligned} H &= \frac{1}{2}\, \chi_4^{*2} + \frac{1}{2}\, \chi_5^{*2} + V(\chi_1^{*2} + \chi_2^{*2}) \\ &= \frac{1}{2}(1+e^2)\, \chi_4^{*2} + V((1+e^2)\chi_1^{*2}). \end{aligned} \quad (0.8)$$

Note that after substitution of the new variables

$$x_* = (1+e^2)^{1/2}\chi_1^*, \quad (9.a)$$

$$p_* = (1+e^2)^{1/2}\chi_4^*, \quad (9.b)$$

tlie projected Hamiltonian can be written as

$$H^* = \frac{1}{2}\, p_*^2 + V(x_*^2). \quad (10)$$

Hence we have obtained tlie canonical equations of motion:

$$\dot{x}_* = \{x_*, H\}_{PB} = p_*, \quad (11.a)$$

$$\dot{p}_* = \{p_*, H^*\}_{PB} = -\frac{\partial V}{\partial x_*}, \quad (11.b)$$

These results are in agreement with Costa-Girotti[8] obtained via DBQP. Thus in this simple example, by tlie use of tlie sympletic projector method we have in a systematical way obtained what in the literature is usually called tlie "star" or "pliysical" variables.

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