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Quantization of dissipative velocity-power forces

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Abstract A method, recently proposed to quantize velocity-power forces $F = -\alpha\dot{x}^n$, $n \geq 1$ integer, $\alpha > 0$, is extended to quantize the force $F_d = -\alpha|\dot{x}|^{n-1}\dot{x}$ (dissipative for even and odd n) and its three-dimensional form $\mathbf{F}_{d3} = -\alpha v^{n-1}\mathbf{v}$. The Hamilton operator thus obtained is dissipative for all positive integer n, i.e. the probability to find the particle in the original one-particle state decreases in time when friction is switched on. The quantization of $\mathbf{F}_{d3}$ is – with analogous results – also discussed for the motion of a charged particle in a magnetic field, as an example where the canonical momentum $\mathbf{p}_c$ of the conservative system ($\alpha = 0$) is different from the mechanical momentum $m\mathbf{v}$ of the particle. Possible connections to the optical model and dissipative nuclear reactions are considered.

1. Introduction

The quantization of the motion of a (point-like) particle with mass m under the influence of a velocity-power force

$$F = -\alpha\dot{x}^n, \quad \alpha > 0, \quad n > 0 \text{ integer} \tag{1a}$$

and possibly other, conservative forces which - unlike F - have a scalar potential $V(x)$ is an old problem of quantum physics. It has been studied theoretically since about half a century ago, in particular for a linearly damped ($n = 1$) harmonic oscillator[1]. The principal method is canonical quantization, where one must find a Lagrange function of the classical equation of motion and then quantize the canonical momentum and the Hamilton function obtained from the Lagrangian. The problems are that a classical (normally not unique) Lagrange function for an equation of motion with the force (1a) can be found only by means of an integrating function and is different from the difference T – V between the kinetic and potential energy of the particle[2]. Correspondingly, the Hamilton function is, at least for $\alpha \neq 0$, not the energy of the particle[2,3] and its physical interpretation not obvious. E.g. the Caldirola-Kanai Hamiltonian[4] appears to describe a harmonic oscillator with a time-dependent mass[2,5] rather than a damped oscillator and (when interpreted as the Hamiltonian of a damped oscillator) leads to contradictions with the uncertainty principle for large times t. Difficulties with the energy and the uncertainty principle appear also when Bateman's dual Hamiltonian is quantized[1]. We mention that classical Hamilton functions exist and have been considered for quantization which do not correspond with the energy even in the limit $\alpha \to 0$. An example is $H = \dot{x} + \alpha x$ for the equation of motion $\ddot{x} + \alpha \dot{x} = 0$ [6]. (The reason for considering it was that the corresponding Lagrange function does not depend explicitly on time.) However one cannot expect the resulting Hamilton operator to produce physical wave functions of the damped quantal (quantized classical) system[3].

For a force quadratic in $\dot{x}$, canonical quantization of one classical Hamilton function can lead to different Hamilton operators in consequence of ordering problems for the operators p and x [7,8]. Apparently these problems can be overcome if one requires that Ehrenfest's theorems be fulfilled. But still, for $\alpha \neq 0$, the Hamilton function is not the energy, and contradictions with the uncertainty principle occur, now for large values of x [8]. Similar problems must be expected for velocity-power forces with higher exponents n.

Also nonlinear Schrodinger equations have been considered among the attempts of quantization. We mention only Kostin's Schrödinger-Langevin equation [3,9,10], which is based on complex and generalized Hamilton-Jacobi formalisms, but has also been found by stochastic quantization[11]. A problem is that the superposition principle is violated.

Recently we proposed a new quantization[12] for equations of motion involving the force (1a). The principal idea (and difference to canonical quantization) is to quantize not (one of) the mathematically correct Hamilton functions of the entire system, but the energy

$$E = \mathcal{H}_0 + \Delta E \tag{1b}$$

$$\mathcal{H}_0 = \frac{1}{2}m\dot{x}^2 + V(x) \tag{1c}$$

$$\Delta E = \alpha \int^x \dot{x}^n \mathrm{d}x \tag{1d}$$

and to use the quantization rules of the corresponding conservative system (CCS) described by the Hamilton function $\mathcal{H}_0$. Throughout the course of this paper the term CCS will denote the system under consideration for a r 0, i.e. without the forces (la), (2a) or (2b), respectively. E.g. in section 3, CCS does not imply the existence of a scalar potential $V(\mathbf{x})$, but of a generalized potential as well as a Lagrange and Hamilton function. The momentum to be quantized according to

$$p_c \rightarrow \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x} = p \tag{1e}$$

is the canonical momentum of the CCS. If the Hamilton function of the CCS has the form (1c) the canonical momentum p_c is identical with the mechanical momentum $m\dot{x}$ of the particle, and quantization of A E – by x $\rightarrow p/m$ – yields the non-Hermitian operator

$$\Delta E \rightarrow \alpha H_i = \alpha \int^x \left(\frac{\hbar}{\mathrm{i}m}\frac{\partial}{\partial x}\right)^n dx = \alpha \left(\frac{\hbar}{\mathrm{i}m}\right)^n \frac{\partial^{n-1}}{\partial x^{n-1}}. \tag{1f}$$

Apart from its simplicity, the method avoids the problems mentioned above, and its physical results are apparently all consistent. The principal one is that the Hamilton operator

$$H, = H_0 + \alpha H_i \tag{1g}$$

with

$$H_0 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) \tag{1h}$$

is dissipative for odd n = 1 and n = 3 while for even n = 2 and n = 4 it can have stationary eigenstates stable in time. Thus αH_i reproduces the property that the force (la) is dissipative only for odd n, and (globally) conservative for even n. This result makes the quantization of F for even n – when F is not just physically realistic – of considerable theoretical interest. Namely, it shows that for non-dissipative velocity-power forces the quantization (1f) can lead to a quantum theory with norm-conserved stable one-particle states.

These results motivate us to consider two extensions of the proposed quantization which both refer to important situations in classical physics. First, a velocity-power force – physically more realistic than F itself – is

$$F_d = -\alpha \mid \dot{x} \mid^{n-1} \dot{x} \quad , \quad \alpha > 0 \quad , \quad n > 0 \text{ integer} . \tag{2a}$$

It is dissipative for all n, and for odd n (and real $\dot{x}$) corresponds to expression (1a). The quantization of F_d and of its three-dimensional form

$$\mathbf{F}_{d3} = -\alpha v^{n-1} \mathbf{v} \quad , \quad v = +(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2} \tag{2b}$$

will be discussed in section 2. We will concentrate more on the general aspects rather than on specific solutions. The result is a quantum interaction operator which is dissipative for even and odd n. Secondly, in section 3 we shall consider the quantization of $\mathbf{F}_{d3}$ for the motion of a charged particle in a magnetic field, as the standard example where the canonical momentum p_c of the CCS is different from the mechanical momentum mv of the particle. In section 4 we show a relationship between the decay rate of a quantum state and its kinetic energy and discuss possible connections of our results to the optical model and dissipative collisions in nuclear physics.

2. Quantization of the classical energy loss

2.1 The operator αH_d

The force (2a) causes the energy loss

$$\Delta E_d = \alpha \int^t \mid \dot{x} \mid^{n-1} \dot{x}^2 \mathrm{d}t = \alpha \int^x \mid \dot{x} \mid^{n-1} \dot{x} \mathrm{d}x . \tag{3a}$$

Differently from formula (1d), the product $\alpha \mid \dot{x} \mid^{n-1} \dot{x} \mathrm{d}x$ in the integral (3a) is always non-negative ($\alpha > 0$, $\mathrm{d}t > 0$). The force (2a) is dissipative and diminishes the energy $\mathcal{H}_0$ of the particle monotonically ($\mathcal{H}_0 + \Delta E_d$ is constant in time and ΔE_d non-negative). The energy los ΔE_d can be quantized analogously as *AE*. (*p* and $|\, p\,|^{n-1}$ commute). The result is the non-Hermitian operator

$$\Delta E_d \to \alpha H_d = \frac{\alpha\hbar}{im^n}\int^x |\, p\,|^{n-1}\frac{\partial}{\partial x}\mathrm{d}x \;=\; \frac{\alpha\hbar}{im^n}\,|\, p\,|^{n-1}$$
$$= \frac{\alpha\hbar}{im^n}\begin{cases}\left(\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x}\right)^{n-1} & ,\ n \text{ odd}\\ \left|\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x}\right|^{n-1} & ,\ n \text{ even}\end{cases} \tag{3b}$$

For odd n, the operator (3b) corresponds with Eq. (lf). Since the momentum operator *p* is self-adjoint ($p = p^*$) its square $p^2 = pp^*$ is a positive (and self-adjoint) operator, i.e.

$$(\phi, p^2\phi) = \|\, p\phi\,\|^2 \geq 0\,. \tag{3c}$$

The operator $|\, p\,|$ is defined as the positive square root of pp* [13–16]

$$|\, p\,| = +\sqrt{pp^*} = +\sqrt{p^2} \tag{3d}$$

and is self-adjoint and positive. The specral decomposition theorem provides an explicit representation of $|\, p\,|$ and $|\, p\,|^{n-1}$. Like $|$ p $|$, the power $|\, p\,|^{n-1}$ is self-adjoint and positive for all integer $n > 0$, since $f(\lambda) = +\sqrt{\lambda^2}^{\,n-1}$ is, for all real arguments, a (real-valued) non-negative function. (*A* runs over the spectrum of p.) These properties are sufficient for the purposes of this paper. We shall not need the explicit representation of $|\, p\,|^{n-1}$ and will not enter into details of the domains of these operators which are dense in the Hilbert space of square-integrable functions.

2.2 Dissipative property of H_d

To discuss the operator αH_d we consider the Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial\psi(x,t)}{\partial t} \;=\; H_{ed}\psi(x,t) \tag{4a}$$

with the Hamilton operator

$$H_{ed} \;=\; H_0 + \alpha H_d. \tag{4b}$$

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H_0 is the conservative Hamilton operator (1h). From Eqs. (4a) and (4b) one can derive the modified continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = -\frac{\alpha}{m^n}\left(\psi^* \mid p \mid^{n-1} \psi + \psi \mid p \mid^{n-1} \psi^*\right) \tag{5a}$$

where

$$\rho(x,t) = \psi^*(x,t)\psi(x,t) \tag{5b}$$

and

$$j(x,t) = \frac{\hbar}{2im}\left(\psi^* \frac{\partial}{\partial x}\psi - \psi \frac{\partial}{\partial x}\psi^*\right) \tag{5c}$$

are the (one-dimensional) one-particle and current density. Integrating Eq. (5a) along the real x-axis one finds

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{+\infty} \rho(x,t)dx = -\frac{2\alpha}{m^n}\int_{-\infty}^{+\infty} \psi^*(x,t) \mid p \mid^{n-1} \psi(x,t)\mathrm{d}x. \tag{5d}$$

The integral on the RHS of Eq. (5d) is non-negative since $\mid p \mid^{n-1}$ is a positive operator. That this integral be zero would require that the (continuous and square-integrable) solution ψ vanish identically on the real x-axis since $+\sqrt{x^2} > 0$ for $x^2 > 0$ and the spectrum of p is continuous. Thus, for nontrivial solutions ψ the integral on the RHS of Eq. (5d) is positive, and for $a > 0$ on gets

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{+\infty} \rho(x,t)\mathrm{d}x < 0. \tag{5e}$$

The solutions of Eqs. (4a) and (4b) decay in time, as a consequence of the interaction operator αH_d. E.g., separable solutions

$$\psi(x,t) = \varphi(x)\exp(Et/i\hbar) \tag{5f}$$

cannot be stationary, and their energy eigenvalues E must have a negative imaginary part (αH_d is non-Hermitian). The quantum operator αH_d is dissipative for all integer (even and odd) n, in full correspondence with the properties of the classical force (2a).

With regard to the operator αH_i (1f), discussed in Ref. 12, the result (5e) shows that αH_i is dissipative for all odd $n \geq 1$ and generalizes our former results for n $= 1$ and n $= 3$. The difference between αH_i and αH_d for even n is that αH_i contains the operator p, instead of $|\,p\,|$ in αH_d, and only $|\,p\,|$, but not p, is a positive operator. Therefore one cannot conclude, for even n, that αH_i is dissipative.

2.3 Interpretation of dissipation

To interpret Eq. (5e) the following comments are in order and refer also to the corresponding result (11e) in three spatial dimensions with ($\mathbf{A} \neq 0$) or without ($\mathbf{A} \equiv 0$) magnetic field.

(i) The dissipation of the solutions to Eqs. (4a) and (4b) means that the probability to find the particle in the original one-particle eigenstate to H_{ed} decreases in time when friction is switched on. The particle, of course, does not disappear, but due to the frictional interaction it loses energy to, or forms more complicated many-particle states with, the particles of the frictional medium. (The latter states are not eigenstates of H_{ed} which is an effective one-particle Hamilton operator. It depends only on the momentum and position operators of the particle under consideration.) The non-Hermitian part αH_d (cf. comment (iii) below) takes into account both types of the above "inelastic reactions" in a summary form (note that already in the classical force F_d the details of the interactions between the single particle and the many particles of the dissipative medium have been lost) leading to a reduction of the probability of the original one-particle state. That quantization does not yield a strict one-particle theory appears also plausible from the qualitative argument that friction arises fom microscopic interactions between the single particle and the many particles of the dissipative medium, and that the classical energy loss is irreversible. Therefore it is not unlikely that the quantum mechanical motion of the particle cannot be separated from the states of motion of the particles in the frictional medium.

(ii) It is evident that from the dissipative Hamilton operators one will not obtain the equations of motion for the expectation values of the one-particle operators of momentum p and position x. To calculate these expectation values and their time derivatives one would need the full wave function into which the one-particle state decays, or the complete Hamilton operators which have these wave functions as eigenstates.

(iii) The following analogy seems interesting and worth mentioning. Our quantization leads to a dissipative operator αH_d which corresponds qualitatively to the absorptive imaginary potential in the empirical optical model of nuclear physics[17,18]. For n = 1 the correspondence is almost complete, but the optical potential depends at least on the radial coordinate r and on the energy (cf. section 4.2). The imaginary part of the optical potential describes in a summary form the decay of the original one-nucleon state into "inelastic channels" when the projectile nucleon transfers energy to the target nucleus or when it forms more compliated states with the nucleons of the target, i.e., is "absorbed" by the target. In such a sense, our Hamilton operator αH_d summarily describes the "inelastic interactions" of the particle with the dissipative medium, i.e. coupling of the particle motion to the internal degrees of freedom of the frictional medium.

2.4 Remarks on the three-dimensional quantization

A particle, moving in three spatial dimensions against the force (2b), loses the energy

$$\Delta E_{d3} = \alpha \int^{\mathbf{x}} \sqrt{v^2}^{\,n-1} \mathbf{v} \cdot d\mathbf{x} \geq 0. \tag{6a}$$

Replacing, in analogy with the rule (1e), the three-dimensional canonical momentum of the CCS, $\mathbf{p}_c = m\mathbf{v}$, by

$$\mathbf{p}_c \to \frac{\hbar}{i}\nabla = \mathbf{p} \tag{6b}$$

one could, in principle, try to quantize ΔE_{d3}. A more direct method is to replace in the one-dimensional quantum operator αH_d (3b) the operator $\mid p \mid$ by its three-dimensional form

$$\mid \mathbf{p} \mid = +\sqrt{\mathbf{p}^2} = +\sqrt{-\hbar^2\Delta} \tag{6c}$$

where $\mathbf{p}^2 = -\hbar^2\Delta$ is positive and self-adjoint. Then $|\,\mathbf{p}\,|$ and $|\,\mathbf{p}\,|^{n-1}$ are positive self-adjoint operators, and the operator

$$\alpha H_{d3} \;=\; \frac{\alpha}{m^n}\frac{\hbar}{\mathrm{i}}\,|\,\mathbf{p}\,|^{n-1} = \frac{\alpha}{m^n}\frac{\hbar}{\mathrm{i}}\sqrt{-\hbar^2\Delta}^{\;n-1} \tag{6d}$$

can be discussed in an analogous way to αH_d earlier on. Rewriting Eqs. (4) and (5) in three spatial dimensions, one sees hat αH_{d3} is dissipative for all integer $n \geq 1$. (Cf. Eqs. (11a)-(11e) in the following section, setting $A \equiv 0$.)

3. Quantization in a magnetic field

3.1 The gauge invariant interaction operator

As an example where the canonical momentum $\mathbf{p}_c$ of the CCS is different from the mechanical momentum $m\mathbf{v}$ we shall quantize the dissipative force (2b) when acting on a nonrelativistic spinless particle, which carries an electrical charge q and moves in an electromagnetic field, $\mathrm{E} = -\nabla A_0 - \partial \mathbf{A}/\partial ct$ and $\mathrm{B} = \nabla \times \mathbf{A}$. $(A_0, \mathbf{A})$ is the four-vector of the electromagnetic potential. Let us write the Hamilton operator in the form

$$H_{eA} \;=\; H_{0A} + \alpha H_{dA} \tag{7a}$$

where

$$H_{0A} = \frac{1}{2m}\left(\frac{\hbar}{\mathrm{i}}\nabla - \frac{q}{c}\mathbf{A}\right)^2 + V + qA_0 \tag{7b}$$

is the Hamiltonian of the CCS. As is well known in quantum mechanics, one can obtain H_{0A} either from the classical Hamilton function of a charged particle in an electromagnetic field (plus $V(\mathbf{x})$), using the quantization rule (6b) and taking into account the relation (7e), or from the Hamilton operator

$$H_{03} = \frac{\mathbf{p}^2}{2m} \;+\; V(\mathbf{x}) \tag{7c}$$

through the replacement

$$\mathbf{p} = \frac{\hbar}{\mathrm{i}}\nabla \rightarrow \frac{\hbar}{\mathrm{i}}\nabla - \frac{q}{c}\mathbf{A} \tag{7d}$$

and addition of the electric potential A_0. The rule (7d), known as minimal coupling between the canonical momentum of the particle and the electromagnetic potential, reflects the classical relationship

$$\mathbf{p}_c = m\mathbf{v} + \frac{q}{c}\mathbf{A} \tag{7e}$$

between canonical and mechanical momentum of the particle.

To find the operator H_{dA} it is almost suggesting itself, in the quantized operator H_{d3}, valid for A $\equiv 0$, to replace the momentum operator p by Eq. (7d) (correspondingly as H_{0A} is obtained from H_{03}). This yields

$$\alpha H_{d3} \underset{\mathbf{A}\neq 0}{\longrightarrow} \alpha H_{dA} = \frac{\alpha\hbar}{im^n}\sqrt{\left(\frac{\hbar}{\mathrm{i}}\nabla - \frac{q}{c}\mathbf{A}\right)^2}^{\,n-1} \tag{7f}$$

The operator (7f) preserves the gauge invariance of the Schrõdinger equation

$$\mathrm{i}\hbar\frac{\partial\psi(\mathbf{x},t)}{\partial t} = H_{eA}\psi(\mathbf{x},\mathrm{t}) \tag{8a}$$

in the usual sense[19] that together with the electromagnetic potential

$$\mathbf{A} \rightarrow \mathbf{A}' = \mathbf{A} + \nabla f \tag{8b}$$

$$A_0 \rightarrow A_0 = A_0 - \frac{1}{c}\frac{\partial f}{\partial t} \tag{8c}$$

the wave function $\psi(\mathrm{x},\mathrm{t})$ is transformed as well:

$$\psi \rightarrow \psi' = \psi\exp(\mathrm{i}f/\hbar c) \tag{8d}$$

where f (x,t) is a scalar, real gauge function.

For the proof we distinguish between even and odd n and define the operator

$$\alpha H_{dA'} = \frac{\mathrm{a}\hbar}{\mathrm{im}^n}C(\nabla f)^{n-1}, \quad C(\nabla f) = +\sqrt{\left(\frac{\hbar}{\mathrm{i}}\nabla - \frac{q}{c}(\mathbf{A}+\nabla f)\right)^2} \tag{9a}$$

$C(\nabla \mathrm{f})$ is self-adjoint and positive, and (7f) can be written as

$$\alpha H_{dA} = \frac{\alpha\hbar}{\mathrm{im}^n}C(0)^{n-1} \tag{9b}$$

(i) n ≥ 1, odd. Then

$$C(\nabla f)^{n-1} = \left[\left(\frac{\hbar}{\mathrm{i}}\nabla - \frac{q}{c}(\mathbf{A}+\nabla f)\right)^2\right]^m, \tag{9c}$$

where m = (n – 1)/2 is integer, and similarly as for H_{0A} [19] one finds directly

$$\alpha H_{dA'}\psi' = \frac{\alpha\hbar}{im^n}C(\nabla f)^{n-1}G\psi = \frac{\alpha\hbar}{im^n}C(\nabla f)^{n-3}GC(0)^2\psi = \cdots$$
$$\cdots = \frac{\alpha\hbar}{im^n}GC(0)^{n-1}\psi = G\alpha H_{dA}\psi \tag{9d}$$

with

$$G = \exp(iqf/\hbar c). \tag{9e}$$

(ii) n ≥ 2, even. Define the operator $\tilde{C}$ by the relation

$$C(\nabla f)G\psi = G\tilde{C}\psi\ , \tag{9f}$$

i.e.

$$\tilde{C} = C(\nabla f) + G^{-1}[C(\nabla f), G]_- = G^{-1}C(\nabla f)G. \tag{9g}$$

From Eq. (9g) and $G^{-1} = G^*$ one sees that with $C(\nabla f)$ also $\tilde{C}$ is self-adjoint and positive. Applying $C(\nabla f)$ once more to Eq. (9f), one gets

$$C(\nabla f)^2G\psi = G\tilde{C}^2\psi. \tag{9h}$$

Comparing Eq. (9d), for n − 1 = 2, with Eq. (9h), one concludes

$$\tilde{C}^2 = C(0)^2, \tag{9i}$$

independently of $C(\nabla f)$ and then.

$$\tilde{C} = +C(0) \tag{9j}$$

since the self-adjoint, positive square root of $C(0)^2$ is unique. Eqs. (9f) and (9j) show the gauge invariance for n = 2. For even n ≥ 4, one obtains from Eqs. (9d), (9f) and (9j)

$$C(\nabla f)^{n-1}G\psi = C(\nabla f)^{n-2}G\tilde{C}\psi = GC(0)^{n-2}\tilde{C}\psi = GC(0)^{n-1}\psi. \tag{9k}$$

This completes the proof of the gauge invariance, for all positive integer n.

We emphasize that the gauge invariance of the Schrõdinger equation (8a) is an important argument in finding the interaction operator αH_{dA}. When quantizing the classical energy loss (6a) in the presence of a magnetic field (instead of applying the minimal coupling rule (7d) to the operator (6d)) one could arrive at a result

$$\Delta E_{d3} \to \frac{\alpha}{m^n} \int^{\mathbf{x}} \sqrt{\left(\frac{\hbar}{\mathrm{i}}\nabla - \frac{q}{c}\mathbf{A}\right)^2}^{\,n-1} \left(\frac{\hbar}{\mathrm{i}}\nabla - \frac{q}{c}\mathbf{A}\right) \cdot \mathrm{d}\mathbf{x}. \tag{10}$$

This operator, as well as αH_{d3} instead of αH_{dA} in the Hamilton operator (7a), would destroy the gauge invariance of the Schrõdinger equation (8a).

Like the operators αH_d and αH_{d3}, also αH_{dA} is dissipative for a > 0 and all positive integer n. Rewriting Eqs. (5a)-(5c) in three dimensions and replacing the momentum operator by formula (7d), one gets

$$\frac{\partial \rho(\mathbf{x},t)}{\partial t} + \mathrm{div}\mathbf{j}(\mathbf{x},t) = -\frac{\alpha}{m^n}\left\{\psi^* \left|\frac{\hbar}{\mathrm{i}}\nabla - \frac{q}{c}\mathbf{A}\right|^{n-1} \psi + \psi \left|\frac{\hbar}{\mathrm{i}}\nabla - \frac{q}{c}\mathbf{A}\right|^{n-1} \psi^*\right\} \tag{11a}$$

where

$$\rho(\mathbf{x},t) = \psi^*(\mathbf{x},t)\psi(\mathbf{x},t) \tag{11b}$$

and

$$\mathbf{j}(\mathbf{x},t) = \frac{\hbar}{2\mathrm{i}m}(\psi^*\nabla\psi - \psi\nabla\psi^*) - \frac{q}{mc}\mathbf{A}\psi^*\psi. \tag{11c}$$

Integrating Eq. (11a) over an infinite volume V of surface S, excluding $\mathbf{x} = 0$ by an infinitesirnal sphere S_0 of radius r $=|\mathbf{x}| \to 0$ [20] and supposing that

$$\int_S \mathbf{j}\cdot \mathrm{d}\mathbf{f} \underset{|\mathbf{x}|\to\infty}{\longrightarrow} 0\,, \quad \int_{S_0} \mathbf{j}\cdot \mathrm{d}\mathbf{f}_0 \underset{r\to 0}{\longrightarrow} 0 \tag{11d}$$

(df and $d\mathbf{f}_0$ are infinitesimal surface elements of S and S_0, respectively) one finds

$$\frac{\mathrm{d}}{\mathrm{d}t} \parallel \psi \parallel^2 = \frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{+\infty} \rho \mathrm{d}\mathbf{x} = -\frac{2\alpha}{m^n}\int_{-\infty}^{+\infty} \psi^* \sqrt{\left(\frac{\hbar}{\mathrm{i}}\nabla - \frac{q}{c}\mathbf{A}\right)^2}^{\,n-1} \psi \mathrm{d}\mathbf{x}. \tag{11e}$$

The RHS of Eq. (11e) is negative for a > 0, n $= 1,2,3,\ldots$ and nontrivial states ψ, since the operator $C(0)^{n-1}$ is self-adjoint and positive. I.e., the probability to find the particle in the original one-particle state ψ which solves the Schrõdinger equation with Hamiltonian (7a) decreases in time. The frictional part αH_{dA} makes the Hamiltonian H_{eA} dissipative.

3.2 Example

As a simple example we consider the Hamilton operatorH_{eA} for V r Ao $\equiv 0$ and $n = 1, 3$. The Schrodinger equation for separable states

$$\psi(\mathbf{x}, t) = \varphi(\mathbf{x})\exp(Et/\mathrm{i}\hbar) \tag{12a}$$

$$\frac{1}{2m}\left(\frac{\hbar}{\mathrm{i}}\nabla - \frac{q}{c}\mathbf{A}\right)^2 \varphi + \frac{\alpha}{m^n}\frac{\hbar}{\mathrm{i}}\sqrt{\left(\frac{\hbar}{\mathrm{i}}\nabla - \frac{q}{c}\mathbf{A}\right)^2}^{\,n-1} \varphi = E\varphi\ . \tag{12b}$$

For a $= 0$, suppose that $\varphi = \hat{\varphi}(\mathbf{x})$ is a solution of Eq. (12b) with a real eigenvalue $E = \hat{E}$. Then, for a > 0, $\hat{\varphi}(\mathbf{x})$ is also a solution of Eq. (12b) with a complex eigenvalue

$$E = E^{(1)} = \hat{E} - \frac{\alpha\hbar\mathrm{i}}{m} \quad \text{if } n = 1 \tag{12c}$$

$$E = E^{(3)} = \hat{E}\left(1 - \frac{2\alpha\hbar\mathrm{i}}{m^2}\right) \quad \text{if } n = 3. \tag{12d}$$

The negative imaginary part of $E^{(1)}$ leads to a decay of the state (12a) in time. The eigenvalue $E^{(3)}$ makes the state (12a) decaying if $\hat{E}$ is positive. This condition is fulfilled since $(\frac{\hbar}{\mathrm{i}}\nabla - \frac{q}{c}\mathbf{A})^2$ is a positive operator. For a homogeneous magnetic field the eigenvalue problem (12b) witha $= 0$ has been solved in Refs. 20 and 21.

In the example above the x-dependent factor $\hat{\varphi}$ of the solution (12a) does not depend on a. Only the eigenvalues E $= E^{(1)}$ and E $= E^{(3)}$ depend on a, and Eqs. (12c) and (12d) show that the limit $\alpha \to 0$ is smooth and yields the solution of the CCS: $E^{(1)} \to \hat{E}$ and $E^{(3)} \to \hat{E}$. The solution (12a)becomes stationary in this limit.

3.3 Uncertainty relation

As the canonical momentum of the CCS is quantized, no contradictions with the uncertainty principle for the mechanical momentum will occur, and, for $\alpha \neq 0$, the mechanical momentum will satisfy the same uncertainty relation as for $\alpha = 0$. Explicitly, since A depends on the coordinates only (and perhaps on time) these commutation rules are

$$[A_k, x_j]_- = 0 \tag{13a}$$

and

$$\left[p_k - \frac{q}{c}A_k, x_j\right]_- = [p_k, x_j]_- = \frac{\hbar}{i}\delta_{kj} \tag{13b}$$

for $j, k = 1, 2, 3$. They imply[15]

$$\Delta p_k \Delta x_k \geq \frac{1}{2}h \tag{13c}$$

and

$$\Delta(p_k - \frac{q}{c}A_k)\Delta x_k = \Delta m v_k \Delta x_k \geq \frac{1}{2}\hbar \tag{13d}$$

for the products of the uncertainties Δp_k or $\Delta m v_k$ with Δx_k.

4. Discussion and conclusions

4.1 Decay rates

From Eq. (12d) one observes that the ground state would not decay if its energy $\hat{E} = \hat{E}_0$ were equal to zero. Now, in a homogeneous magnetic field one finds $\hat{E}_0 = M_B \mid B \mid$ [20] where $M_B = q\hbar/2mc$ and an additive positive constant corresponding to the kinetic energy of the particle's motion parallel to the magnetic field B has been neglected. Thus the value of $\hat{E}_0$ is equal to the zero-point energy of a harmonic oscillator of frequency $w = q \mid B \mid /mc$ indicating that the decay of the ground state is a consequence of the non-vanishing zero-point energy, or of the uncertainty principle.

To generalize this result for all $n \geq 2$ we rewrite Eq. (11e) in the form

$$\frac{d}{dt} \parallel \psi \parallel^2 = -\left(\frac{2}{m}\right)^{\frac{n+1}{2}} \alpha\epsilon_n(t) \parallel \psi \parallel^2 . \tag{14a}$$

The expectation value

$$\epsilon_n(t) = \int_{-\infty}^{+\infty} \psi^* \left(\frac{C(0)^2}{2m}\right)^{\frac{n-1}{2}} \psi dx / \parallel \psi \parallel^2 \tag{14b}$$

is positive and does not contain the time-dependence related with the decay

$$\parallel \psi(t) \parallel^2 = \exp\left(-\left(\frac{2}{m}\right)^{\frac{n+1}{2}} \alpha \int_0^t \epsilon_n(t) dt\right) . \tag{14c}$$

Since $C(0)2/2m$ is the positive and self-adjoint operator of the kinetic energy, whether $\vec{A}$ vanishes or not it follows from the uncertainty relation that $\epsilon_n(t)$ can tend to zero only when the uncertainties Δx, Δy and Δz of the particle's coordinates tend to infinity. Therefore states of (or close to) lowest energy decay with a minimum rate compatible with the uncertainty principle while states of higher (kinetic) energies decay faster. For n $= 1$ all states decay with the same rate $\exp(-2\alpha t/m)$. Note that this is equal to the decay rate of the kinetic energy of a particle moving against a classical force $F = -\alpha\dot{x}$ (without any other potential).

4.2 Friction in nuclear physics

In section 2.3 we mentioned a correspondence between the absorptive operator αH_d or αH_{d3}, in particular for n $= 1$, and the (imaginary part of the) optical potential in nuclear physics. The differences are, first that the constant $\alpha\hbar/m$ in our approach corresponds to a radially symmetric function W(r) in the optical model. This can be explained by the geometry of the nucleus, i.e. that it has a finite spatial extension and approximate spherical symmetry, but is not homogeneous and surface effects may be important as well. With these restrictions, inelastic scattering of a nucleon by an (energy-independent) optical potential can be considered as a quantum mechanical analogue to the motion of a classical particle under the influence of a linear frictional force (and a conservative force). Secondly, the empirical optical potential has a marked energy dependence[17,18] which is, apart from an intrinsic energy dependence (as nuclear matter is dispersive), the consequence of the non-locality of the nuclear potential. (A non-local potential can be obtained by projecting out the ground state of the target nucleus[18].) Thus one has to expect a very wide and general variety of coordinate- and energy- (velocity-)dependent functions for empirical local optical potentials. However, for an infinite homogeneous nuclear "target matter" the non-locality can be simulated by potentials depending on the velocity only[18]. This indicates some connection (and its limitations) between inelastic nuclear processes and the quantization of dissipative velocity-power forces also for n > 1.

Quantization of dissipative velocity-power forces

In connection with the optical model and nuclear processes the quantization in the presence of a magnetic field (cf. section 3) is of direct interest, even though of only minor quantitative importance, since nuclei in general have a magnetic moment. In the leading order $n = 1$ the absorptive part of the optical potential (if corresponding to quantized linear friction) should not be influenced by the magnetic field resulting from the nuclear magnetic moment.

Experimental evidence for frictional effects like loss of relative kinetic energy and angular momentum has been observed in heavy-ion collisions, i.e. deep-inelastic collisions, fusion and capture above the barrier, which have been described with some success in classical models as in the surface friction model[22]. This model considers – besides the Coulomb and nuclear potential – the harmonic deformation (low-lying quadrupole) modes of both the projectile and the target nuclei in the conservative Lagrangian L. Radial and tangential friction as well as intrinsic damping of the deformation modes (vibrators) are described through a Rayleigh dissipation function R which is derived from Brownian motion theory and contains three universal parameters. The deformation modes are important because both nuclei experience strong deformations when they collide. Already therefore, these processes and the corresponding models are much more complex than the motion of a single particle without interna1 degrees of freedom in a homogeneous dissipative medium which we are quantizing, and we shall not attempt an extension of our approach to the surface friction model. (From L and R (Eqs. (1) and (2) in Ref. 22) one should find the Hamilton function $\mathcal{H}_0'$ and the energy loss AE^1 and then quantize $\mathcal{H}_0' + \Delta E'$ by $\pi_i \rightarrow \frac{\hbar}{i}\frac{\partial}{\partial \alpha_i}$ were π_i and α_i are the momenta and deformation parameters for the vibrators. However, the non-diagonal frictional coupling terms between radial relative motion and the vibrators and between the vibrators present in R could lead to a cumbersome Schrödinger equation.) Anyway, dissipative heavy-ion collisions appear to be interesting examples that frictional interactions can mediate the formation of quantum states between the particle and the frictional medium, and thus support the interpretation of our approach. Furthermore, that these reactions can be described by models based on classical friction might have a connection with the result of our quantization which – for simpler reactions – suggests that some relationship exists between the absorptive part of the optical potential and friction.

4.3 Summarizing remarks

Following the quantization[12] of the velocity-power force (1a), we have quantized the dissipative forces (2a) and (2b) in one and three space dimensions, respectively. The quantum mechanical operators αH_d and αH_{d3} thus obtained are, for $\alpha > 0$, dissipative for even and odd n, in full correspondence with the classical forces. Having in one's mind the optical model, one can interpret the decay of one-particle states as "absorption" of the particle (or the one-particle state) by the dissipative medium.

Once the quantization of the three-dimensional force $\mathbf{F}_{d3}$ is achieved, the motion of a charged particle in a magnetic field and under the influence of $\mathbf{F}_{d3}$ itself, can be studied. This is another important extension of the method proposed in Ref. 12, since the mechanical momentum now is different from the canonical momentum of the CCS. Gauge invariance suggests that te momentum operator $\mathbf{p}$ in H_{d3} be replaced by the minimal coupling (7d), in analogy with the procedure to find the Hamilton operator H_{0A} from H_{03} in standard quantum mechanics. Generalizing, we expect that quantization of velocity-power forces (1a), (2a), and (2b) (and any linear combination of them, with different exponents n) is possible in sytems which, for $\alpha = 0$, have a known Lagrange and Hamilton function, the latter one being identical with the energy of the particle. The momentum to be quantized, $\mathbf{p}_c \rightarrow \frac{\hbar}{i}\nabla$, is the canonical momentum of the CCS.

We summarize that for a sufficiently large class of the examples considered the quantization of frictional velocity-power forces and the results allow a consistent interpretation. That the Hamilton operators obtained are not norm-conserving and "only" describe the decay of the original one-particle state in time indicates that the particle forms more complicated quantum states of motion with the particles of the dissipative medium. This may be one reason for the difficulties one meets in the attempts to obtain a strict one-particle quantum theory for frictional velocity-power forces, e.g. in canonical quantization of $F = -\alpha\dot{x}$.

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Resumo

Um método, recentemente sugerido para quantizar forças proporcionais a potências da velocidade, $F = -\alpha x^n$, $n \geq 1$ inteiro, $\alpha > 0$, é estendido para forças $F_d = -\alpha \mid \dot{x} \mid^{n-1} \dot{x}$ (dissipativas para n par e ímpar) e para sua versão tridimensional $\mathbf{F}_{d3} = -\alpha v^{n-1}\mathbf{v}$. O hamiltoniano obtido é dissipativo para todos os n, inteiros e positivos, i. e., a probabilidade de encontrar a partícula no estado inicial de uma partícula é reduzida pelo atrito. A quantização de $\mathbf{F}_{d3}$ é discutida, com resultados análogos, também para o movimento de uma partícula com carga elétrica num campo magnético, como um exemplo no qual o momentum canônico $\mathbf{p}_c$ do sistema conservativo ($\alpha = 0$) é diferente do momentum mecânico mv da partícula. Possíveis conexões com o modelo ótico e com reações nucleares dissipativas são consideradas.