Gravitational Aharonov-Bohm Effect for a Spinor Particle

Ivonete B. dos Santos and Valdir B. Bezerra
Departamento de Física, Centro de Ciências Exatas e da Natureza
Universidade Federal da Paraiba, 58059 João Pessoa, Pb, Brasil

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We consider a spinor particle in the background spacetime generated by a cosmic string. Some physical effects associated with the non-trivial conical topology of this spacetime are investigated.

Topological defects of spacetime can be characterized by a spacetime metric with null Riemann-Christoffel curvature tensor everywhere except on the defects, that is, by conic type of curvature singularities. Recent attempts to marry the grand unified theories of particle physics with general relativistic models of the early evolution of the universe have predicted the existente of such topological defects. One example of these topological defects are the cosmic strings[1] which appear naturally in gauge theories with spontaneous symmetry breaking.

Cosmic strings are expected to be created during the phase transitions. Some may still exist and may even be observable; others may have collapsed long ago, and have served as the seeds of the galaxies[1,2]. Straight cosmic strings are long and exceedingly fine objects. Their thickness is comparable to the Compton wavelength $\hbar/m$ (in units c = 1) of typical particles when the string was formed and their tension were enormous, numerically equal to their linear mass density times the square of the speed of light. The tension, for example, for grand unified strings with mass per unit length on the order of 10^{22}g/cm would be 10^{43} dynes.

The line element of the spacetime described by an infinite, straight and static cylindrically symmetric cosmic string[2], lying along the z-axis, is given by[3]

$$ds^2 = dt^2 - d\rho^2 - \alpha^2\rho^2 d\varphi^2 - dz^2, \qquad (1)$$

in a cylindrical coordinate system (t,ρ,φ,z) with $\rho \geq 0$ and $0 \leq \varphi \leq 2\pi$, the hypersurface $\varphi = 0$ and $\varphi = 2\pi$ being identified. The parameter a is related to the linear mass density μ of the string by $a = 1 - 4\mu$. This metric describes the spacetime which is locally flat (for $\rho \neq 0$) but has conelike singularity at $\rho = 0$ with the angle deficit $8\pi\mu$. Then, the spacetime around an infinite straight and static cosmic string is locally flat but of course not globally flat; it does not differ from Minkowski spacetime locally, it does differ globally. The local flatness of the spacetime surrounding a straight cosmic string means that there is no local gravity due to the string. There is no Newtonian gravitational potencial around the string and consequently a particle placed at rest in the spacetime of a cosmic string will not be attracted to it, even though the string may have a linear mass density on the order of 10^{22}g/cm. There is no Newtonian gravitational potential around the string, however we have some very interesting gravitational effects associated with the non-trivial topology of the space-like sections around the cosmic string. Among these effects, a cosmic string can act as a gravitational lens[2] and can induce a repulsive force on an electric charge at rest[4]. Others effects include pair production by a high energy photon when it is placed in the spacetime around a cosmic string[5] and a gravitational analogue[6] of the electromagnetic Aharonov-Bohm effect[7]. Clearly all those gravitational effects of a cosmic string are due to global (topological) features of this spacetime.

In this paper we study some effects of the global features of the spacetime of a straight cosmic string on a spinor particle. To do this we use the Dirac equation in covariant form.

Let us consider a spinor quantum particle imbedded in a classical background gravitational field. Its behavior is described by the covariant Dirac equation

$$\left[i\gamma^\mu(x)\frac{\partial}{\partial x^\mu} - i\gamma^\mu(x)\Gamma_\mu(x)\right]\psi(x) = m\psi(x), \qquad (2)$$

where $\gamma^\mu(x)$ are the generalized Dirac matrices and are given in terms of the standard flat spacetime gammas $\gamma^{(a)}$ by the relation

$$\gamma^\mu(x) = e^\mu_{(a)}(x)\gamma^{(a)}, \qquad (3)$$

where $e^\mu_{(a)}$(x) are vierbeins defined by the relations

$$e^{\mu}_{(a)}e^{\nu}_{(b)}\eta^{(a)(b)} = g^{\mu\nu}. \quad (4)$$

The product $\gamma^{\mu}\Gamma_{\mu}$ that appears in the Dirac equation can be written as[8]

$$\gamma^{\mu}\Gamma_{\mu} = \gamma^{(a)}(A_{(a)}(x^{\mu}) + i\gamma^{(5)}B_{(a)}(x^{\mu})), \quad (5)$$

where $\gamma^{(5)} := i\gamma^{(0)}\gamma^{(1)}\gamma^{(2)}\gamma^{(3)}$ and $A_{(a)}$ and $B_{(a)}$ are given by

$$A_{(a)} = \frac{1}{2}\left(e^{\rho}_{(a),\mu} + e^{\rho}_{(a)}\Gamma^{\mu}_{\rho\mu}\right), \quad (6)$$

and

$$B_{(a)} = \frac{1}{2}\epsilon_{(a)(b)(c)(d)}e^{(b)\mu}e^{(c)\nu}e^{(d)}_{\nu,\mu}, \quad (7)$$

where $\epsilon_{(a)(b)(c)(d)}$ is the completely antisymmetric fourth-order unit tensor and the comma denotes $\partial/\partial x^{\mu}$.

For the nietric corresponding to a cosmic string we shall use the following vierbeins:

$$e^{\mu}_{(0)} = \delta^{\mu}_{0},\ e^{\mu}_{(1)} = \cos\varphi\ \delta^{\mu}_{(1)} - \frac{1}{\alpha\rho}\sin\varphi\ \delta^{\mu}_{2},$$
$$e^{\mu}_{(2)} = \sin\varphi\ \delta^{\mu}_{(1)} + \frac{1}{\alpha\rho}\cos\varphi\ \delta^{\mu}_{2},\ e^{\mu}_{(3)} = \delta^{\mu}_{3}, \quad (8)$$

which yields the proper flat spacetime limit ($\alpha = 1$). Using Eqs. (5), (6), (7) and the above set of vierbeins we get

$$\Gamma_0 = \Gamma_1 = \Gamma_3 = 0$$

and

$$\Gamma_2 = \frac{1}{2}(1-\alpha)\gamma^{(1)}\gamma^{(2)}, \quad (9)$$

and consequently, the following Dirac equation

$$\left\{i\gamma^{(0)}\partial_t + i\gamma^{(\rho)}\left[\partial_{\rho} - \frac{1}{2\rho}\left(\frac{1-\alpha}{\alpha}\right)\right] + \frac{i}{\alpha\rho}\gamma^{(\varphi)}\partial_{\varphi} + i\gamma^{(3)}\partial_3 - m\right\}\psi = 0, \quad (10)$$

where $\gamma^{(\rho)} = \cos\varphi\ \gamma^{(1)} + \sin\varphi\ \gamma^{(2)}$ and $\gamma^{(\varphi)} = -\sin\varphi\gamma^{(1)} + \cos\varphi\ \gamma^{(2)}$.

To exploit the symmetry of the string under z translation, we use the following representation of the Dirac matrices:

$$\gamma^{(0)} = \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}; \quad \gamma^{(1)} = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix};$$
$$\gamma^{(2)} = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & i\sigma^1 \end{pmatrix}; \quad \gamma^{(3)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (11)$$

Since there is no z term in Dirac equation, it separates into two independent equations for two-component spinors.

The spacetime corresponding to a cosmic string is time independent, so the time dependence of the wave function that solves Eq.(2) may be separated as e^{-iEt} and one is led to a stationary problem at fixed energy E. Moreover, rotational invariance of the metric allow us to separate the φ dependence. In view of these we choose the solutions of Eq.(2) in the form

$$\psi = \begin{pmatrix} \sqrt{E+m} & u_1(\rho) \\ i\sqrt{E-m} & u_2(\rho)e^{i\varphi} \end{pmatrix}\exp(-iEt + i\ell\varphi). \quad (12)$$

The equations of motion become

$$\begin{pmatrix} (E-m) & i\left[\left(\partial_{\rho} + \frac{1}{2\rho}\right) + \frac{1}{\alpha\rho}\left(\ell + \frac{1}{2}\right)\right] \\ i\left[-\left(\partial_{\rho} + \frac{1}{2\rho}\right) + \frac{1}{\alpha\rho}\left(\ell + \frac{1}{2}\right)\right] & -(E+m) \end{pmatrix}\begin{pmatrix} \sqrt{E+m} & u_1 \\ i\sqrt{E-m} & u_2 \end{pmatrix} = 0. \quad (13)$$

The general solutions of the above equations are given by

$$u_i(\rho) = C^{(1)}_{i,\ell}J_{|\nu+(i-1)|}(\lambda\rho) + C^{(2)}_{i,\ell}N_{|\nu+(i-1)|}(\lambda\rho), \quad (14)$$

where $i = 1,2$, $\lambda^2 = E^2 - m^2$ and $\nu = \frac{\ell+\frac{1}{2}}{a} - \frac{1}{2}$, $C^{(1)}_{i,\ell}$ and $C^{(2)}_{i,\ell}$ are constant two-component spinors and $J_{|\nu+(i-1)|}(\lambda\rho)$ and $N_{|\nu+(i-1)|}(\lambda\rho)$ are Bessel functions of the first and second kind, respectively.

The equations of this section are similar to the obtained in the analysis of the $(2+1)$- dimensional case[9].

Let us brirfly study the dependence of on a. This dependence can be see also from the expression for the rnean energy density ($\langle H \rangle = \int d^3r J\psi^{\dagger}H\psi$) of a particle in a state with wave function ψ. In the expression for $\langle H \rangle$, J is the Jacobian and is given by $J = \sqrt{det\ g_{ij}} = \alpha\rho$ and

$$H = -i\gamma^{(0)}\gamma^{(\rho)}\left[\partial_{\rho} - \frac{1}{2\rho}\left(\frac{1-\alpha}{\alpha}\right)\right] - \frac{i}{\alpha\rho} - i\gamma^{(0)}\gamma^{(\varphi)}\partial_{\varphi} - i\gamma^{(0)}\gamma^{(3)}\partial_3 + \gamma^{(0)}m \quad (14)$$

is the Harniltonian.

From the expressions for H and < H > it is easy to see that the energy levels will depend strongly on a and thus they will be different from the Minkowski case by this parameter.

Now, let us compute the current, which is defined as

$$j^\mu = \bar{\psi}\gamma^\mu\psi. \tag{15}$$

If ψ is a massive field, j^μ can be written as

$$\begin{aligned} j^\mu &= \frac{1}{2m}(\psi\sigma^{\mu\lambda}\psi)_{,\lambda} + \frac{i}{4m}g^{\mu\lambda}\bar{\psi}\overleftrightarrow{\partial}_\lambda\psi \\ &+ \frac{i}{4m}\bar{\psi}([\gamma^\lambda_{,\lambda},\gamma^\mu] + [\gamma^\lambda,\gamma^\mu_{,\lambda}]\psi \\ &+ \frac{i}{2m}\bar{\psi}[\gamma^\lambda\Gamma_\lambda,\gamma^\mu]\psi, \end{aligned} \tag{16}$$

which represents the Gordon decomposition of the Dirac probability current j^μ.

In the spacetime of a cosmic string, we have

$$[\gamma^\lambda_{,\lambda},\gamma^0] = \frac{2}{\alpha\rho}\gamma^{(0)}\gamma^{(\rho)};\ [\gamma^\lambda,\gamma^0_{,\lambda}] = 0;\ [\gamma^\lambda,\Gamma_\lambda,\gamma^0] = \frac{1}{\rho}\left(\frac{1}{\alpha}-1\right)\gamma^{(0)}\gamma^{(\rho)}; \tag{17a}$$

$$[\gamma^\lambda_{,\lambda},\gamma^1] = [\gamma^\lambda,\gamma^1_{,\lambda}] = [\gamma^\lambda\Gamma_\lambda,\gamma^1] = 0; [\gamma^\lambda_{,\lambda},\gamma^2] = -\frac{1}{\alpha^2\rho^2}[\gamma^{(1)},\gamma^{(2)}]; \tag{17b}$$

$$[\gamma^\lambda,\gamma^2_{,\lambda}] = 2[\gamma^\lambda\Gamma_\lambda,\gamma^2] = \frac{1}{\alpha\rho^2}\left(\frac{1}{\alpha}-1\right)[\gamma^{(1)},\gamma^{(2)}]; \tag{17c}$$

$$[\gamma^\lambda_{,\lambda},\gamma^3] = -\frac{1}{\alpha\rho}[\gamma^{(\rho)},\gamma^{(3)}]; [\gamma^\lambda,\gamma^{(3)}_{,\lambda}] = 0; [\gamma^\lambda\Gamma_\lambda,\gamma^3] = \frac{1}{2\rho}\left(\frac{1}{\alpha}-1\right)[\gamma^{(\rho)},\gamma^{(3)}]. \tag{17d}$$

$$\begin{aligned} \sigma^{01} &= \frac{i}{2}[\gamma^0,\gamma^1] = i\gamma^{(0)}\gamma^{(\rho)}; \\ \sigma^{02} &= \frac{i}{\alpha\rho}\gamma^{(0)}\gamma^{(\varphi)};\ \sigma^{03} = i\gamma^{(0)}\gamma^{(3)}; \\ \sigma^{12} &= \frac{i}{2\alpha\rho}[\gamma^{(1)},\gamma^{(2)}];\ \sigma^{13} = \frac{i}{2}[\gamma^{(\rho)}\gamma^{(3)}]; \\ \sigma^{23} &= \frac{i}{2\alpha\rho}[\gamma^{(\varphi)},\gamma^{(3)}]; \\ \gamma^\mu\Gamma_\mu &= \frac{1}{2\rho}\left(\frac{1}{\alpha}-1\right)\gamma^{(\rho)}. \end{aligned}$$

Then, using Eq. (16) and the results in (17a-d), we obtain,

$$j^0 = \nabla\,.\,\mathbf{P} + \rho_{\text{convective}}, \tag{18a}$$

with

$$j_{(\rho)} = -\partial_t P_{(\rho)} + (\nabla\times\mathbf{M})_{(\rho)} + j_{(\rho),\ \text{convective}}, \tag{18b}$$

$$j_{(\varphi)} = -\partial_t P_{(\varphi)} + (\nabla\times\mathbf{M})_{(\varphi)} + j_{(\varphi),\ \text{convective}} + \frac{1}{\rho}\left(\frac{1-\alpha}{\alpha}\right)M_z, \tag{18c}$$

and

$$j_{(z)} = -\partial_t P_{(z)} + (\nabla\times\mathbf{M})_{(z)} + j_{(z),\ \text{convective}}, \tag{18d}$$

where the convective parts are derived from $\frac{i}{4m}g^{\mu\lambda}\bar{\psi}\overleftrightarrow{\partial}_\lambda\psi$. The polarization densities are given by

$$P_{(\rho)} = \frac{i}{2m}\bar{\psi}\gamma_{(0)}\gamma_{(\rho)}\psi, \tag{19a}$$

$$P_{(\varphi)} = \frac{i}{2m}\bar{\psi}\gamma_{(0)}\gamma_{(\varphi)}\psi, \tag{19b}$$

$$P_{(z)} = \frac{i}{2m}\bar{\psi}\gamma_{(0)}\gamma_{(3)}\psi, \tag{19c}$$

and the corrponents of M are given by

$$M_{(\rho)} = \frac{i}{4m}\bar{\psi}[\gamma_{(\varphi)}, \gamma_{(3)}]\psi, \tag{20a}$$

$$M_{(\varphi)} = \frac{i}{4m}\bar{\psi}[\gamma_{(3)}, \gamma_{(\rho)}]\psi, \tag{20b}$$

$$M_{(z)} = \frac{i}{4m}\bar{\psi}[\gamma_{(1)}\gamma_{(2)}]\psi \tag{20c}$$

The vector M has the meaning of a magnetization current density if we consider to an externa1 electromagnetic field.

Note the dependence of j^{μ}, thought the component $j_{(\varphi)}$, on the parameter a. Then, the current differs from the Minkowski spacetime case by a term containing a dependence on a. So, the fact that the spacetime corresponding to a cosmic string is locally flat but not globally is also ccded into the probability current. There is a physical effect on the current relative to Minkowski spacetime wliich comes out from the topological features of the spacetime surrounding cosmic string.

A quantum spinor particle in the background spacetime around a cosmic string is affected by the fact that this spacetime is locally flat but not globally. This can be seen from the dependence of the wave function and the energy or the parameter α, as well as on the influente of the parameter a on the Dirac probability current. All these interesting effects, codified into the wave function, energy and current, are due to a non-trivial conical topolcgy of the spacetime generated by a cosrnic string.

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