High-Intensity Multiphoton Effects in Electron-Ion Scattering

H. Torres-Silva: P. H. Sakanaka and L. C. Braga
Instituto de Física, Universidade Estadual de Campinas
Caixa Postal 6165, Campinas, 13081, SP, Brasil

Received May 4, 1992

The differential cross-section for inelastic scattering in the presence of an intense laser field is discussed. We have also calculated the absorption coefficient $\bar{\alpha}$ for a monoenergetic beam of electrons scattered by a static potential. Here we have derived ir starting under the framework of quantum mechanics, making the classical correspondence (h → 0) according to the kinetic theory, and show that the absorption coefficient is always positive for all values of the particle incoming velocity, $\vec{v}_i$. Numerical calculation of the total cross-section is also reported, showing that the well known sum rule for multiphoton free-free transitions does not hold, at small scattering angles.

I. Introduction

The investigation of the absorption of intense electromagnetic radiation in a fully ionized plasma with collisions of electrons and ions still is, at present, incomplete. The main difficulty at high intensity radiation is that a large number of photons must be included in the calculation. Thus the direct evaluation of the contribution from many photons to obtain the total absorption coefficient α is difficult. In the quantum mechanical approach, the inverse bremsstrahlung (I.B.) problem is solved by: a) calculating the transition probability (or cross section, a and b) calculating the rate at which energy is absorbed[1,2,5]. However, for high intensities of the laser beam, i.e., in the range of 10^{16} to 10^{18} W/cm^2, relativistic effects on scattering potential must be considered.

The differential cross section for inelastic scattering in the presence of an intense laser field, obtained by us[1] was recently confirmed[6,7], when the kibble parameter ($\epsilon = v_0/c$) is less than unity. The main result is that the Kroll-Watson expression[4]

$$\left(\frac{d\sigma}{d\Omega}\right)_L = (1+\epsilon^2)^2\left(\frac{d\sigma}{d\Omega}\right)_R = \sum_{n=-\infty}^{\infty}\left(\frac{d\sigma}{d\Omega}\right)_n \qquad (1)$$

breaks down. Here $(d\sigma/d\Omega)_R$ is the Rutherford differential cross section. In addition, there are great difficulties when it is necessary to calculate the global absorption coefficient ir, because equation (1) is independent of the photon number n and of the distribution function of electrons. Similar difficulties appear on the energy balance and heating by multiphoton process when a monoenergetic beam of electrons is scattered by a static potential in the presence of a strong laser field. Some authors[8] have shown that $\bar{\alpha}$ can be negative if $v_0 < v_i$, ($\vec{v}_i$ = particle incoming velocity, $\vec{v}_0$ = amplitude of the oscillatory velocity). In Section II, we present a proper way to calculate the absorption coefficient $\bar{\alpha}$ from the cross section treatment, and we are able to show that $\bar{\alpha} \geq 0$, for all $\vec{v}_i$. Section III contains some numerical results and Section 4 is dedicated to discussions and conclusion.

*Universidad de Tarapacá, Dto de Electrónica, Arica, Chile.

II. Quantum Mechanical Treatment

II.1. Cross Section and Transition Rate

We begin by considering the problem of a non-relativistic particle of mass m and charge e being scattered by a static, local potential $V(r)$, in the presence of a strong laser field. Here we consider that the electrons interact with infinitely heavy ions via the Coulomb potential, and the first Born approximation is used for the scattering of electrons by ions. The transition probability $T_{nk}(\vec{p}_i \rightarrow \vec{p}_f)$ from a initial state $\vec{p}_i$ to a final state $\vec{p}_f$, under the perturbation $\phi(\vec{k})$ due to the n-photon process is given by[3,9]

$$T_{nk}(\vec{p}_i - \vec{p}_f) = \frac{2\pi}{\hbar}\sum_n |\phi(\vec{k})|^2 J_n^2(x)\delta(E_f - E_i - n\hbar\omega), \qquad (2)$$

where $\phi(\vec{k})$ is the Fourier transform of the potential interaction $\phi(\vec{r})$, $E_{f,i} = p_{f,i}^2/2m$ and x $= \vec{k}\cdot\vec{v}_0/\omega = \left(e\vec{E}_0/m\omega\right)(\vec{p}_f - \vec{p}_i)/\hbar\omega$. The differential cross section for I.B. is obtained from equation (??)

$$d\sigma_{nk}^{a,e} = \frac{4Z^2e^4}{v_i}\frac{|R(k)|^2}{\hbar^2k^4} J_n^2(x)\delta\left(\vec{k}\cdot\vec{v}_i - n\omega \pm \frac{\hbar k^2}{2m}\right), \quad (3)$$

where, Z, $\vec{v}_i$, and n are respectively, the ionic charge number, the velocity of the incoming particle beam and the number of photons exchanged with the assisting field, with plus for absorption, superscript a, and minus for emission, superscript e. $R(k)$ is the form factor which takes into account the range of the static screened Coulomb potential.

II.2. Absorption Coefficient

In this subsection we deal with the calculation of the absorption coefficient $\bar{\alpha}$. This quantity is defined as

$$\bar{\alpha} = \sum_n \alpha_n \quad (4)$$

with

$$\alpha_n = \frac{N_eN_iv_i}{I_0}\int d^3k(n\hbar\omega + \hbar\vec{k}\cdot\vec{v}_i)(d\sigma_{nk}^a - d\sigma_{nk}^e) \quad (5)$$

where N_e, N_i and I_0 are, respectively, the concentration of the incoming beam and of the scattering centers, and the intensity of the laser field. The term nhw + $\hbar\vec{k}\cdot\vec{v}_i$ corresponds to the final energy of the electrons so that $\bar{\alpha}I_0$ r dW/dt is the total energy decay rate. From equations (3), (4) and (5) we obtain

$$\bar{\alpha} = \sum_n \frac{4Z^2e^4N_eN_i}{\hbar^2I_0}\int\frac{d^3k}{k^4}|R(k)|^2 \left(n\hbar\omega + \hbar\vec{k}\cdot\vec{v}_i\right)J_n^2\left(\frac{\vec{k}\cdot\vec{v}_0}{\omega}\right) \left\{\delta\left(\vec{k}\cdot\vec{v}_i - n\omega + \frac{\hbar k^2}{2m}\right) - \delta\left(\vec{k}\cdot\vec{v}_i - n\omega - \frac{\hbar k^2}{2m}\right)\right\}. \quad (6)$$

When the electric field is parallel to $\vec{v}_i$ we can perform the integration over the total solid angle and get

$$\bar{\alpha}(v_i, v_0, \hbar) = \sum_n \frac{8\pi Z^2e^4N_iN_e}{\hbar v_iI_0}\int\frac{dk}{k^3}|R(k)|^2 \left\{\left[2n\omega + \frac{\hbar k^2}{2m}\right]J_n^2\left[\frac{v_0}{v_i}\left(\frac{n\omega + \hbar k^2/2m}{\omega}\right)\right] - \left[2n\omega - \frac{\hbar k^2}{2m}\right]J_n^2\left[\frac{v_0}{v_i}\left(\frac{n\omega - \hbar k^2/2m}{\omega}\right)\right]\right\}. \quad (7)$$

Finally, taking the limit $h \rightarrow 0$ and considering that $J_n^2(nv_0/v_i) = J_{-n}^2(-nv_0/v_i)$, we obtain

$$\bar{\alpha}(v_i, v_0) = \sum_{n=1}^{\infty}\frac{8\pi Z^2e^4N_iN_e}{mv_iI_0}\int\frac{dk}{k}|R(k)|^2 J_n^2\left(n\frac{v_0}{v_i}\right). \quad (8)$$

For $v_0/v_i \ll 1$, laser intensity is small so we consider only the single photon process, that is, n = 1. Naming $\bar{\alpha}_1$ the absorption coefficient for a single photon process, we have

$$\bar{\alpha}_1 = \frac{2\mathrm{x}Z^2e^4N_iN_e}{mv_iI_0}\left(\frac{v_0}{v_i}\right)^2 \ln\Lambda \quad (9)$$

where ln A is the generalized Coulomb logarithm. This expression is always positive for any value of v_0. We now normalize the multi-photon absorption coefficient in units of its weak-field value $\bar{\alpha}_1$

$$\frac{\bar{\alpha}}{\bar{\alpha}_1} = \frac{4}{(v_0/v_i)^2}\sum_{n=1}^{n_{max}} J_n^2\left(n\frac{v_0}{v_i}\right), \quad (10)$$

where $n_{max} = mv_0^2/\hbar\omega$. We note that the number of terms in the sum is limited to a maximum number of photons, n_{max}, which comes naturally from the integration over the k value which is bounded by a k_{max} that is related to n_{max}.

For $v_0/v_i \gg 1$ we can take for the Bessel function, $J_n(nv_0/v_i)$ the asymptotic approximation for large arguments, with fixed order n. Such approximation gives $J_n^2(nv_0/v_i) \doteq (\pi nv_0/v_i)^{-1}$, therefore

$$\frac{\bar{\alpha}}{\bar{\alpha}_1} = \frac{1}{\pi}\frac{4}{(v_0/v_i)^3}\sum_{n=1}^{n_{max}}\frac{1}{n} \approx \frac{4}{\pi}\frac{1}{(v_0/v_i)^3}\ln(n_{max}). \quad (11)$$

In this form we have recovered an expression which is similar to the Silin's expression[10], which contains a product of two logarithms and $\alpha \sim v_0 \sim I_0^{-3/2}$.

For $v_0/v_i \approx 1$, $\bar{\alpha}$ can become very large (see Section III) if the number of photons becomes large. With this range of parameters and for n large we can approximate the Bessel function $J_n(n)$ as $an^{-1/3}$, where a = 0.4473. With this approximation we obtain

$$\frac{\bar{\alpha}}{\bar{\alpha}_1} = \frac{12a^2}{(v_0/v_i)^2}n^{1/3}max. \quad (12)$$

An important point in this paper is that we do not calculate $\bar{\alpha}$ on the basis of the total cross-section $\sigma_{Tn} = \sigma_n^a - \sigma_n^e$, as done by Bivona et al.[8]. However, if we take this approach and go on calculating $\bar{\alpha}$, we obtain the diverging absorption coefficient given as

$$\bar{\alpha_d} = \sum_{n=1}^{\infty}\frac{N_iN_e\hbar\omega v_i}{I_0}n\left(\sigma_n^a - \sigma_n^e\right), \quad (13)$$

where n has no imposed upper bound, from the mathematical point of view. For parallel geometry and small scattering angles and taking the peaking approximation we obtain

$$\tilde{\alpha}_d = \frac{32\pi Z^2 e^4 N_i N_e}{v_i I_0} \sum_{n=1}^{\infty} J_n^2\left(n\frac{v_0}{v_i}\right). \qquad (14)$$

This expression can also be obtained from ref. 5. However such formalism has two problems: i) the Coulomb logarithm is lost and ii) this expression is divergent when n $\rightarrow \infty$ and $v_0 \rightarrow v_i$ [11]. Another way to obtain expression (14) is to take σ_T given as equation (3.6) of Daniele et al.[11], or equation (19) of reference 12. For absorption process we make $\bar{\alpha} = (\omega_p^2/\omega^2)\nu/c$, where $\nu = N_i \sigma_T v_i$ is the collision frequency and w_p is the plasma frequency. Here also, $\ln A$ do not appear and σ_T can be divergent when n $\rightarrow \infty$ and v_0 = vi. All these problems reflect the limitation of the cross-section treatment. What we did is to avoid taking the total cross section to calculate $\bar{\alpha}$ but, instead, we have taken α_n as given in (5). This resulted in expressions (6) through (8) which are more physically justifiable than expression (14).

III. Numerical Calculation

The main purpose of the present calculation is to discuss the multiphoton exchanges, responsible for the resulting value of $\bar{\alpha}$ and σ_T for a monoenergetic beam of electrons scattered by a static potential in the presence of a strong laser field. To compare our results with those of the references 8, 11, 12 we take $V(r) = (Ze^2/r)e^{-r/r_o}$ with $r_0 \gg a_0$, a_0 being the Bohr radius. This approximation gives $R(k) \simeq 1$. Since $\ln\Lambda = \int k^{-1}dk$, k_{max} and k_{min} are related to n_{max} and n_{min} respectively. Then, $A = n_{max}$ if we take $n_{min} = 1$. Also when $v_0 \rightarrow$ v, $k_{max} \rightarrow mv_0/\hbar$, then $n_{max} \rightarrow mv_0^2/\hbar\omega$. In order to prevent the divergence of σ_T, from equation (8), we take $\sigma_T = \sigma_0 + \sigma_0 F$, where $F = \sum_{n=1}^{n_{max}} (1 - \ln n/\ln n_{max})\, J_n^2(nv_0/v_i)$ is a normalized factor which takes into account the presence of the laser field. Figure 1 shows F as a function of v_0/v_i. When $v_0 \approx v_i$ the usual sum rule for multiphoton transitions breaks down and a cross-section larger than that of the field-free case is obtained. Thus, the multiphoton free-free transitions does not hold, contrary to expectations, at small scattering angles. Compared with the field-free results, significant enhancement occurs in the total cross-section in the presence of a laser taken as a single-mode homogeneous field in the dipole approximation. This effect increases with $A = n_{max}$. We have calculated F up to n = 10^4 photons. At low photon processes (n < 10) F has its maximum at $v_0 > v_i$ and it shifts towards $v_0/v_i = 1$ as the intensity increases (or frequency is lowered). Since $F(\Lambda, v_0/v_i)$ is a normalized curve, it allows us to easily obtain a complete or partial cross section. If $hw = 1\ eV$ and the incident particle energy is 100 eV, then $F(100, v_0/v_i)$, $F(10, v_0/v_i)$ represent the total (or partial) contribution over the multiphoton exchanges respectively.

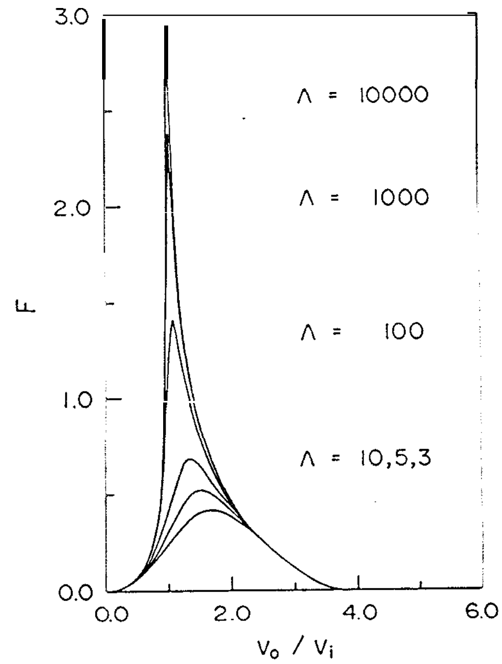

Figure 1: Total difference cross section [F = $(\sigma_T - \sigma_0)/\sigma_0$, in σ_0 units] versus v_0/v_i for six values of the incident particle energy [$A = n_{max} = (mv_i^2/\hbar\omega)_{v_i \rightarrow v_0}$ in units of $\hbar\omega$]. The oscillatory velocity v_0 is parallel to the incoming particle velocity vi. The range of the potential is $r_0 \gg a_0$.

To compare figure 1 of reference 12 with our results, we take F_{max} for different values of E_i and obtain the results given on table I.

Table I - Total cross section, σ_T, as function of the incident particle energy.

E_i [eV]	v_0/v_i	Λ	F_{max}	I_0 [W/cm^2]	σ_T/σ_0
100	1.40	10	0.70	4.50 10^{14}	1.70
250	1.10	25	0.97	9.42 10^{14}	1.97
500	1.00	50	1.19	16.10 10^{14}	2.19

We can see on table I that our results agree, in essence, with the final results of ref. 12, that is, as σ_T/σ_0 becomes larger with increasing incident particle energy (i.e., strengthening the inequality $E_i \gg \hbar\omega_0$). Our expression $\sigma_T = \sigma_0(1 + F)$, which is convergent when $v_0 \rightarrow$ vi, may be compared with equation (3.10) of ref. 11. Clearly, it diverges at $v_0/v_i = 1$ as on appendix of ref. 11.

This rapid increase of $\sigma_T = \Sigma\sigma_n$ can also be shown numerically. It is well known that $J_n^2(x)$ has its first maximum at $x \doteq n$ (the departure of its maximum from the $x = n$ becomes larger as n becomes smaller). Fig. 2 shows x_{max}, the argument of the first maximum of $J_n^2(x)$, versus n. The initial slope of the curve is not equal to one but as n increases, such as $n \geq 100$, it approaches one and it can be represented by $x_{max} = n^q$, $q = 1$. If we take the usual approximation, $J_n^2 \sim 1/\pi n$, we find that σ_T is proportional to $\int n^{-q} dn$, which gives a logarithm divergence. But at $x \doteq n$, where the peaking approximation is allowed, we have $J_n^2(n) \sim a^2/n^{2/3}$ which is also confirmed in the numerical calculation shown in Figure 3. In this case (for $n \gg 1$) we have $\sigma_T = C\sum_1^\infty J_n^2(x_{max}) = C_1 \int_1^\infty n^{-2/3} dn \sim C_1 n^{1/3}$ which yields a fast increase of σ_T when v_0 approaches v_i and $n \to \infty$.

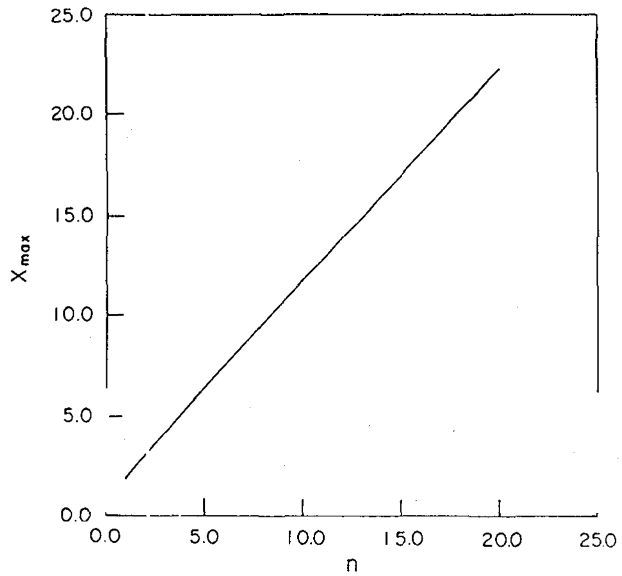

Figure 2: Arguments of the square of the Bessel function against the number of exchanged photons n which gives the maximum of $J_n^2(x)$ at $v_0 \simeq v_i$ for n moderately large ($1 < n < 25$).

Up to now, experiments have reached values of $E_i < 1\ keV$, $\hbar\omega \geq 1\ eV$ and $I_0 \leq 10^{16}\ W/cm^2$. For these experimental parameters the number of exchanged photons (n) is only moderately large so equation (3.10) of ref. 11 can be used to estimate σ_T/σ_0. If, however, n is large ($n > 10^5$) [for $E_i > 1keV$, $hw < 1eV$ and $I > 10^{16}$ W/cm^2], where it is likely that computational limitations appear, the problem of divergence discussed above must be taken into account.

If the geometry chosen is $\vec{v}_0 \perp \vec{v}_i$, the usual sum rules are recovered ($F = 0$)[13,14]. The numerical calculation of short-range potential ($r_0 \sim a_0$, not reported here) will be discussed in a future paper.

Similar considerations can be made for the absorption coefficient. Fig. 4 shows $\bar{\alpha}/\alpha_1$ versus v_0/v_i, for n up to $A = n_{max} = 10^4$. In the region $v_0/v_i > 1$,

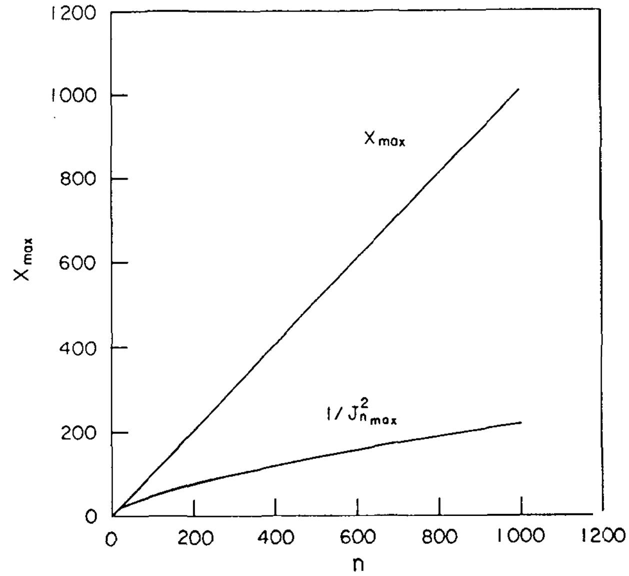

Figure 3: As in figure 2 except that $100 < n < 1200$ and the function $1/J_n^2(x)$ versus n for maximum multiphoton exchanges at $v_0 \simeq v_i$ for n large.

the results of ref. 8 are confirmed ($6 \sim I_0^{-3/2}$). For $v_0/v_i \ll 1$ $\bar{\alpha}/\alpha_1 \to 1$; as v_0/v_i passes over the value 1 the absorption coefficient reaches its maximum; for values of v_0/v_i greater than the latter value $\bar{\alpha}$ decreases monotonically, except for a periodic peak which will be explored later. For any value of v_0/v_i $\bar{\alpha}$ is always positive. This is so because we have defined $\bar{\alpha}$ as the overall absorption coefficient of the system (electrons + laser + ions). Reports of $\bar{\alpha} < 0$ given by many authors correspond to the energy loss of the electron beams obtained within the framework of the kinetic theory[15,16,17].

In this sense, $a < 0$ (for A large) would correspond to Figure 2 of ref. 5, which is obtained from the classical treatment. However, this agreement is only partial. At $v_0 = v_i$ the classical treatment has a significant flaw, because the instantaneous collision assumption is violated[8], and $\bar{\alpha}$ would be divergent. It is easy to show this by just making $v_0/v_i = 1$ and $\omega t = \frac{3}{2}\pi$ in the expression (2.29) of ref. 5. Also the classical result does not show the shift of the maximum of $\bar{\alpha}(n_{max}, v_0/v_i)$ towards $v_0/v_i = 1$ when $n_{max} = A$ increases. The quantum mechanical correspondence of the classical result[5] would be equation (14) with $n_{max} \to \infty$.

Table II gives the values of $\bar{\alpha}(n_{max}, 1)$ (equation 8) and the numerically calculated values of expression (14) of the text.

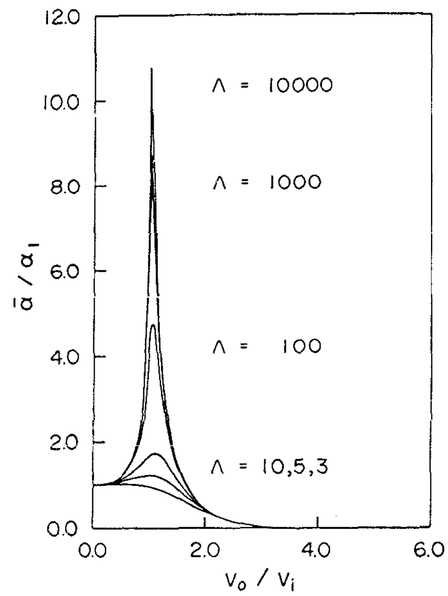

Figure 4: Absorption coefficient in units of α_1 versus v_o/v_i, $r_0 \gg$ ao, for six values of A = n_{max} (3, 5, 10, 100, 1000, 10000). The laser field is linearly polarized along the direction of the incoming electron velocity.

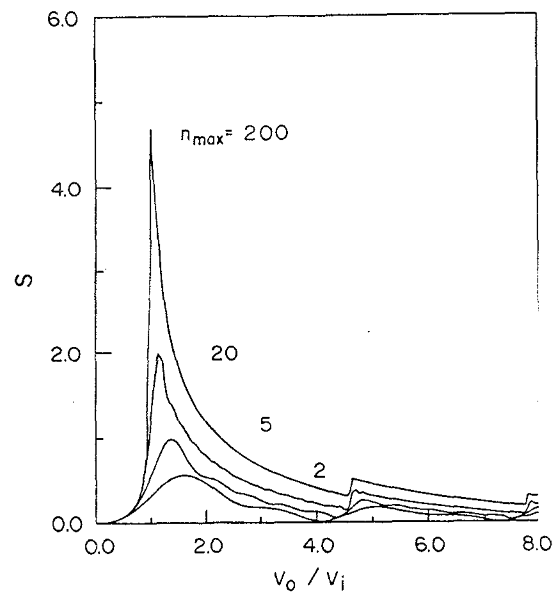

Figure 5: Plots of $S = \sum_{n=1}^{n_{max}} J_n^2(nv_0/v_i)$ versus (v_0/v_i) having as parameter n_{max}. As n_{max} increases S exibits saw-tooth like oscillations.

Table II - Absorption coefiicients as function of n_{max}

n_{max}	$\bar{\alpha}_c/\alpha_1$ [equation (8)]	$\bar{\alpha}_d/\alpha_1$ [equation (14)]
5	1.2	1.9
10	1.7	2.8
100	4.7	6.0
1000	8.4	11.0
10^4	10.8	24.9

For n > 1000 equation (14) begins to overestimate $\bar{\alpha}$, so it is necessary to introduce some saturation mechanism for $\bar{a}$. Clearly the field inhomogeneity introduced by Bivona et al[8] reduces $\bar{\alpha}_d/\alpha_1$ and may be compared with $\bar{\alpha}_c/\alpha_1$.

Finally we note that at the high-intensity range $(v_0/v_i > 1)$, the calculation of the partial a and α exhibits oscillations with maxima at $v_0/v_i \sim 4.65$ and $v_0/v_i \sim 7.80$ (Fig. 5). Bivona et al[8] have mentioned these oscillations. At higher values of n_{max} , the oscillations resemble saw-teeth. Here we confirm that both the cross-section and the absorption coefficients exhibit this saw-tooth like oscillations which are lost in the classical instantaneous approximation[5].

IV. Conclusion

We have studied in detail the processes which occur in the scattering of a monoenergetic electron beam. The analysis has been carried out for a geometry in which the laser electric field is parallel to the incoming electron velocity. Contrary to the results of Bivona et al[8] we find that $\bar{\alpha}$ is always positive, so this situation could not be a reminiscence of a well-known process occurring in plasma physics, namely the two stream instability[17].

The difference between our results and those of other authors[8], is due to the form by which α_n is calculated. With expression (21) of ref. 8 the Coulomb logarithm is lost and we get $\bar{\alpha} < 0$ if $v_0/v_i < 1$. Equations (4) and (5) of our paper, obtained within the framework of the kinetic theory[17,18,19], give $\bar{\alpha} > 0$ for all values of vi.

Also, in the instantaneous approximation and within the cross-section treatment, Ehlotzky[20] finds that $\bar{\alpha}(v_0/v_i < 1)$ is negative if $\vec{v}_0 \perp \vec{v}_i$ while Bivona et al. show that $\bar{\alpha}(v_0/v_i < 1) < 0$, if $\vec{v}_0 \parallel \vec{v}_i$. All these points reflect the limitations of the cross section treatment and it seems adequate to consider only the magnitude of $\alpha = |\alpha|$.

Acknowledgements

The authors would like to thank to Professor Abdus Salam, the IAEA and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste, Italy, (1991). This work was also supported by Fundação de Amparo à Pesquisa do Estado de São Paulo, FAPESP, and Conselho Nacional de Desenvolvimento Científico e Tecnológico, CNPq (Brazil).

References

1. H. Torres-Silva and P.H. Sakanaka, "Charged Plasma Particle Collision in the Presence of a

Strong Laser Field", Spring College on Plasma Physics, Trieste (1985).
2. H. Torres-Silva and P. H. Sakanaka, to be published in the J. Phys. Soc. of Japan, (1992).
3. H. Torres-Silva, L. C. Braga and P. H. Sakanaka, "Multiphoton Effects in Electron-Ion Scattering: a limitation of the cross-section treatment", Internal Publication, IC/91/171, ICTP, Trieste (1991).
4. H. M. Kroll and K. M. Watson, Phys. Rev. A8 804 (1973).
5. F. B. Bunkin, A. E. Kazakov, and M. V. Federov, Sov. Phys. Usp. 15, 1416 (1973).
6. F. Ehlotzky, Opt. Commun. 66, 265 (1988).
7. P. Krstic and D. Miloševic, Phys. Rev. A39, 1783 (1989).
8. S. Bivona, R. Zangara, and G. Ferrante, Nuovo Cimento, 7D, 113 (1986), Phys. Lett. **A110**, 375 (1985).
9. J. Seely and E. G. Harris, Phys. Rev. A7, 1064 (1973).
10. V. P. Silin, S. A. Uryupin, Sov. Phys. JETP 54, 485 (1981).
11. R. Daniele, F. Trombetta, G. Ferrante, C. Cavaliere and F. Morales, Phys. Rev. A36, 1156 (1987).
12. R. Daniele, G. Ferrante, F. Morales, and F. Trombetta, J. Phys. B19, L133 (1986).
13. N. Kroll, K. Watson, Phys. Rev. **A8**, 804 (1973).
14. H. Kruger and C. Jung, Phys. Rev. **A17**, 1706 (1978).
15. H. Torres Silva and P. H. Sakanaka "Laser Field Effects on the Transport Phenomena: Energy Loss and Stopping Power", Proc. of the IV Latin-American Workshop on Plasma Physics, Buenos Aires, (1990), pg. 207.
16. H. Torres-Silva, Ph.D. Thesis, Universidade Estadual de Campinas (1990).
17. H. Torres-Silva and P. H. Sakanaka, J. Phys. Soc. (Japan), 60, 2985, (1991).
18. R. Jones and K. Lee, Phys. Fluids 25, 2307 (1982).
19. S. A. Sapogov and V. P. Seminozhenko, Solid State Comumm. 41, 399 (1982).
20. F. Ehlotzky, Can. J. Phys. 63, 907 (1985).
21. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (National Bureau of Standards, Washington 1968).