Loop Variables and Holonomy Transformations for a Class of Space-Times

V. B. Bezerra
Departamento de Física, Centro de Ciências Exatas e da Natureza
Universidade Federal da Paraíba, 58059, João Pessoa, PB, Brasil

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We show that tlie loop variables for static spherically symmetric space-times are elements of the Lorcntz group SO(3,1), or more generally, they are elements of the covering group of the Lorentz group in order to include ferinions. Tlie analogous results concerning tlie cylindrically symmetric space-time are given. In tliis case we particularize our results to tlie (2+1)-dimensional space-time sliowing tliat tlie loop variables are elements of SO(2,1) or its covering group. Some examplcs and applications are discussed.

I. Introduction

In tlie loop space formalism for gauge theories[1] the fields depend on paths rather than on space-time points, and a gauge field is described by associating with each path in space-time an element of tlie corresponding gauge group. The fundamental quantity that arises from tliis path-dependent approach, the non-integrable phase factor[2] (or loop variable, in our terminology) represents the electromagnetic field or a general gauge field more adequately than the field strength or tlie integral of the vector potentia1[2]. In the electromagnetic case, for example, as observed by Wu and Yang[2], in a situation where global aspects are taken into consideration the field strength underdescribes the theory and tlie integral of the vector potential for every loop overdescribes it. The exact description is given by the factor $\exp(\frac{ie}{\hbar c}\oint_c A_\mu dx^\mu)$.

The extension of the loop space formalism to the theory of gravity was first considered by Mandelstam[3] who established several equations involving the loop variables, and also by Yang[4], Menskii[5] and Voronov and Makeenko[6]. Recently, Bollini et al.[7] computed the loop variables for the gravitational field corresponding to the Kerr metric.

Einstein gravity in (2 + 1)-dimensional space-time has recently developed into an area of active research[8,9]. One reason for this interest is that there are systems whose symmetry properties reduce the effective number of dimensions. In gravity this occurs for the space-time created by an infinite cosmic string[10], which we shall consider here. On the other hand, this interest has been stimulated by the peculiar and non-generic properties of this field theory.

Space-time is flat outside matter in three-dimensional gravity as well as outside a cosmic string and lience tliere can exist no static interaction between sources. The effects of the sources show up in global aspects of tlie geomctry and we find topology assuming the role played by curvature in the (3+1)-dimensional theory. Although tlie local curvature of source free regions in (2+1)-dimensional gravity is unaffected by any matter in the space-time, it is important to understand that matter can still produce nontrivial global effects. In order to study these effects we shall use the only possible observables in this theory which must come from non-local variables such as the loop variables matrices.

The loop variables in the theory of gravity are matrices representing parallel transport along contours in a space-time with a given affine connection. They are connected with the holonomy transformations whicli contain important topological information. These mathematical objects contain information, for example, about how vectors change when parallel transported around a closed curve. They also can be thought of as measuring the failure of a single coordinate patcli to extend all the way around a closed curve.

Suppose that we have a vector v^a at a point p of a closed curve C in a space-time. Then, one can produce a vector $\bar{v}^\alpha$ at p which, in general, will be different from v^a, by parallel transporting v^a around C. In this case, we associate with tlie point p and the curve C a linear map U^α_β such that for any vector v^a at p, the vector $\bar{v}^\alpha$ at p results from parallel transporting v^α around C and is given by $\bar{v}^\alpha = U^\alpha_\beta v^\beta$. The linear map U^α_β is called the holonomy transformation associated with the point p and the curve C. If we choose a tetrad frame and a parameter $\lambda\epsilon[0,1]$ for the curve C such that $C(0) = C(1) = p$, then in parallel transporting a vector v^a from $C(\lambda)$ to C(X $+ d\lambda$), the vector components change by $\delta v^\alpha = M^\alpha_\beta[x(\lambda)]v^\beta\lambda$, where M^α_β is a linear map which depends on the tetrad, the afine connection of tlie space-time and tlie value of A. Then, it follows tliat tlie liolonomy transformation U^α_β is given by tlie ordercd matrix product of the N linear maps as

$$U^\alpha_\beta - \lim_{N\to\infty}\prod_{i=1}^{N}\left\{\delta^\alpha_\beta + \frac{1}{N}M^\alpha_\beta[x(\lambda)]\Big|_{\lambda=i/N}\right\} . \qquad (I.1)$$

One oftcn writes the expression iii Eq. (1.1) as

$$U(C) = P\exp\left(\int_C M\right) . \qquad (I.2)$$

where P means ordered product along a curve C. Equation (1.2) should be understood as simply an abbreviation for tlie expression in Eq.(1). Note that if M^α_β is independent of A, then it follows from Eq.(1) tliat U^α_β is given by $U^\alpha_\beta = [\exp(M)]^\alpha_\beta$. Under a change of coordinates x $\to x' =$ Lx, U^α_β transforms as $L(U^\alpha_\beta)L^{-1}$.

In this paper we shall use tlie notation

$$, \quad = P\exp\left(\int_A^B \quad \frac{dx^\mu}{\cdots} \right), \qquad (I.3)$$

where Γ^μ is the tetradic connection and A,B are the initial and final points of the path. Tlien, associated with every path C from poiiit A to poiiit B, we have a loop variable given by Eq.(I.3) which is a function of tlie patli C as a geometrical object.

Tlie aim of this paper is to study the theory of gravity using loop variables on the basis of a metric formalism. In Section II we compute the loop variables for a static spherically symmetric space-time and the results are applied to tlie black hole-string metric for an uncharged non-rotating hole. Section III contains similar results concerriing the cylindrically symmetric space-time in (2+1) and 3+1) dimensions and a bricf discussion on the gravitational analogue[11] of the Aharonov-Bohm effect[12] and on tlie study of space-time configuration fiom tlie global point of view. Finally, in Section IV, we add some concluding remarks.

II. Loop variables in a sphcrically symmetric spacc-time

Tlie space-time metric which represents a static spherically synimetric solution of the Einstein's field equations can be written as

$$ds^2 = e^{2\Phi(r)}dt^2 - e^{2(\Lambda(r))}dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\varphi^2 , \qquad (II.1)$$

where $\Phi(r)$ and $\Lambda(r)$ are functions of r only, t is tlie time-like coordinate ($-\infty < t < \infty$) and r,θ and φ are spherical coordinates.

We wish to incorporate a string defect in this metric because we are interested in the effect of the string in this background space-time. We can easily introduce a conical singularity describing a straight cosmic string assuming that a string is a defect in space-time and is to be iiitroduced by removing a sector of angle, say $8\pi\mu$ (μ is the linear mass density of tlie string) and identifying the sides of tlie sector, that is, identifying φ with $\varphi + 2\pi(1-4\mu)$ rather than with $\varphi + 2\pi$, so making the periodicity arbitrary. Tlius, the (r,φ) plane is topologically equivalent to a cone of angle $\sin^{-1}(1-4\mu)$.

The static spherically symmetric metric with a string passing through is simply given by

$$ds^2 = e^{2\Phi(r)}dt^2 - e^{2\Lambda(r)}dr^2 - r^2 d\theta^2 - r^2(1-4\mu)^2\sin^2\theta d\varphi^2, \qquad (11.2)$$

where $0 \le \varphi \le 2\pi$.

In ordcr to compute the loop variables we have to write an explicit expression for the tetradic connection Γ_μ.

Let us introduce a set of four vectors $e^\mu_{(a)}$(a = 0,1,2,3 is a tetradic index) which are orthonormal at each point with respect to the metric with Minkowski signature, tliat is, $g_{\mu\nu}e^\mu_{(a)}e^\nu_{(b)} = \eta_{ab} =$ diag(+1, −1, −1, −1). We assume tliat tlie $e^\mu_{(a)}$'s are matrix invertible, tliat is, tliat tliere exists an inverse frame $e^{(a)}_\mu$ given by $e^{(a)}_\mu e^\nu_{(a)} = \delta^\nu_\mu$ and $e^{(a)}_\mu e^\mu_{(b)} = \delta^a_b$.

Define tlie one-forms $\omega^a (a = 1,2,3,4)$ as

$$\begin{aligned}
\omega^0 &= e^{\Phi(r)}dt , \\
\omega^1 &= e^{\Lambda(r)}dr , \\
\omega^2 &= rd\theta , \\
\omega^3 &= (1-4\mu)r\ \sin\theta d\varphi . \qquad (II.3)
\end{aligned}$$

Then, in a coordinate system ($x^0 = t, x^1 =$ r, $x^2 = \theta$ and $x^3 = \varphi$) the tetrad frame defined by $\omega^a = e^{(a)}_\mu dx^\mu$ is given by

$$e^{(a)}_\mu = \begin{pmatrix} e^{\Lambda(r)} & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & (1-4\mu)r\ \sin\theta & 0 \\ 0 & 0 & 0 & e^{\Phi(r)} \end{pmatrix} .$$

Using the Cartan's structure equations $d\omega^a = e^{(a)}_{\mu\|\nu}dx^\nu\Lambda dx^\mu = -\omega^a_b\Lambda\omega^b$, we get the following expressions for the tetradic connections $\Gamma^a_{\mu b}$ (a, b are tetradic indices)

$$\begin{aligned}
\Gamma^1_{t0} &= \Gamma^0_{t1} = e^{-\Lambda(r)}\frac{d}{dr}(e^{\Phi(r)}), \\
\Gamma^1_{\theta 2} &= -\Gamma^2_{\theta 1} = e^{-\Lambda(r)}, \\
\Gamma^1_{\varphi 3} &= -\Gamma^3_{\varphi 1} = -(1-4\mu)\sin\theta e^{-\Lambda(r)}, \\
\Gamma^2_{\varphi 3} &= -\Gamma^3_{\varphi 2} = -(1-4\mu)\cos\theta. \qquad (11.4)
\end{aligned}$$

First of all we shall consider gciicrol curves in tlic xy plane (and planes parallel to it), at fixed times. Iii this case we have

$$\Gamma_\mu dx^\mu = \Gamma_\varphi d\varphi, \tag{II.5}$$

where

$$\begin{aligned}\Gamma_\varphi &= -i(1-4\mu)\begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\sin\theta e^{-\Lambda(r)} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\cos\theta \\ &= -i(1-4\mu)(\sin\theta e^{-\Lambda(r)}J_{13} + \cos\theta J_{23}).\end{aligned} \tag{II.6}$$

In Eq.(II.6), J_{13} aiid J_{23} are, respectively, tlie generators of rotatioiis about tlic y– aiid x–axis iii $\Re^3$. Therefore, for a general curve in tlic xy-plane, the loop variable is givcii by

$$\begin{aligned}U_{\varphi_2\varphi_1}(C) &= \exp[-i(\varphi_2-\varphi_1)(1-4\mu) \\ &\quad (\sin\theta e^{-\Lambda(r)}J_{13} + \cos\theta J_{23})]\ .\end{aligned} \tag{II.7}$$

When tlie curve is closed we get from Eq.(II.7) the following expression for tlic holonomy transformatioii

$$U_{2\pi,0}(C) = \exp[-2\pi i(1-4\mu)(\sin\theta e^{\Lambda(r)}J_{13} + \cos\theta J_{23})] \tag{II.8}$$

Now, consider a curve $r(\lambda)$, with $\theta(\lambda)$ coiitaiiicd in a meridian plane. Iii this case we have

$$\Gamma_\lambda d\lambda = \left(\Gamma_\theta\frac{d\theta}{d\lambda} + \Gamma_r\frac{dr}{d\lambda}\right)d\lambda. \tag{II.9}$$

From Eqs.(II.4) we see that $\Gamma_r = 0$ aiid $\Gamma_\theta = ie^{-\Lambda(r)}J_{12}$, independent of 0, and then tlic loop variables for a general curve iii tlie meridian plane is givcii by

$$U_{\theta_2\theta_1}(C) = \exp[ie^{-\Lambda(r)}(\theta_2-\theta_1)J_{12}], \tag{II.10}$$

where

$$J_{12} = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is tlie generator of rotations about tlic local z-axis in $\Re^3$.

From Eq.(II.10) we get for a closed curve

$$U_{2\pi,0}(C) = \exp[-2\pi i(1-e^{-\Lambda})J_{12}] \tag{11.11}$$

$$= \begin{pmatrix} \cos[2\pi(1-\text{e-''})] & \sin[2\pi(1-\text{e-''})] & 0 & 0 \\ -\sin[2\pi(1-\text{e-''})] & \cos[2\pi(1-e^{-\Lambda})] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{11.12}$$

Eq.(II.11) represents a rotation about tlic Oz axis through an angle $2\pi(1-e^{-\Lambda})$. Finally coiisidcr a translation in time. In this case $\Gamma_\mu dx^\mu = \Gamma_t dt$ where

$$\begin{aligned}\Gamma_t &= -i\begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} e^{-\Lambda(r)}\frac{d}{dr}(e^{\Phi(r)}) \\ &= -ie^{-\Lambda(r)}\frac{d}{dr}(e^{\Phi(r)})J_{01}.\end{aligned} \tag{II.13}$$

J_{01} bciiig tlic generator of a boost in tlie Ox-direction. Usiiig Eq.(II.12) we get for a time translation between t_1 aiid t_2, the following expression for tlie loop variable

$$\begin{aligned}U_{t_2t_1}(C) &= \exp\left[-ie^{-\Lambda(r)}\frac{d}{dr}\left(e^{\Phi(r)}\right)(t_2-t_1)J_{01}\right] \\ &= \begin{pmatrix} \cosh\gamma & 0 & 0 & \sinh\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\gamma & 0 & 0 & \cosh\gamma \end{pmatrix},\end{aligned} \tag{II.14}$$

where $\gamma = e^{-\Lambda(r)}\frac{d}{dr}(e^{\Phi(r)})$ is tlie boost parameter. Eq.(II.13) represeiits a boost in tlie (x,t) direction.

Usiiig tlie above results we can write a general expression for $U(C)$. In tlie general case $U(C)$ reads

$$U(C) = P\exp\left(\frac{i}{2}\int_c \Gamma^{ab}_{\mu}(x)J_{ab}dx^{\mu}\right), \qquad \text{(II.15)}$$

where J_{ab} are tlie generators of llic Lorciitz groiip $SO(3,1)$ aiid Γ^{ab}_{μ} are tlic appropriate tetradic connections. From tlie above result we conclude that tlic lioloiiomy for $(3+1)$-dimensional static spacc-tiiiic spherically syrnmetric, is tlic homomorphism tliat maps tlic homotopy class of all tlic curves to tlic rotations and Loosts in $SO(3,1)$. As ordinary vectors live in tangent space to tlic manifold and for static space-times there is no sli ft iii tiiiie aiid consequently no translations, tlic transformations that act oii tliis spacc are tlie Lorentz ones and therefore tlie parallel transport matrices (loop variables) must be elements of tlic Lorciilz group. In general, tlie J_{ab}'s generate tlic representation of tlic Lorentz groiip wliicli acts oii tlic traiisported quantity wliicli can be a vector or a spinor. In tlic spinor case, instead of tlic group $SO(3,1)$ we have a coveriiig groiip of tliis one. Therefore, when we have fermions, tlic loop variables are elements of tlic covering groiip of the Lorentz groiip.

From tlic previous results we see tliat tlic wedge removal affects tlic loop variable in tlic xy-plane only, so that, a vector parallel transported along a curve iii tlic xy plane will detect tlie presciice of tlic striiig.

As an example consider the black hole-string metric for an uncharged non-rotating hole wliicli is given by

$$\begin{aligned} ds^2 &= \left(1-\frac{2M}{r}\right)dt^2 - \left(1-\frac{2M}{r}\right)^{-1}dr^2 \\ &- r^2(d\theta^2 + (1+4\mu)\sin^2\theta d\varphi^2), \end{aligned} \qquad \text{(II.16)}$$

where $0 \leq \varphi \leq 2\pi$.

Iii view of t ie previous results we should expect tlic presence of tlic striiig through tlic role to modify tlie lioloiiomy transformation for general curves in tlic xy plane, wliicli is given, in tliis case, by

$$U_{2\pi,0}(C) = \exp\left\{+2\pi i(1-4\mu)\left[\left(1-\frac{2M}{r}\right)^{1/2}\sin\theta J_{13} + \cos\theta J_{23}\right]\right\}. \qquad \text{(II.17)}$$

Consider a path formed by two beams which is assumed to circulate tlie z axis aiid to be for fixed θ, say $\pi/2$. Then, tlie relevant phase is

$$U_{2\pi,0}(C) = \exp\left\{+2\pi i\left[1-(1-4\mu)\left(1-\frac{2M}{r}\right)^{1/2}\right]J_{13}\right\}, \qquad \text{(II.18)}$$

where we have introduced a factor $\exp(2\pi i J_{13})$ in order to take into account tlie rotation of tlie local tetrad frame with respect to a tetrad of fixed orientation aiid wliicli is equal to tlic 4×4 identity matrix.

Eq.(II.18) give us llic pliase acquired by a vector when parallel traiisported aroiiiid tlic source for $\theta = \pi/2$, wliicli is associated with tlic non-triviality of tlic lioloiiomy transformation for all values of $r \neq 2M$. Note that tlic wedge removal produced by tlic presence of a striiig appears in tlic phase through tlic parameter μ.

We can obtain a similar result iii tlic spinor case. However, in order to incorporate fermions we have to use tlic spinorial representation of tlic Lorciitz group. So, we change J_{13} by $\sum_{13} = \frac{1}{2}[\gamma_1,\gamma_3]$, where γ_1 and γ_3 are Dirac matrices in tlic standard representation. Then, for a path in tlic xy plane, for $\theta = \pi/2$, we get

$$U_{2\pi,0}(C) = \exp\left\{+2\pi i\left[1-(1-4\mu)\left(1-\frac{2M}{r}\right)^{1/2}\right]\Sigma_{13}\right\}. \qquad \text{(II.19)}$$

From Eq.(II.19) we see tliat tlic lioloiiomy transformation for tlic black liole-striiig metric and for Sclimarzscliild metric ($\mu = 0$) also, is trivial only for $r = 2M$, where the metric is infinite. For physical sources however, tliis singularity occurs inside tlic source where tlic exterior solution does iiot apply. Tlicii, tlic phase can never reach tlic trivial value, or in other words, tlie observability of tliis value is limited by physical considerations.

III. Loop variables in a cylindrically symmetric space-time and applications

Tlic niost general static cylindrically symmetric metric may be espressed in tlie form

$$ds^2 = e^{2\nu}dt^2 - e^{2\lambda}(d\rho^2 + dz^2) - e^{2\psi}d\varphi^2, \qquad \text{(III.1)}$$

where t is tlie time-like coordinate ($-\infty < t < \infty$), ρ, φ aiid z are ordinary cylindrical coordinates with $0 \leq \rho < \infty$, $0 \leq \varphi \leq 2\pi$ aiid $-\infty < z < \infty$ aiid ν, λ aiid ψ are functions of ρ.

Proceeding as in Sectioii II we define the forms

$$\begin{aligned} \omega^0 &= e^{\nu}dt, \\ \omega^1 &= e^{\lambda}\cos\varphi d\rho - e^{\psi}\sin\varphi d\varphi, \\ \omega^2 &= e^{\lambda}\sin\varphi d\rho + e^{\psi}\sin\varphi d\varphi, \\ \omega^3 &= e^{\lambda}dz. \end{aligned} \qquad \text{(III.2)}$$

Then, in a coordinate system ($x^1 = \rho, x^2 = \varphi, x^3 = z$ and $x^0 = t$), the tetrad frame defined by $e^{(a)}_\mu dx^\mu$ is given by[13]

$$\begin{aligned} e^{(1)}_1 &= e^{\lambda}\cos\varphi \quad , \quad e^{(1)}_2 = -e^{\psi}\sin\varphi, \\ e^{(2)}_1 &= e^{\lambda}\sin\varphi \quad , \quad e^{(2)}_2 = -e^{\psi}\cos\varphi, \\ e^{(3)}_3 &= e^{\lambda} \quad , \quad e^{(0)}_0 = +e^{\nu}. \end{aligned} \tag{III.3}$$

Proceeding analogously to the spherically symmetric case we can show that the loop variables are given by Eq.(II.14), with the tetradic connections given by[13]

$$\begin{aligned} \Gamma^1_{t0} &= \Gamma^1_{t1} = e^{-\lambda}\frac{d}{d\rho}(e^{\nu}) \cos\varphi, \\ \Gamma^2_{t0} &= \Gamma^1_{t2} = e^{-\lambda}\frac{d}{d\rho}(e^{\nu})\sin\varphi, \\ \Gamma^1_{\varphi 2} &= -\Gamma^2_{\varphi 1} = \left[1 - e^{-\lambda}\frac{d}{d\rho}(e^{\psi})\right], \\ \Gamma^1_{z3} &= \quad = -e^{-\lambda}\frac{d}{d\rho}(e^{\lambda})\cos\varphi, \\ &= -\Gamma^3_{z2} = e^{-\lambda}\frac{d}{d\rho}(e^{\lambda})\sin\varphi. \end{aligned} \tag{III.4}$$

Consider now the (2 + 1)-dimensional case. Then Eqs.(III.4) reduces to

$$\begin{aligned} \Gamma^1_{t0} &= \Gamma^0_{t1} = e^{-\lambda}\frac{d}{d\rho}(e^{\nu})\cos\varphi, \\ \Gamma^2_{t0} &= \Gamma^0_{t2} = e^{-\lambda}\frac{d}{d\rho}(e^{\nu})\sin\varphi, \\ \Gamma^1_{\varphi 2} &= -\Gamma^2_{\varphi 1} = \left[1 - e^{-\lambda}\frac{d}{d\rho}(e^{\psi})\right]. \end{aligned} \tag{III.5}$$

Using these connections and considering general curves in the xy-plane, translation in time and radial segments it is easy to show[14] that the loop variables are given by Eq.(II.14) where now the J_{ab}'s are generators of the group $SO(2,1)$ or in general, J_{ab}'s are generators of the covering group of the group $SO(2,1)$.

Now let us define the deficit angle and establish its connection with the holonomy transformation. The deficit angle is one number and the holonomy transformation is a set of linear maps (one for each point and closed curve). One must then obtain from the linear map a single number, the deficit angle which is a property of axially symmetric, asymptotically conical space-times (at infinity, these space-times are asymptotically a cone rather than a plane). To obtain the single linear map we consider a point p on the curve C. Since the space-time is axially symmetric, it does not matter which point we choose. Then U^α_β, as defined previously, is the holonomy transformation associated with the point p and a curve C, where C is an integral curve of the axial Killing field in the asymptotic region. With U^α_β, the deficit angle χ can be defined by

$$\cos\chi = U^\alpha_\beta \hat{A}_\alpha \hat{A}_\beta, \tag{III.6}$$

where $\hat{A}_\alpha$ is the unit vector in the direction of the axial Killing field. Using tetradic indices we can write

$$\cos\chi = \tilde{A}^a \eta A_a, \tag{III.7}$$

where $\tilde{A}^a = U^a_b A^b$.

As $\hat{A}^b$ is a unit vector, the elements of U are the components of the parallel translated vector. From this and Eq. (III.7), it follows that, the corresponding diagonal element of U is the cosine of the angle between the vectors. Then, we can write in this case $\cos\chi_a = U^a_a$, where a is a tetradic index.

Considering a = 1, we have

$$\cos\chi_1 = \cos\left[2\pi\left(1 - e^{-\lambda}\frac{d}{d\rho}(e^{\psi})\right)\right], \tag{III.8}$$

or

$$|\chi_1| = \left|2\pi\left(1 - e^{-\lambda}\frac{d}{d\rho}(e^{\psi})\right) + 2\pi n\right|.$$

As $e^{-\lambda}\frac{d}{d\rho}(e^{\psi}) \to 0$, we must have $\chi_1 \to 0$, and we choose n = 0 so that

$$|\chi_1| = \left|2\pi\left(1 - e^{-\lambda}\frac{d}{d\rho}(e^{\psi})\right)\right|. \tag{III.9}$$

Eq.(III.9) corresponds to the general formula for the angular deficit for a class of static cylindrically symmetric space-times metric given by Eq.(III.1).

Now, let us apply our results to the change in a vector as well as in a spinor when parallel transported along a closed curve in the space-time of a static cylindrically symmetric cosmic string[10]. As we know, the space-time corresponding to this solution has the geometry of a cone $\Re^2$. The curvature vanishes everywhere except in the vertex. Then, if a vector (or a spinor) is carried around a closed curve encircling the vertex, after the transport is completed, the vector (or the spinor) changes due to the global effect of the enclosed curvature.

For the cosmic string solution, the metric is a particular case of the one given by Eq.(III.1) with $e^{\nu} = e^{\lambda} = 1$ and $e^{\psi} = (1-4\mu)\rho$, where μ is the linear mass density of the string and we have considered Newton's constant G = 1. Explicitly, the line element of the space-time described by an infinite, straight and static cylindrically symmetric cosmic string, lying along the z-axis, is given by[10]

$$ds^2 = dt^2 - d\rho^2 - (1-4\mu)^2\rho^2\varphi^2 - dz^2, \tag{III.10}$$

with the deficit angle $\chi = 8\pi\mu$, obtained from Eq.(III.9).

For general curves in tlie xy plane me find, using Eqs.(II.14) aiid (III.4) witli e" = e' = 1 and e* = $(1-4\mu)$.?, tliat in the cosmic string case, $U(C)$ is given by

$$U(C) = \exp(-8\pi i\mu J_{12}) = \begin{pmatrix} \cos(8\pi\mu) & \sin(8\pi\mu) & 0 & 0 \\ -\sin(8\pi\mu) & \cos(8\pi\mu) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{(III.11)}$$

where $8\pi\mu$ is tlie deficit angle associated witli tlie space-time of a cosrric string. We can obtain a similar rcsult in tlie case of transport of spinors but in order to incorporate ferrrions we have to use tlie spinorial representation of tl e Lorentz group. So, we cliange J_{12} by $\Sigma_{12} = \frac{1}{2}[\gamma_1, \gamma_2$, where γ_1 aiid γ_2 are Dirac matrices in tlie standard representation. Tlien, for general closed curves in the x y-plane we get

$$U(C) = \exp(-4\pi i\mu\Sigma_{12}). \quad \text{(III.12)}$$

After tlie parallel transport, tlie spinor $\psi(\varphi = 2\pi)$ will be given in terms of tlie original one $\psi(\varphi = 0)$ by tlie relation

$$\psi(\varphi = 2\pi) = e^{-4\pi i\mu\Sigma_{12}}\psi(\varphi = 0). \quad \text{(III.13)}$$

From Eq.(I 1.13) we conclude tliat there will be no Aharonov-Bohm effect if aiid only if $4\pi\mu$ is an even integer. However, tliis condition is not always satisfied. Tlien, we have shown that if we parallel transport a spinor around a closed path in tlie xy-plane lying in tlie flat region, tlie transported one does not necessarily coincide witli tlie original. Therefore, when we parallel transport a spinor in a region in which the curvature vanishes, it exl ibits a physical effect arising from tlie enclosed non-zero curvature associated with tlie presente of tlie cosmic string. This is an example of tlie gravitational aiialogue of tlie Aharonov-Bohm effect. Tliis effect should be regarded as basically classical, and it is associated with the non-triviality of tlie liolonomy transformation for general curves in the xy-plane, due to tlie presence of the cosmic string. As in this case tlie geometry is locally flut the phase sliift acquired by tlie spinor when parallel transported around tlie source may be regarded as due to tlie coupling of its energy-momentum to the global geoinetrical properties of this space-time. The same analysis can be applied to a vector. In this case we use the expression for tlie liolonomy transformation given by Eq.(III.11) concluding that the same effect occurs.

A similar result can be obtained in tlie case of a spinless point particle solution[8] (three-dimensional case). In tlie spinor case, the spinor acquires a pliase given by $\exp(-4\pi i m\Sigma_{12})$, where $\Sigma_{12} = \frac{1}{2}[\sigma_1, \sigma_2]$ with σ_1 and σ_2 being Pauli's matrices, aiid m is tlie mass of tlie particle tliat generates tlie gravitational field. Following tlie arguments of tlie string case, we conclude that we have an Aharonov-Bohm effect in this case. Tlie same analysis can be extended to tlie transport of vectors, with a similar conclusion.

Similarly, we can consider tlie metric[15]

$$ds^2 = dt^2 - d\rho^2 - G_0^2\rho^2 d\varphi^2 - (B_0 t + B_1)^2 dz^2, \quad \text{(III.14)}$$

where G_0, B_0 are B_1 aiid integration constants.

The metric given by Eq.(III.14) corresponds to a Minkowski space-time minus a wedge as we see by defining tlie coordinates Φ, Z and T, for $B_0 \neq 0$, by

$$\begin{aligned} \Phi &\equiv G_0\varphi , \\ Z &\equiv \left(t + \frac{B_1}{B_0}\right)\sinh(B_0 z) , \\ T &\equiv \left(t + \frac{B_1}{B_0}\right)\cosh(B_0 z). \end{aligned} \quad \text{(III.15)}$$

In tlie new coordinates, tlie metric given by Eq.(III.14) reads

$$ds^2 = dT^2 - d\rho^2 - \rho^2 d\Phi^2 - dZ^2. \quad \text{(III.16)}$$

Tlius tlie above metric is locally flat but not globally. Tlie deficit angle is $2\pi G_0$.

We can do the previous analysis slioming that we have a gravitational analogue of tlie Aharonov-Bohm effect also, in tlie vector and spinor cases, witli tlie holonomies in tlie xy plane given by $U(C) = \exp(-2\pi i G_0 J_{12})$ and $U(C) = \exp(-\pi i G_0 \Sigma_{12})$, respectively.

As another application we sliall study tlie space-time configuration of two moving cosmic strings. To do tliis we shall use Eq.(III.10) and a result[13] tliat only strings enclosed by tlie circles contribute to tlie phase factor acquired by a vector when parallel transported in the background space-time of tlie multiple cosmic string solution[16].

Tlien, suppose tliat we transport a vector around a string 2 localized at $(a_2, 0, 0, 0)$. The pliase factor acquired by this vector is $U_2 = \exp(-8\pi i\mu_2 J_{12})$. Now, carrying tlie resultiiig vector along a circle around string 1 localized at $(a_1, 0, 0, 0)$, it is easy to conclude that tlie resulting vector will have a phase given by the product U_1, U_2, where $U_1 = \exp(-8\pi i\mu_1 J_{12})$. Note tliat we can continue tliis process involving N strings. After tliis, the vector will have acquired a pliase given by $U_1 U_2 \ldots U_{K-1} U_K U_{K+1} \ldots U_{N-1} U_N$, where $U_K = \exp(-8\pi i\mu_K J_{12})$, μ_K being tlie linear mass density of the K th string.

Now consider a system of two moving strings. Consider string 1, initially at the origin with velocity $\vec{v}_1$

and string 2 located along the r-direction at $(a_2, 0, 0, O)$, witli velocity $\vec{v}_2$, in the xy-plane. These strings can be vicwed as strings at rest tliat were boostcd. Tlien, if we take a vector and carry it alonga circle around string 2, instead of tlie pliase U_2 tlie vector will acquire a phase $L_2 U L_2^{-1}$, whicli corresponds to the transformation of tlic loop variable under tlie cliange of coordinate corresponding to the boost L_2 wliich is given by

$$L_2 = \begin{pmatrix} \cosh\gamma_2 & 0 & 0 & \sinh\gamma_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\gamma_2 & 0 & 0 & \cosh\gamma_2 \end{pmatrix}, \qquad \text{(III.17)}$$

where γ_2 is the boost parameter and sucli that $|\vec{v}_2| = \tanh\gamma_2$.

If now, tlie resulting vector is parallel transported around string 1, along a circle, tlie final pliase will be $L_1 U_1 L_1^{-1} L_2 U_2 L_2^{-1}$, where L_1 is given by tlie same expression for L_2 with the intercliange of γ_2 by $\gamma_1 (|\vec{v}_1| = \tanh\gamma_1)$.

Let us now consider a tliird string that behaves globally like tlicsc two. Tliis string can be viewed as one boosted by

$$L_3(\varphi_3, \gamma_3) = \begin{pmatrix} 1 - \cos^2\varphi_3(1 - \cosh\gamma_3) & -\cos\varphi_3 \sin\varphi_3(1 - \cosh\gamma_3) & 0 & \cos\varphi_3 \sinh\gamma_3 \\ -\cos\varphi_3 \sin\varphi_3(1 - \cosh\gamma_3) & 1 - \sin^2\varphi_3(1 - \cosh\gamma_3) & 0 & \sin\varphi_3 \sinh\gamma_3 \\ 0 & 0 & 1 & 0 \\ -\cos\varphi_3 \sinh\gamma_3 & \sin\varphi_3 \sinh\gamma_3 & 0 & \cosh\gamma_3 \end{pmatrix} \qquad \text{(III.18)}$$

The form of $L_3(\varphi_3, \gamma_3)$ comes out from tlie fact tliat every homogeneous Lorentz transformation can be decomposed in tlie following way: $L(\varphi, y) = R(\varphi) L(0, \gamma) S(\varphi)$, wliere R and S are rotations and we choose $S = R^{-1}$ and in tliis case $L^{-1}(\varphi, y) = R(\varphi) L(0, -\gamma) S(\varphi)$.

We want tlie third string to be globally equivalent to tlie two previous ones. Tlien, we have to eqiiate tlie pliase factor acqiiired in both situations, tliat is $L_1 U_1 L_1^{-1} L_2 U_2 L_2^{-1} = L_3 U_3 L_3^{-1}$. Taking the trace of tliis relation we find tlie solutions

$$\begin{aligned} \pm\cos(4\pi\mu_3) \; = \; & \cos(4\pi\mu_1)\cos(4\pi\mu_2) \\ & -\sin(4\pi\mu_1)\sin(4\pi\mu_2) \times \\ & (\cosh\gamma_1 \cosh\gamma_2 - \sinh\gamma_1 \sinh\gamma_2) \end{aligned} \qquad \text{(111.19)}$$

Equation (111.19) is tlie relation between tlie deficit angles produced by the system of strings, the velocities and the deficit angle produced by tlie tliird string. From tliis equation we see tliat tlie angle $4\pi\mu_3$ depends on the linear mass densities of tlie strings 1 and 2, and on its velocities.

In the (2+1)-dimensional case we obtain the same Eq.(III.19) for a system of particles, intercliangíng μ by m (mass of the particlc).

Equation (111.19) and the similar one in the case of point particles give us information about tlie global features of the space-time gcnerated by a system of strings and point particlcs, respectively.

IV. Concluding remarks

We have shown by explicit computation for metrics corresponding to spherically symmctric space-times, tliat tlie pliase acqiiired by a particle (vector or spinor), when parallel transported along a given curve C in tlicsc background gravitational fields is givcn by the loop vaiables $U(C) = P\exp(\int_c \Gamma_\mu dx^\mu)$ with $\Gamma_\mu = \frac{i}{2}\Gamma_\mu^{ab} J_{ab}$, wliere J_{ab} are tlie generators of tlie Lie algebra of tlie Lorentz group $SO(3,1)$ or of its covering group. Tlien, for a given curve in these space-times, tlie phase sliift acquired by a particle is an element of tlie Lorentz group, or in general, the phase factor is an element of tlie covering group of the Lorentz group, in order to include fermions.

For tlie metrics corresponding to cylindrically symmetric space-times, tlie loop variablcs are elements of tlie Lorentz group $SO(3,1)$ or of its covering group[13], and in the (2 + 1)-dimensional case, they are elements of the $SO(2,1)$ group or of its covering group also in order to include fermions. These results permit us to study tlie gravitational analogue of the Aharanov-Bolim effect[11].

As tlie loop variables for the static geometric structures under considerations are elements of tlie Lorentz group $SO(3,1)$, tliis means that these quantities are related to the holonomies of a flat $SO(3,1)$ connections and consequently that the space-time geometry is encoded in tlie liolonomies of these flat $SO(3,1)$ connections.

The configuration of a space-time corresponding to two moving strings or particles ((2 + 1)-dimensional case) sliows that there is a linking between tlie parameters that describe this space-time and tlie space-times generated by each of the two strings or particles, respectively.

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