Revista Brasileira de Física, Vol. 21, nº 4, 1991

Low-energy theorems in non-Abelian Compton scattering on targets of arbitrary spin

Silvestre Ragusa
Departamento de Física e Ciência, Instituto de Física e Química de São Carlos, Universidade de São Paulo, Caixa Postal 369, São Carlos, 13560, SP, Brasil

Received January 7, 1991, in final form June 30, 1991

Abstract The low-energy theorems in non-Abelian Compton scattering previously obtained for spin-1/2 targets are extended to the case of targets of arbitrary spin.

1. Introduction

In a previous paper[1] we have obtained four new isospin-antisymmetric low-energy theorems in nucleon non-Abelian Compton scattering by using a second lemma for the Compton amplitude. In this note we shall use it to discuss the scattering on targets of arbitrary spin. Lin[2] has considered the case of normal Abelian Compton scattering. We shall then concentrate only in the isospin-antisymmetric part of the amplitude and show that seven new low-energy theorems can be obtained to second order in the frequency of the incoming photon. The basic equation is Eq.(4) of Ref.1, which reads

$$k'_m(E^{\alpha\beta}_{mn} - \omega'\ \Gamma^{\alpha\beta}_{mn}) = -k'_m\ U^{\alpha\beta}_{mn} + \omega'\ U^{\alpha\beta}_{on}$$
$$+ i\ (2\pi)^3 \left(\frac{EE'}{M^2}\right)^{1/2} \epsilon^{\alpha\beta\gamma} < \vec{p}\ '|J\gamma_n|\vec{p}\ ' > \ . \qquad (1)$$

In this equation $E^{\alpha\beta}_{mn}$ is the space-space component of the excited part of the amplitude tensor and $U^{\alpha\beta}_{\mu\nu}$ is the unexcited part. $\vec{k}'(\omega')$ is the momentum (energy) of the outgoing photon, $\vec{k},(w)$ will designate the corresponding quantities for the incoming photon and $\vec{p}, E(\vec{p}', E')$ are the initial (final) nucleon momentum and energy respectively. J_ν^γ is the electromagnetic current and $\Gamma_{mn}^{\alpha\beta}$ is a quantity which is odd under crossing in the Breit frame, where

$$\vec{p}\,' = -\vec{p} = -\frac{\vec{k}' - \vec{k}}{2} \quad E' = E \quad \omega' = \omega. \tag{2}$$

2. Expansion of $E_{mn}^{\alpha\beta}$ in the Breit frarne

The Breit frame is the frame where T-reversal invariance achieves its simplest form as used by Pais[3] to write down the minimal basics $B_{mn}^{(i)}$ in which $E_{mn}^{\alpha\beta}$ is to be expanded. To order ω^2 we have, for the isospin-antisymmetric part,

$$E_{mn}^{[\alpha\beta]} = [I^\alpha, I^\beta] \sum_i a_i B_{mn}^{(i)}$$

$$= [I^\alpha, I^\beta]\{\omega a_{1,1}\delta_{mn} + a_2(0) + a_{2,1}\vec{k}\vec{k}' + a_{2,2}\omega^2\epsilon_{mnj}S_j$$

$$+ \omega a_{3,1}\left(S_m S_n + S_n S_m - \frac{2}{3}\vec{S}^2\delta_{mn}\right)$$

$$+ a_4(0)[\delta_{mn}\vec{S}.(\vec{k}' \times \vec{k}) - \vec{k}.\vec{k}'\epsilon_{mnj}S_j]$$

$$+ a_5(0)[k_m(\vec{S} \times \vec{k})_n - k'_n(\vec{S} \times \vec{k}')_m]$$

$$+ a_6(0)[k'_m(\vec{S} \times \vec{k}')_n - k_n(\vec{S} \times \vec{k})_m - 2\omega^2\epsilon_{mnj}S_j]$$

$$+ a_7(0)[k_m(\vec{S} \times \vec{k}')_n - k'_n(\vec{S} \times \vec{k})_m - 2\vec{k}.\vec{k}'\epsilon_{mnj}S_j]$$

$$+ a_8(0)[k'_m(\vec{S} \times \vec{k})_n - k_n(\vec{S} \times \vec{k}')_m]$$

$$+ a_9(0)[< S_m, S_n, \vec{S}.(\vec{k}' \times \vec{k} > + < S_n, S_m, \vec{S}.(\vec{k}' \times \vec{k}) > -$$

$$- \frac{2}{5}(3\vec{S}^2 - 1)\vec{S}.(\vec{k}' \times \vec{k})\delta_{mn}]$$

$$+ a_{10}(0)\epsilon_{mnj}[< \vec{S}.\vec{k}', \vec{S}.\vec{k}', S_j > + < \vec{S}.\vec{k}, \vec{S}.\vec{k}, S_j >]$$

$$+ a_{11}(0)\epsilon_{mnj}[< \vec{S}.\vec{k}^t, \vec{S}.\vec{k}, S_j > + < \vec{S}.\vec{k}, \vec{S}.\vec{k}', S_j > -$$

$$- \frac{2}{5}(3\vec{S}^2 - 1)\vec{S}.(\vec{k}' \times \vec{k})\delta_{mn}]$$

$$+ a_{12}(0)[< S_m, (\vec{S} \times \vec{k})_n, \vec{S}.\vec{k} > - < S_{n'}(\vec{S} \times \vec{k}')_m, \vec{S}.\vec{k}' >]$$

$$+ a_{13}(0)[< S_m, \vec{S}.\vec{k}, (\vec{S} \times \vec{k})_n > - < S_n, \vec{S}.\vec{k}', (\vec{S} \times \vec{k}')_m >]$$

$$+ a_{14}(0)[< S_m, (S \times C)_n, \vec{S}.\vec{k}' > - < S_n, (\vec{S} \times L)_m, \vec{S}.\vec{k}$$

$$- (3\vec{S}^2 - 1)(B^{(7)}_{mn} + B^{(8)}_{mn})]$$

$$+ a_{15}(0)[< S_m, \vec{S}.\vec{k}', (\vec{S} \times \vec{k})_n > - < S_n, \vec{S}.\vec{k}, (\vec{S} \times \vec{k}')_m > -$$

$$- (3\vec{S}^2 - 1)(B^{(7)}_{mn} + B^{(8)}_{mn})\}, \tag{3}$$

where

$$< A, B, C >= ABC + CAB + BCA \tag{4}$$

$\vec{S}^2 = S(S+1)$, I^a is the ath component of isospin, and the expansion of the coefficients, to order w^2 , is an accordance with the even crossing symmetry property of $E^{\alpha\beta}_{mn}$. In Ref.4 the expansion in Eq.(12) has been done in the Lab frame and the last four basis elements $B^{12}_{mn} - B^{15}_{mn}$ are actually missing in that equation. However, the conclusion of that paper continues to be valid because they are based on a relation for $k'_m k_n E_{mn}$ which does not contain $B^{12}_{mn} - B^{15}_{mn}$, since these basis elements are orthogonal to $k'_m k_n$. One can show that the other possible parity-and time-reversal invariant basis elements obtained by interchanging $\vec{k}$ and $\vec{k}'$ in $B^{(12)}_{mn}$ to $B^{(15)}_{mn}$ are not linearly independent,as we discuss in the Appendix. For convenience we have used parity-and time-reversal-invariant amplitudes basis elements B^i_{mn} for i = 5-8 of which the corresponding basis elements $E^{(i)}_{\alpha}$ of Pais,[3] there numbered i = 8-11, are linear combinations, as given by Eqs.(A.7) to (A.lO) of the Appendix.

3. The Low-Energy Theorems

The usual procedure to obtain low-energy theorems uses a relation involving $k'_m k_n E^{\alpha\beta}_{mn}$ and then it cannot give information on the partia1 amplitudes for which the corresponding basis elements are orthogonal to $k'_m k_n$ that is, $k'_m k_n B^{(i)}_{mn} = 0$. This is the case for i = 4-8 and i = 12-15 in Eq.(3), which will now be accessible to us. We shall then concentrate only on this part of the amplitude.

As $\Gamma^{\alpha\beta}_{mn}$ is a three-tensor as is $E^{\alpha\beta}_{mn}$, it can be expanded in the same minimal basis. To get information about $E^{[\alpha\beta]}_{mn}$ to order w^2 we see from Eq.(1) that we need to consider $\Gamma^{[\alpha\beta]}_{mn}$ to order w. As $\Gamma^{\alpha\beta}_{mn}$ is odd under crossing we have, to order w ,

$$\Gamma^{\alpha\beta}_{mn} = [I^\alpha, I^\beta]\left\{b_1(0)\delta_{mn} + \omega b_{21}\epsilon_{mnj}S_j + b_3(0)\left(S_mS_n + S_nS_m - \frac{2}{3}\vec{S}^2\delta_{mn}\right)\right\} \quad (5)$$

Using Eqs. (3) and (5) and the identity (A.15) of the Appendix the isospin - antisymmetric part of the left-hand side of Eq.(3), which contains the terms that are of interest us (i = 4-8 and i = 12-15), become

$$\begin{aligned} k'_m(E^{[\alpha\beta]}_{mn} - \omega\Gamma^{[\alpha\beta]}_{mn}) \rightarrow\ & [I^\alpha, I^\beta]\{k'_n\vec{S}.(\vec{k}' \times \vec{k})[a_4(0) + a_7(0) \\ & \frac{2}{5}(3\vec{S}^2 - 1)(a_9(0) + a_{11}(0))] + (\vec{S} \times \vec{k})_n\vec{k}.\vec{k}'a_5(0) + \omega^2 a_8(0) \\ & + k_n\vec{S}.(\vec{k}' \times \vec{k})a_6(0) + \\ & [< \vec{S}.\vec{k}', S_n, \vec{S}.(\vec{k}' \times \vec{k}) > + < S_n, \vec{S}.\vec{k}', \vec{S}.(\vec{k}' \times \vec{k}) >][a_9(0) - \frac{1}{2}(a_{14}(0) + \\ & a_{15}(0))] + [< \vec{S}.\vec{k}', \vec{S}.\vec{k}', (\vec{S} \times \vec{k}')_n + < \vec{S}.\vec{k}, \vec{S}.\vec{k}, ((\vec{S} \times \vec{k}')_n >]a_{10}(0) \\ & + [< \vec{S}.\vec{k}', \vec{S}.\vec{k}, (\vec{S} \times \vec{k}')_n > + < \vec{S}.\vec{k}, \vec{S}.\vec{k}', (\vec{S} \times \vec{k}')_n >][a_{11}(0) + \frac{1}{2}(a_{14}(0) + \\ & a_{15}(0))] + [< \vec{S}.\vec{k}', (\vec{S} \times \vec{k})_n, \vec{S}.\vec{k} > a_{12}(0) + < \vec{S}.\vec{k}', \vec{S}.\vec{k}, (\vec{S} \times \vec{k})_n > a_{13}(0)\} \end{aligned} \quad (6)$$

All the coefficients that are present in this equation will be determined by the right-hand side of Eq.(1), giving rise to nine low-energy theorems. The coefficients $a_{10}(0)$ and the sum $a_9(0) + a_{11}(0)$ have been determined before[4] by considering a relation for $k'_m k_n E^{[\alpha\beta]}_{mn}$. The rest constitutes the new seven low-energy theorems and we shall concentrate only on them.

To establish the expression of the new lowenergy theorems we have to calculate the right-hand side of Eq.(1) in the Breit frame. For that purpose we need the expression of the current matrix element for arbitrary spin. This has been determined before[4] to second order in terms of the related form factors and multipole moments, which are the charge and magnetic moment mean square radii $< r^2 >^V$ and $< R^2 >^V$, respectively, the isovector magnetic dipole moment μ^V (in units of $1/2M$), the electric quadrupole moment Q^V (in units of $1/M^2$) and the magnetic octupole moment Ω^V (in units of $\sqrt{6}/2M^3$) , defined by Eqs. (17a-e) of Ref.4. Calculating the right-hand side of Eq.(1) and comparing with Eq.(6) we obtain as new isospin-antisymmetric low-energy theorems,

$$a_4(0) + a_7(0) = \frac{i}{8m^3}\left(\frac{3\mu^V}{2S} - 2\right) + \frac{i}{4M^3S(2S-1)}\left[\frac{(2S^2-1)\mu^V Q^V}{2S} - \frac{7(3\vec{S}^2-1)Q^V}{15(S-1)}\right] \quad , \qquad (7)$$

$$a_5(0) = \frac{1}{i}\left[\frac{\mu^V}{16M^3S}\left(\frac{\mu^V}{S} - 1\right) - \frac{< R^2 >}{5S} + \frac{(3S^2-1)\Omega^V}{30M^3S(S-1)(2S-1)}\right] \quad , \qquad (8)$$

$$a_6(0) = \frac{\mu^V}{16iM^3S}\left(\frac{\mu^V}{S} - 1\right) \quad , \qquad (9)$$

$$a_8(0) = \frac{< R^2 >^V}{10iS} + \frac{i\mu^V}{2MS}\left(\frac{< r^2 >^V}{6} + \frac{1}{8M^2}\right) + + \frac{i}{3M^3S(2S-1)}\left[\frac{(2\vec{S}^2-1)\mu^V Q^V}{4S} - \frac{(3\vec{S}^2-1)\Omega^V}{5(S-1)}\right] \quad , \qquad (10)$$

$$a_9(0) - \frac{1}{2}(a_{14}(0) + a_{15}(0)) = \frac{i}{24M^3S(2S-1)}\left(\frac{\Omega^V}{S-1} - \frac{\mu^V Q^V}{S}\right) \quad , \qquad (11)$$

and

$$a_{12}(0) = a_{13}(0) = \frac{i\Omega^V}{12M^3S(S-1)(2S-1)} \quad , \qquad (12)$$

where use has been made of the identities (A.17) and (A.18) of the Appendix.

For S = 1/2 it will be present only the results without the terms proportional to Q^V and Ω^V. It is easy to see that they agree with those already derived for spin-1/2 targets[1] by expressing $< r^2 >^V$ and $< R^2 >^V$ for S = 1/2 in terms of μ^V and the nucleon form factors.

Acknowledgement

The author wishes to thank the referee for useful suggestions that helped the presentation of the paper.

Appendix

By expanding $\vec{S}.[(\vec{S} \times \vec{k}) \times (\vec{p} \times \vec{q})]$ and $\vec{S}.(\vec{k} \times [\vec{q} \times (\vec{a} \times \vec{b}])$ in two different ways one easily establishes the identities

$$\vec{S}.\vec{k}\vec{S}.(\vec{p} \times \vec{q}) - \vec{S}^2\vec{k}.(\vec{p} \times \vec{q} = \vec{S}.\vec{p}\vec{S}.(\vec{k} \times \vec{q}) - \vec{S}.\vec{q}\vec{S}.(\vec{k} \times \vec{p}) \qquad (A.1)$$

and

$$\vec{S}.\vec{q}\vec{k}.(\vec{a} \times \vec{b}) - \vec{S}.(\vec{a} \times \vec{b})\vec{k}.\vec{q} = \vec{S}.(\vec{k} \times \vec{a})\vec{q}.\vec{b} - \vec{S}(\vec{k} \times \vec{b})\vec{q}.\vec{a} \ . \qquad (A.2)$$

Using

$$S_m S_n - S_n S_m = i\epsilon_{mnj} S_j \qquad (A.3)$$

we also have, from (A.1) ,

$$\vec{S}.(\vec{p} \times \vec{q})\vec{S}.\vec{k} - S^2\vec{k}.(\vec{p} \times \vec{q}) = \vec{S}.(\vec{k} \times \vec{q})\vec{S}.\vec{p} - \vec{S}.(\vec{k} \times \vec{p})\vec{S}.\vec{q} \ . \qquad (A.4)$$

a. Using these identities one easily shows that the other possible basis elements obtained by interchanging $\vec{k}$ and $\vec{k}'$ in $B_{mn}^{(i)}$ for i = 12 - 25 are not linearly independent basis elements. Calling them $A_{mn}^{(i)}$ we have for instance for i = 12 and with a convenient contraction with arbitrary vectors a, and b_n ,

$$a_m b_n A_{mn}^{(12)} =< \vec{S}.\vec{a}, \vec{S}.(\vec{k}' \times \vec{b}), \vec{S}.\vec{k}' > - < \vec{S}.\vec{b}, \vec{S}.(\vec{k} \times \vec{a}), \vec{S}.\vec{k} > \ . \qquad (A.5)$$

Using Eqs. (A.1), (A.3) and (A.4) the first term on the right-hand side of Eq.(A.5) can be written as

$$< \vec{S}.\vec{a}, \vec{S}.(\vec{k}' \times \vec{b}), \vec{S}.\vec{k}' > =< \vec{S}.\vec{b}, \vec{S}.(\vec{k}' \times \vec{a}), \vec{S}.\vec{k}' > + < \vec{S}.\vec{k}', \vec{S}(\vec{a} \times \vec{b}), \vec{S}.\vec{k}' >$$
$$- (3\vec{S}^2 - 1)\vec{S}.\vec{k}'\vec{k}'.(\vec{a} \times \vec{b})$$

A similar relation holds for the second term on the right-hand side of Eq.(A.5). Adding the two terms and making use of (A.2) one obtains

$$A_{mn}^{(12)} = B_{mn}^{(10)} - B_{mn}^{(12)} + (3\vec{S}^2 - 1)(B_{mn}^{(5)} + B_{mn}^{(6)}) \ . \qquad (A.6)$$

Similar considerations applies to the other cases, i = 13 - 15.

b. By contracting $E^{(8)}$ of Pais[6] with $a_m b_n$ for convenience and using Eq.(A.2) it is easy to see that

$$E^{(8)}_{mn} = -B^{(5)}_{mn} - B^{(6)}_{mn} \quad , \qquad (A.7)$$

Also

$$E^{(9)}_{mn} = -B^{(7)}_{mn} - B^{(9)}_{mn} \qquad (A.8)$$

As mentioned before[1] the plus ans minus signs in both equations for i = 10 and i = 11 of Pais should be interchanged otherwise they would not be independent basis elements. Correcting these misprints they are related to our elements by the equations

$$E^{(10)}_{mn} = B^{(5)}_{mn} - B^{(6)}_{mn} - (\omega^2 + \omega'^2)B^{(2)}_{mn} \qquad (A.9)$$

and

$$E^{(11)}_{mn} = B^{(7)}_{mn} - B^{(8)}_{mn} + 2\vec{k}.\vec{k}' B^{(2)}_{mn} \quad . \qquad (A.10)$$

c. Callnow

$$X_n = < \vec{S}.\vec{k}', \vec{S}.\vec{k}, (\vec{S} \times \vec{k}')_n > + < \vec{S}.\vec{k}, \vec{S}.\vec{k}', (\vec{S} \times \vec{k}')_n > \qquad (A.11)$$

and

$$Y_n = < S_n, \vec{S}.\vec{k}', \vec{S}.\vec{k}, \vec{S}.(\vec{k}' \times \vec{k}) > + < \vec{S}.\vec{k}', S_n, \vec{S}.(\vec{k}' \times \vec{k}) > \quad . \qquad (A.12)$$

Using Eqs.(A.1) and (A.3) it follows the relation

$$X_n - Y_n = 2 < \vec{S}.\vec{k}', \vec{S}.\vec{k}', (\vec{S} \times \vec{k})_n > -2(3S^2 - 1)\vec{S}.\vec{k}'(\vec{k}' \times \vec{k})_n \quad . \qquad (A.13)$$

From Eq.(A.2) we also have

$$\vec{S}.\vec{k}'(\vec{k}' \times \vec{k})_n = k'_m(B^{(8)}_{mn} + B^{(7)}_{mn}) \quad . \qquad (A.14)$$

Therefore,

$$< \vec{S}.\vec{k}', \vec{S}.\vec{k}', (\vec{S} \times \vec{k})_n > -(3\vec{S}^{\,2} - 1)\vec{k}'_m(B^{(8)}_{mn} + B^{(7)}_{mn} = \frac{1}{2}(X_n - Y_n) \quad . \qquad (A.15)$$

d. Using *(A.3)* we can write

$$2\vec{S}.\vec{k}'(\vec{S} \times \vec{k})_n \vec{S}.\vec{k}' = \{(\vec{S} \times \vec{k})_n, (\vec{S}.\vec{k}')^2\} + \vec{S}.(\vec{k}' \times \vec{k})\vec{k}'_n - \vec{k}.\vec{k}'(S \times \vec{k}')_n \ . \quad (A.16)$$

From here we have

$$< \vec{S}.\vec{k}', \vec{S}.\vec{k}', (\vec{S} \times \vec{k})_n > = \frac{3}{2}\{(\vec{S} \times \vec{k})_n, (\vec{S}.\vec{k}')^2\} + \frac{1}{2}(\vec{S}.(\vec{k}' \times \vec{k})k'_n - \vec{k}.\vec{k}'(\vec{S} \times R),) \ . \quad (A.17)$$

Using *(A.15)* in (A.17) we obtain

$$\{(S \times \vec{k})_n, (\vec{S}.\vec{k}')^2\} = \frac{1}{3}(X_n - Y_n) + (2\vec{S}^2 - \frac{2}{3})w^2(S \times \vec{k})_n$$
$$(2S^2 - 1)(\vec{S}.(\vec{k}' \times \vec{k})\vec{k}'_n - \vec{k}.\vec{k}'(\vec{S}. \times \vec{k}')_n) \quad (A.18)$$

References

1. *S*. Ragusa, Phys. Rev. D38, *2178 (1988).*
2. Lin, Phys. Rev. D24, *1014 (1983).*
3. *A*. Pais, Nuovo Cimento 53A, 433 (1968).
4. S. Ragusa, Phys. Rev. D8, 1190 *(1973).* In the third line of Eq.(21) one should read $q^2/16M^2$ intead of $q^2/16M^3$.

Resumo

Os teoremos de baixa energia em espalhamento Compton não-abeliano, previamente obtidos para alvos de spin 1/2, são estendidos para o caso de alvos com spin arbitrário.