Harmonic Oscillator with Time-Dependent Mass and Frequency

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A general treatment of the quanta1 harmonic oscillator with time-dependent mass and frequency is presented. The treatment is based on the use of some time-dependent transformatiois and in the method of invariants of Lewis and Riesenfeld. Exact coherent states for such a system are also constructed.

I. Introduction

The study of problems involving harmonic oscillators with time-dependent frequencies or with time-dependent masses (or both simultaneously) has attracted considerable interest in the last few years[1-17]. Apart from their intrinsic mathematical interest, these problems have invoked much attention because of their connection with many other problemas belonging to different areas of physics like plasma physics, gravitation, quantum optics etc. For example, Colegrave and Abdalla[18] studied the harmonic oscillator with a constant frequency and a time-dependent mass in order to describe the electromagnetic field intensities in a Fabry-Pérot cavity. Also, Lemos and Natividade[19] studied a harmonic oscillator with a time-dependent frequency and a constant mass in an expanding universe.

In this paper we are mainly concerned with the harmonic ascillator with both frequency and mass being arbitrary given functions of time. Our main purpase is to exhibit, in a simple way, an alternative treatment for such a system. The treatment is based on the use of some time-dependent transformations and on the method of invariants of Lewis and Riesenfeld[20]. Furthermore, we also construct exact coherent states for the harmonic oscillator with time-dependent mass and frequency.

A brief outline of the present paper is as follows. In Sec. II, we outline our treatment. In Sec. III, we construct exact coherent states. In Sec. IV, we discuss the uncertainty relatiori. Finally, some concluding remarks are added in Sec. V.

II. The Treatment

A. Time-dependent harmonic oscillator

We start with the time-dependent harmonic oscillator Hamiltonian

$$H(t)=\frac{p^2}{2M(t)}+\frac{1}{2}M(t)w^2(t)q^2 \qquad (2.1)$$

where q and p are canonically conjugate and $M(t)$ and $w(t)$ are, respectively, the mass and frequency associated to the oscillator which are arbitrary real functions of time. The variables q and p satisfy the canonical commutation relation

$$[q,p]=i\hbar \qquad (2.2)$$

The canonical equations of motion are

$$\dot{q}=\frac{1}{i\hbar}[q,H(t)]=\frac{p}{M(t)}, \qquad (2.3a)$$

$$\dot{p}=\frac{1}{i\hbar}[p,H(t)]=-M(t)w^2(t)q, \qquad (2.3b)$$

which when combined yield the equation

$$\ddot{q}+\gamma(t)\dot{q}+w^2(t)q=0\ , \qquad (2.4)$$

where

$$\gamma(t)=\frac{d}{dt}[lnM(t)]\ . \qquad (2.5)$$

Next we consider the time-dependent canonical transformation given by the generating function[21]

$$F(q,P,t)=\frac{1}{2}(qP+Pq)\left(\frac{m}{M(t)}\right)^{-1/2}-\frac{M(t)\gamma(t)}{4}q^2, \qquad (2.6)$$

where m is a constant mass. The transformation equations are Q = $\partial F/\partial P$, p = $\partial F/\partial q$, from which we obtain the new canonical variables

$$Q=\left(\frac{m}{M(t)}\right)^{-1/2}q \qquad (2.7.a)$$

$$P=\left(\frac{m}{M(t)}\right)^{1/2}p+(mM(t))^{1/2}\frac{\gamma(t)}{2}q \qquad (2.7.b)$$

Here we remark that the well-know Kanai-Caldirola Hamiltonian[22] is recovered when $M(t)$ = m $\exp(\gamma t)$

with constant y. Also, note that $[Q, P] = [q, p]$ which implies that the commutation relations remain the same in both coordinates. Then, under this transformation the Hamiltonian (2.1) is transformed in a new Hamiltonian $H_1(t) = H(t) + \partial F/\partial t$ which, in terms of the new variables, is expressed as

$$H_1(t) = \frac{P^2}{2m} + \frac{m\Omega^2(t)}{2}Q^2 , \tag{2.8}$$

where

$$\Omega^2(t) = w^2(t) - \left(\frac{\gamma^2(t)}{4} + \frac{\dot{\gamma}(t)}{2}\right), \tag{2.9}$$

is the modified time-dependent frequency. Note that the Hamiltonian (2.8) is of the form of that considered by Lewis and Riesenfeld[20]. Here, let us recall that Lewis and Riesenfeld have developed a general theory of explicity time-dependent invariants for quantum systems characterized by explicitly time-dependent Hamiltonians. They have derived a simple relation between eigenstates of such an invariant and solutions of the corresponding Schrodinger equation and have aplied it to the case of a harmonic oscillator with time-dependent frequency. In the next subsection we briefly review the theory of Lewis and Riesenfeld for the system characterized by the transformed Hamiltonian (2.8).

B. Time-dependent invariants and Schrodinger equation

Let us now consider the Hamiltonian (2.8). It is well-known that an exact invariant for (2.8) is given by[1,21]

$$I(t) = \frac{m}{2}\left[\left(\frac{Q}{\rho}\right)^2 + (\rho\dot{Q} - \dot{\rho}Q)^2\right], \tag{2.10}$$

which can be rewritten as

$$I(t) = \frac{1}{2m}\left[m^2Q^2\rho^{-2} + (P\rho - m\dot{\rho}Q)^2\right], \tag{2.11}$$

where $Q(t)$ satisfies the equation

$$\ddot{Q} + \Omega^2(t)Q = 0 \tag{2.12}$$

and $\rho(t)$ is a c-number quantity satisfying the auxiliary equation

$$\ddot{\rho} + \Omega^2(t)\rho = 1/\rho^3 . \tag{2.13}$$

The equations (2.12) and (2.13) together are know as Ermakov pairs. The invariant (2.11) was first derived by Ermakov[23] by eliminating $R^2(t)$ between these two equations. It is clear that $I(t)$ satisfies the equation[20,24]

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + \frac{1}{i\hbar}[I, H_1] = 0 \tag{2.14}$$

and $I^+ = I$. In order to make $I(t)$ Hermitian, we choose only the real solutions of (2.13). Further, the eigenfunctions $\phi_n(Q,t)$ of $I(t)$ are assumed to form a complete orthonormal set corresponding to the time-independent eigenvalues λ_n. Thus

$$I\phi_n(Q,t) = \lambda_n\phi_n(Q,t) , \tag{2.15}$$

$$(\phi_{n'}, \phi_n) = \delta_{n',n}. \tag{2.16}$$

Now consider the time-dependent Schrodinger equations

$$i\hbar\frac{\partial\Psi}{\partial t} = H_1(t)\Psi, \tag{2.17}$$

with

$$H_1(t) = -\frac{\hbar}{2m}\frac{\partial^2}{\partial Q^2} + \frac{m\Omega^2(t)}{2}Q^2 \tag{2.18}$$

where $P = -i\hbar\partial/\partial Q$ has been used. The solutions $\Psi_n(Q,t)$ of the Schrodinger equation (2.17) are related to $\phi_n(Q,t)$ by the relation[20]

$$\Psi_n(Q,t) = e^{i\alpha_n(t)}\phi_n(Q,t), \tag{2.19}$$

where the phase functions $\alpha_n(t)$ satisfy the equation

$$\hbar\frac{d\alpha_n(t)}{dt} = \left\langle\phi_n\left|i\hbar\frac{\partial}{\partial t} - H_1(t)\right|\phi_n\right\rangle. \tag{2.20}$$

Then, since each of Ψ_n satisfies the Schrodinger equation, the general solution of (2.17) may be written as

$$\Psi(Q,t) = \sum_n c_n e^{i\alpha_n(t)}\phi_n(Q,t), \tag{2.21}$$

where the c_n are time-independent coefficients.

C. Solution of the Schrodinger equation

In this subsection we are mainly interested in solving, by a particular direct method, the Schrodinger equation (2.17). To this end we proceed as follows. Consider the unitary transformation[25,26]

$$U = e^{-im\dot{\rho}Q^2/(2\hbar\rho)} \tag{2.22}$$

Under this unitary transformation the operator I changes into I' according to

$$I' = UIU^+, \tag{2.23}$$

where we find by straightforward calculation that

$$I' = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial\sigma^2} + \frac{m\sigma^2}{2} \tag{2.24}$$

with

$$\sigma = Q/\rho \tag{2.25}$$

Then, the eigenvalue equation (2.15) is mapped into

$$I'\phi'_n(\sigma) = \lambda_n\phi'_n(\sigma), \tag{2.26}$$

where

$$\phi'_n = \rho^{1/2}U\phi_n \tag{2.27}$$

The factor $\rho^{1/2}$ is introduced into (2.27) so that the normalization conditions

$$\int \phi_n^* \phi_n dQ = \int \phi_n'^* \phi_n' d\sigma = 1 \qquad (2.28)$$

hold. By using (2.24) the eigenvalue problem (2.26) becomes

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial\sigma^2} + \frac{m}{2}\sigma^2\right]\phi_n'(\sigma) = \lambda_n \phi_n'(\sigma) \qquad (2.29)$$

which is an ordinary one-dimensional Schrodinger equation whose solution is given by

$$\phi_n'(Q,t) := \left[\frac{m^{1/2}}{\pi^{1/2}\hbar^{1/2}n!2^n}\right]^{1/2} \times \exp\left(-\frac{m}{2\hbar}\left(\frac{Q}{\rho}\right)^2\right) H_n\left[\left(\frac{m}{\hbar}\right)^{1/2}\frac{Q}{\rho}\right], \qquad (2.30)$$

where

$$\lambda_n = \hbar\left(n+\frac{1}{2}\right)$$

and $H_n(Q)$ is the usual Hermite polynomial of order n. Thus, by using (2.22), (2.27) and (2.30) we find that

$$\phi_n(Q,t) = \left[\frac{m^{1/2}}{\pi^{1/2}\hbar^{1/2}n!2^n\rho}\right]^{1/2} \times \exp\left(\frac{im}{2\hbar}\left(\frac{\dot\rho}{\rho}+\frac{i}{\rho^2}\right)Q^2\right) \times H_n\left[\left(\frac{m}{\hbar}\right)^{1/2}\frac{Q}{\rho}\right]. \qquad (2.31)$$

Hence, the solutions $\Psi_n(Q,t)$ of the transformed Schrodinger equation (2.18) is given by

$$\Phi_n(Q,t) = e^{i\alpha_n(t)}\left[\frac{m^{1/2}}{\pi^{1/2}\hbar^{1/2}n!2^n\rho}\right]^{1/2} \times \exp\left(\frac{im}{2\hbar}\left(\frac{\dot\rho}{\rho}+\frac{i}{\rho^2}\right)Q^2\right) \times H_n\left[\left(\frac{m}{\hbar}\right)^{1/2}\frac{Q}{\rho}\right]. \qquad (2.32)$$

where the phase functions $\alpha_n(t)$ are given by[20,26]

$$\alpha_n(t) = -\left(n+\frac{1}{2}\right)\int_0^t \frac{dt'}{\rho^2(t')}. \qquad (2.33)$$

Here it is interesting to observe that the solutions (2.32) for the equation (2.17) have also been obtained by Khandekav and Lawande[27] by using Feynman path integral~.

Let us now introduce the time-dependent transformation[21]

$$\rho(t) = \left(\frac{m}{M(t)}\right)^{-1/2} x(t), \qquad (2.34)$$

where $x(t)$ is a real function of time which is to be determined. Then using (2.7), (2.9) and (2.34) the equation of motion (2.12) is converted into the original equation (2.4) and the auxiliary equation (2.13) into the equation

$$\ddot x + \gamma(t)\dot x + w^2(t)x = \left(\frac{m}{M(t)}\right)^2 / x^3 \qquad (2.35)$$

The exact invariant (2.11) is transformed to the form

$$I(t) = \frac{1}{2m}[m^2q^2x^{-2} + (px - M\dot x q)^2]. \qquad (2.36)$$

Thus, (2.36) is an exact invariant for the Hamiltonian (2.1) with p given by (2.3a) and $q(t)$ and $x(t)$ satisfying, respectively, the equations (2.4) and (2.35). For $M(t)$ = m = cte we recover the invariant for the time-dependent Hamiltonian where only the frequency is allowed to change with time[21]. Note that in this case the function (2.6) generates the identity transformation.

On the other hand, by using (2.7a) and (2.34) the unitary operator (2.22) is converted into

$$U = \exp\left(-iM(t)\left[\dot x + \frac{\gamma(t)x}{2}\right]q^2/(2\hbar x)\right) \qquad (2.37)$$

and the invariant (2.24) into the form

$$I' = -\frac{\hbar^2}{2m}x^2\frac{\partial^2}{\partial q^2} + \frac{m}{2}\left(\frac{q}{x}\right)^2, \qquad (2.38)$$

which has the same form of (2.24). Also, in terms of the original variables, the eigenfunctions $\phi_n(q,t)$ of $I(t)$ are given by

$$\phi_n(q,t) = \left[\frac{1}{2^n n!\hbar^{1/2}}\left(\frac{m^2}{\pi M(t)x^2}\right)^{1/2}\right]^{1/2} \times \exp\left[\frac{iM(t)}{2\hbar}\left(\frac{\dot x}{x}+\frac{\gamma(t)}{2}+\frac{im}{M(t)x^2}\right)q^2\right] \times H_n\left[\left(\frac{m}{\hbar x^2}\right)^{1/2}q\right]. \qquad (2.39)$$

Note that for $M(t) = me^{\gamma t}$ the above solution reduces to that obtained by Khandekav and Lawande[28]. Now, the solutions $\Psi_n(q,t)$ of the Schrodinger equation for the original system may be written as

$$\Psi_n(q,t) = e^{i\alpha_n(t)}\phi_n(q,t), \qquad (2.40)$$

where the phase functions $\alpha_n(t)$ are now given by

$$\alpha_n(t) = -\left(n+\frac{1}{2}\right)\int_0^t \frac{m}{M(t')x^2(t')}dt'. \qquad (2.41)$$

Here it is interesting to note that when $M(t) \to m$, $w(t) \to w_0$ = constant and $x(t) \to x_0$ = constant =

$1/w_0^{1/2}$ (which is a particular solution of the auxiliar equation (2.35)) the solutions (2.39) become

$$\phi_n(q) = \left[\frac{(mw_0)^2}{(\pi\hbar)^{1/2}2^n n!}\right]^{1/2} \exp\left[-\left(\frac{mw_0}{2\hbar}\right)q^2\right] \times H_n\left[\left(\frac{mw_0}{\hbar}\right)^{1/2} q\right], \qquad (2.42)$$

which is the solution of the Schrodinger equation for the time-dependent harmonic oscillator of mass m and frequency w_0. Further, in this case the phase functions (2.41) are given by

$$\alpha_n(t) = -\left(n+\frac{1}{2}\right) w_0 t, \qquad (2.43)$$

so that by substituting (2.42) and (2.43) into (2.40) we recover the time-dependent solutions of the Schrodinger equation for the usual time-independent oscillator.

III. Coherent States

To obtain coherent states for the harmonic oscillator with time-dependent mass and frequency we proceed as follows. Consider the operators A and $A^\dagger$ given by

$$A = \left(\frac{1}{2m\hbar}\right)^{1/2}\left[m\left(\frac{Q}{\rho}\right) + i\rho P\right], \quad (3.1.a)$$

$$A^\dagger = \left(\frac{1}{2m\hbar}\right)^{1/2}\left[m\left(\frac{Q}{\rho}\right) - i\rho P\right], \quad (3.1.b)$$

It may be easily verified that A and $A^\dagger$ satisfy the commutation relation

$$[A, A^\dagger] = AA^\dagger - A^\dagger A = 1. \qquad (3.2)$$

The invariant operator given by (2.24) can now be written as

$$I' = \hbar\left(AA^\dagger + \frac{1}{2}\right). \qquad (3.3)$$

Now, Hartley and Ray[29] have shown that coherent states for I' have the form

$$\phi'_\alpha(\sigma,t) = e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{(n!)^{1/2}} e^{i\alpha_n(t)}\phi'_n(\sigma), \qquad (3.4)$$

where $\alpha_n(t)$ is given by (2.33) and a is an arbitrary complex number. Note that when $w(t) \to w_0$ and $\rho(t) \to \rho_0 = 1/w_0^{1/2}$ the phases $\alpha_n(t)$ are given by (2.43). In this cases, the coherent states $\phi'_\alpha(Q,t)$ become the correct coherent states for the usual time-independent harmonic oscillator. The states $\phi'_\alpha(\sigma,t)$ are eigenstates of A with eigenvalue $\alpha(t)$, i.e.,

$$A\phi'_\alpha(\sigma,t) = \alpha(t)\phi'_\alpha(\sigma,t), \qquad (3.5)$$

with $\alpha(t)$ given by[29-31]

$$\alpha(t) = \alpha e^{2i\alpha_0(t)}, \qquad (3.6)$$

where

$$\alpha_0(t) = -\frac{1}{2}\int_0^t \frac{dt'}{\rho^2(t')}. \qquad (3.7)$$

Note that when $w(t) \to w_0$ and $\rho(t) \to \rho_0 = 1/w_0^{1/2}$ the eigenvalue (3.6) becomes $\alpha(t) = a\exp(-iw_0 t)$ which is the usual result.

The coherent states for the time-dependent harmonic oscillator (2.18) are obtained by the inverse transformation on $\phi'_\alpha(\sigma,t)$. They are given by

$$\phi_\alpha(Q,t) = \rho^{-1/2} e^{im\dot\rho Q^2/(2\hbar\rho)}\phi'_\alpha(\sigma,t). \qquad (3.8)$$

Next, following Ray[26], we show that (3.8) are coherent states for the time-dependent oscillator (2.18). Transforming (3.5) via the inverse transformation we find that

$$a\phi_\alpha(Q,t) = \alpha(t)\phi_\alpha(Q,t), \qquad (3.9)$$

where

$$a = U^\dagger A U \qquad (3.10)$$

A straightforward calculation reveals that

$$a = \left(\frac{1}{2m\hbar}\right)^{1/2}\left[m\left(\frac{Q}{\rho}\right) + i(\rho Q - m\dot\rho Q)\right], \qquad (3.11.a)$$

$$a^\dagger = \left(\frac{1}{2m\hbar}\right)^{1/2}\left[m\left(\frac{Q}{\rho}\right) - i(\rho Q - m\dot\rho Q)\right]. \qquad (3.11.b)$$

which are exactly the operators associated with the invariant (2.11) which was originally introduced by Lewis[24]. These operators factor the invariant (2.11) as

$$I = \hbar\left(a^\dagger a + \frac{1}{2}\right). \qquad (3.12)$$

This result may also be obtained by applying the inverse transformation on (3.3).

$$a = \left(\frac{1}{2m\hbar}\right)^{1/2}\left[m\left(\frac{q}{x}\right) + i(xp - M\dot x q)\right], \qquad (3.13.a)$$

$$a^\dagger = \left(\frac{1}{2m\hbar}\right)^{1/2}\left[m\left(\frac{q}{x}\right) - i(xp - M\dot x q)\right]. \qquad (3.13.b)$$

In terms of the original variables we also may express the coherent states (3.8) as

$$\phi_\alpha(q,t) = \left[\left(\frac{m}{M(t)}\right)^{1/2} x\right]^{-1/2} \times \exp\left(-iM(t)\left[\dot x + \frac{\gamma(t)}{2}x\right] q^2/(2hr)\right)\ \phi'_\alpha(\sigma,t) \qquad (3.14)$$

where we now have that $a = q/x$ (see eqs. (2.24), (2.25) and (2.38)). The states (3.14) are coherent states for the time-dependect system described by the Hamiltonian (2.1). These states satisfy the eigenvalue equation

$$a\phi_\alpha(q,t) = \alpha(t)\phi_\alpha(q,t) \tag{3.15}$$

where a and $\alpha(t)$ are given respectively, by (3.13a) and (3.6). In this case, the equation (3.7) becomes

$$\alpha_0(t) = -\frac{1}{2}\int_0^t \frac{m}{M(t')x^2(t')}dt'. \tag{3.16}$$

Note that when $M(t)$ = m = cte the states reduce to the coherent states of the time-dependent harmonic oscillator where only the frequency is allowed to change with time[29].

IV. The uncertainty relation

By using (3.13) we may express q and p as

$$q = x\left(\frac{\hbar}{2m}\right)^{1/2}(a^\dagger + a), \tag{4.1.a}$$

$$p = i\left(\frac{m\hbar}{2m}\right)^{1/2}\left[\left(\frac{1}{x} - i\frac{M}{m}\dot{x}\right)a^\dagger - \left(\frac{1}{x} + i\frac{M}{m}\dot{x}\right)a\right]. \tag{4.1.b}$$

From (3.15) and (4.1a) we find that the expectation value of q in the state $\phi_\alpha(q,t)$ is given by

$$< q >= \left[\frac{2\hbar|\alpha|^2x^2}{m}\right]\sin[\varphi(t)+\delta], \tag{4.2}$$

where 6 is the argument of the complex number a and

$$\varphi(t) = -2\alpha_0(t) = \int_0^t \frac{m}{M(t')x^2(t')}dt'. \tag{4.3}$$

We also get that the expectation value of the invariant (3.12) in the state $\phi_\alpha(q,t)$ is given by

$$< I >= \hbar\left(|\alpha|^2 + \frac{1}{2}\right). \tag{4.4}$$

On the other hand, it is known[17,27] that the solution to the equation of motion for the classical time-dependent harmonic oscillator

$$\ddot{q}_{cl} + \gamma(t)\dot{q}_{cl} + w^2(t)q_{cl} = 0 \tag{4.5}$$

can be expressed as

$$q_{cl} = B_0 x(t)\sin[\varphi(t)+\delta], \tag{4.6}$$

where B_0 is a constant and $x(t)$ satisfies the auxiliary equation (2.35). In this case, the invariant $I(t)$ is defined in the same way as (2.36) but using classical variables. Now in the Glauber limit[29,32] h → 0, $|\alpha| \to \infty$, such that $\hbar|\alpha|^2 \to$ finite. Thus, the expectation value of q in (4.2) is exactly the solution for the classical time-dependent oscillator with invariant (see (4.4) $\hbar|\alpha|^2 =< I > -\hbar/2$.

In what follows we wish to obtain the uncertainty relation. After some calculation we find that the uncertainties in q and p is the state $\phi_\alpha(q,t)$ are

$$(\Delta q)^2 = \frac{\hbar}{2m}x^2, \tag{4.7}$$

$$(\Delta p)^2 = \frac{m\hbar}{2}\left(\frac{1}{x^2} + \frac{M^2}{m^2}\dot{x}^2\right). \tag{4.8}$$

Thus, the uncertainty product is expressed as

$$(\Delta q)(\Delta p) = \frac{\hbar}{2}\left(1 + \frac{M^2(t)}{m^2}x^2\dot{x}^2\right)^{1/2}. \tag{4.9}$$

and, in general, does not attain the minimum value. However, for a time-dependent oscillator, we cannot expect to find strictly coherent states, i.e., $(\Delta q)(\Delta p) = \hbar/2$ for all time t. On the other hand, we have already shorn[30,31] that the states $\phi_\alpha(q,t)$ are equivalent to well-known squeezed states whose characteristic feature is the squeezing. Furthermore, the fundamental property of squeezing is that it is a time-dependent phenomenon[34]. Now for $M(t)$ = m = const. the uncertainty product (4.9) reduces to that obtained in Ref. 29. Also, when $M(t) = m$ and $x(t) \to x_0 = 1/w_0^{1/2}$ the uncertainty relation (4.9) attain its minimum value. In this case, the operators a and $a^\dagger$ given in (3.13) reduce, respectively, to the usual annihilation and creation operators and the states $\phi_\alpha(q,t)$ become the correct coherent states for the time-independent harmonic oscillator.

V. Concluding Remarks

In this paper we have presented an alternative treatment for the problem of the quantal harmonic oscillator with time-dependent mass and frequency. The present treatment is based on the use of a time-dependent canonical transformation, an unitary transformation, an auxiliary time-dependent transformation and in the method of invariants of Lewis and Riesenfeld. We also have used the procedure developed in Ref. 26 to construct coherent states for such a system. These coherent states have been expressed in terms of the eigenstates of the invariant $I(t)$ and are more general than those obtained in Ref. 29.

The treatment discussed here may also be applied to other time-dependent systems. As an example, we consider the nonharmonic system described by the Hamiltonian

$$H(t) = \frac{p^2}{2M(t)} + \frac{1}{2}M(t)w^2(t)q^2 + \frac{m^2}{M(t)x^2}g\left(\frac{q}{x}\right) \tag{5.1}$$

which possesses an invariant given by[21]

$$I(t) = \frac{1}{2m}[(px - M\dot{x}q)^2 + m^2q^2x^2 + 2mg(q/x)]. \tag{5.2}$$

where $x(t)$ satisfies the auxiliary equation *(2.35)*. Then, following the same steps as those of Sec. *2*, we convert the Hamiltonian *(5.1)* in the form

$$H_2(t) = \frac{P^2}{2m} + \frac{m\Omega^2(t)}{2}Q^2 + \frac{m}{\rho^2}g(Q/\rho) \tag{5.3}$$

where $\Omega(t)$ is given by *(2.9)* and $\rho(t)$ satisfies the equation *(2.13)*. The invariant *(5.2)* is converted in the form

$$I(t) = \frac{1}{2m}[(P\rho - m\dot{\rho}Q)^2 + m^2Q^2\rho^{-2} + 2mg(Q/\rho)]. \tag{5.4}$$

On the other hand, Hartley and Ray[25] have used the procedure employed in the subsection B and C to solve the Schrödinger equation for the transformed Hamiltonian *(5.3)*. Thus, we can use the transformations *(2.7)* and *(2.34)* in their results for obtaining the solutions for the originul system. i.e., the system described by the Hamiltonian *(5.1)*. Also, Khandekav and Lawande[27] have used Feynman path integrals and the method of Lewis and Riesenfeld to solve the quantum problem described by *(5.3)* for the case when $g = \rho^2/Q^2$. Furthermore, Ray[26] has constructed coherent states for time-dependent systems described by Hamiltonians of the form *(5.3)*. Then, it seems that would not be any problem to construct coherent states for the time-dependent systems associated with *(5.1)* using the same technique presented in this paper and that of Ref. *26*.

We should mention that Jannussis and Bartzis[33] have also constructed coherent states for the harmonic oscillator with time-dependent mass and frequency. However, the approach used by these authors is considerably different from that presented in this paper. We also mention that it may be interesting to compare our treatment with those developed by Leach[g], Abdalla[13], Collegrave and Abdalla[a] and Dodonov and Man'ko[35].

As a concluding remark we point out that in our treatment of the system described by the Hamiltonians *(2.1)* and *(2.8)* the requirements for $\Omega(t)$ are the same as those in the paper by Lewis and Riesenfeld[20], i.e., all that is necessary is that $\Omega^2(t)$ be real, either being positive or negative due to the hermicity of the Hamiltonian. Then, from *(2.5)* and *(2.9)* we see that $M(t)$ must be real. For further details about these considerations on $\Omega(t)$ and $M(t)$ see the discussion in Section *3* of Ref. *20*.

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