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Field copies of the Weyl conformal tensor

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Abstract Field copies associated with the Weyl curvature tensor are considered in the context of both Riemann and Weyl manifolds. Geometrical aspects and physical implications of the examples presented here are discussed.

1. Introduction

Gravitational Lagrangian models with the property of scale invariance have attracted renewed attention in the current literature[1-8]. One of the arguments that justify the ressurection of Weyl's original idea of conformal invariance[9] is that this property, imposed on gravitational Lagrangians, leads to dimensionless coupling constants such as those appearing in strong and eletroweak interactions, which are renormalizable. On the other hand, Einstein's Lagrangian involves a dimensional constant (length^{-2}) and General Relativity is not renormalizable. Accordingly, it would be desirable to modify Einstein's Lagrangian to make it scale invariant. Considerable effort has been dedicated, for instance, to an understanding of the role of the Weyl tensor in the construction of conformally covariant Lagrangians. Indeed, this tensor has remarkable algebraic properties, such as the decomposition introduced by Lanczos[10] and developed by other authors[11,12].

Another interesting point concerning the conformal tensor $C_{\alpha\beta\mu\nu}$ is, in my opinion, the investigation of conformal copies in metric and semi-metric manifolds. One can show that under specific conditions, different affine connections of a given space-time generate the same $C_{\alpha\beta\mu\nu}$. Connections with this property will be called

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be called here conformal copies *(CC)*. This is in some respects similar to the study of field strength copies in non-abelian gauge theories[13], where different potentials A_u^L and $\bar{A}_\mu^L$, not related by any gauge transformation, can give rise to the same field strength $F_{\mu\nu}^L$.

The question of field copies of the curvature tensor in Riemann-Cartan space was treated in Ref. 14. Since the conformal tensor of a Riemannian manifold V_n can be written in terms of the curvature tensor, it is obvious that curvature copies in V_n are also conformal copies in this space. But the converse in not always true, as will be seen in Sec. 2. Furthermore, in semi-metric manifolds W_n the conditions for the existence of CC may involve the Weyl 1-form Q_α, which expresses the change of length of vectors ("dilations") in parallel displacements, and which Weyl expected to be the representation of the electromagnetic potential.

In the subsequent sections of this paper some examples of CC will be presented and discussed. The influence of these copies on couplings of matter and gravitation will be emphasized.

2. The condition for conformal copies

The conventions used here are those of Schouten (Ref. 15). L, an n-dimensional manifold in which a rotation curvature and a curvature of segmentation exist.

The invariance of the Weyl conformal tensor is expressed by

$$C_{\alpha\beta\gamma\cdot}^{\ \ \ \ \delta}(\Gamma^{\cdot}) = \bar{C}_{\alpha\beta\gamma\cdot}^{\ \ \ \ \delta}(\bar{\Gamma}), \tag{1}$$

where $\Gamma_{\alpha\beta}^{\ \ \gamma}$ belongs to L, and $\bar{\Gamma}_{\alpha\beta}^{\ \ \gamma}$ belongs to $\bar{L}_n$, with

$$\bar{\Gamma}_{\alpha\beta}^{\ \ \gamma} = \Gamma_{\alpha\beta}^{\ \ \gamma} P_{\alpha\beta\cdot}^{\ \ \ \gamma}, \tag{2}$$

$P_{\alpha\beta\cdot}^{\ \ \ \gamma}$ being a third-rank tensor which will be named P-field. **A** set of connections of the type **(2)** satisfying **(1)** will be called a set of field copies of the conformal tensor, or – for short – a set of conformal copies (CC).

Equation (1), with $\bar{\Gamma}$ given by (2), leads to the following condition for the P-field

$$\overset{\star}{\nabla}_{[\nu}P_{\mu]\lambda\cdot}{}^{\rho} - \frac{1}{n-2}(\delta^{\rho}_{\nu}\overset{\star}{\nabla}_{[\sigma}P_{\mu]\lambda\cdot}{}^{\sigma} - \delta^{\rho}_{\mu}\overset{\star}{\nabla}_{[\sigma}P_{\nu]\lambda\cdot}{}^{\sigma} +$$
$$+ g_{\lambda\mu}g^{\tau\sigma}\overset{\star}{\nabla}_{[\sigma}P_{\nu]\tau\cdot}{}^{\sigma} - g_{\lambda\nu}g^{\tau\rho}\overset{\star}{\nabla}_{[\sigma}P_{\mu]\tau\cdot}{}^{\sigma}) +$$
$$+ \frac{g^{\tau\omega}}{(n-1)(n-2)}\overset{\star}{\nabla}_{[\sigma}P_{\omega]\tau}{}^{\sigma}(\delta_{\nu}{}^{\sigma}g_{\lambda\mu} - \delta_{\mu}{}^{\rho}g_{\lambda\nu}) = 0, \tag{3}$$

$$\overset{\star}{\nabla}_{\nu}P_{\mu\lambda\cdot}{}^{\rho} \equiv \nabla_{\nu}P_{\mu\lambda}{}^{\rho} - P_{\nu\lambda}{}^{\tau}P_{\mu\tau\cdot}{}^{\rho}, \tag{4}$$

where V, is the covariant derivative associated with $\Gamma_{\alpha\beta\cdot}{}^{\lambda}$, the square brackets around indices denote antisymmetrization, and n > 2. Note that in L_n one has $\nabla_{\alpha}\, g_{\beta\gamma} = -Q_{\alpha\beta\gamma}$ (non-metricity condition). In a semi-metric space W_n the non-metricity tensor becomes $Q_{\alpha\beta\gamma} = -Q_{\alpha}g_{\beta\gamma}$.

Condition (3) leads to the existence of CC that are not curvature copies in the manifold, even though all tensor fields related to curvature copies must satisfy (3).

Copies for which the P-field is generated by the mathematical structure of the initial (base) manifold itself are of particular interest. They will be referred to as intrinsic conformal copies (ICC), and the corresponding transformations for the connection, as intrinsic copy transformations (ICT). Examples of ICC will be presented below.

3. Conforma1 copies *in* W_n

The most evident I C T in a semi-metric space W_n is the usual conforma1 transformation. In this case the transformed Weyl vector $\tilde{Q}_{\alpha}$ differs from the original Q_{α} by a vector $\partial_{\alpha}\log A$, where A is an arbitrary scalar and $\tilde{g}_{\alpha\beta} = Ag_{\alpha\beta}$. As a consequence, $\tilde{\Gamma}_{\alpha\beta}{}^{\gamma} = \Gamma\ {}^{\gamma}$ and $\tilde{C}_{\alpha\beta\gamma\cdot}{}^{\delta} = C_{\alpha\beta\gamma\cdot}{}^{\delta}$. Accordingly one can state:

<u>Theorem 1</u> – The usual conformal transformation is an I C T with vanishing P-field, i.e., the identity copy mapping, if one takes the Weyl 1-form fixed only to within an arbitrary scalar factor.

An example of a non-vanishing P-field which generates an ICC in W_n is

$$P_{\alpha\beta}{}^{\gamma} = g_{\alpha\beta}Q^{\gamma}. \tag{5}$$

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By sybstituting *(5)* into **(3)** one gets

$$(n-1)g_{\lambda[\mu}\nabla_{\nu]}Q^{\rho} - \delta^{\rho}_{[\nu}g_{\mu]\lambda}\nabla_{\sigma}Q^{\sigma} + g_{\lambda\sigma}\delta^{\rho}_{[\nu}\nabla_{\mu]}Q^{\sigma} = 0, \tag{6}$$

which imposes the propagation of the Weyl vector.

<u>Theorem 2</u> – The P-field expressed by *(5)* is a P-field of an ICC if (6) is taken into account.

<u>Corollarv 1</u> – If V, $Q^{\beta} = 0$ then eq. *(5)* guarantees the existence of an ICC in W_n.

Corollary 1 follows irnmediately from the irnposition of the restriction $\nabla_\alpha Q^{\beta} = 0$ on (6). This particular ICC is also a curvature copy (see Ref. **14).**

Another copy is provided by

$$P_{\alpha\beta\cdot}{}^{\gamma} = -2\delta_{(\alpha}{}^{\gamma}Q_{\beta)}, \tag{7}$$

where round brackets mean symnietrization. And from **(3)** one arrîves at

$$\begin{aligned} &2\delta^{\rho}_{[\mu}\nabla_{\nu]}Q_{\lambda} + (n-2)\delta_{\lambda}{}^{\rho}\nabla_{[\mu}Q_{\nu]} + \nabla_{\lambda}Q_{[\mu}\delta_{\nu]}{}^{\rho} + \\ &+ g_{\lambda[\mu}\delta_{\nu]}{}^{\rho}g^{\sigma\tau}\nabla_{\sigma}Q_{\tau} - g^{\sigma\rho}(\nabla_{\sigma}Q_{[\mu})g_{\nu]\lambda} - ng^{\sigma\rho}g_{\lambda[\mu}\nabla_{\nu]}Q_{\sigma} = 0. \end{aligned} \tag{8}$$

<u>Theorem 3</u> – The P-field introduced by *(7)* is associated with an ICC if (8) is satisfied.

<u>Corollarv 1</u> – The P-field of **(7)** together with the assumption V, $Q^{\beta} = 0$ generates an ICC in W_n.

These results follow by accomplishing the same type of substitutions referred to above.

One also has:

<u>Theorem 4</u> – The conditions V, $Q^{\lambda} = 0$ and Q, $Q^{\sigma} - 0$ applied to the tensor field $[g_{\alpha\beta}Q^{\gamma} - 26_{(\alpha}{}^{\gamma}Q_{\beta)}]$ define a set of ICT in W_n.

I notice that the two restrictions mentioned in Theor. 4 are independent and compatible, as one easily verifies.

In semi-metric manifolds without torsion the symmetry $P_{\mu\lambda\rho} = P_{(\mu\lambda)\rho}$ holds for the P-field. Thus assumptions on the trace of the third-rank tensor $\underline{P}$ can also lead to copies. For instance,

$$P_{\mu\lambda\rho} = \frac{1}{n}\left(2\delta^{\rho}_{(\lambda}Q_{\mu)} - Q^{\rho}g_{\mu\lambda}\right), \tag{9}$$

with

$$P_{\mu\sigma}{}^{\sigma} = Q_{\mu} - \frac{n}{2-n}P^{\sigma}_{\cdot\sigma\mu}. \tag{10}$$

Theorem 5 – The tensor defined by relations (9) and (10) expresses an intrinsic conformal mapping in W_n.

Now examples of ICC in a metric n-dimensional space V_n will be considered.

4. Conformal copies in V_n

The identity transformation for copies associated with the invariance of the Weyl tensor in a Riemannian manifold is represented by the conformal mapping for the affine connection, namely

$$\tilde{\Gamma}_{\mu\lambda}{}^{\rho} = \Gamma_{\mu\lambda}{}^{\rho} + \frac{1}{2}(\delta_{\lambda}{}^{\rho}S_{\mu} + \delta_{\mu}{}^{\rho}S_{\lambda} - g_{\mu\lambda}S^{\rho}), \tag{11}$$

$$S_{\mu} \equiv \partial_{\mu}\log A. \tag{12}$$

From (11) and (12) it follows that $\tilde{C}_{\mu\nu\lambda}{}^{\rho} = C_{\mu\nu\lambda}{}^{\rho}$ in V_n. Another result is:

Theorem 6 – The P-field given by (13) and (14) furnishes a conformal copy in V_n.

$$P_{\alpha\beta}{}^{\gamma} = a(g_{\alpha\beta}P^{\gamma} \pm 2\delta^{\gamma}_{(\alpha}P_{\beta)}), \tag{13}$$

$$P_{\lambda}P^{\lambda} = 0, \tag{14}$$

where $\underline{a}$ is a real number.

Corollary 3 – The form of the copy transformation (13) is preserved under a conformal mapping of the type (11), applied to the copy connection, if the supplementary assumption (15) is made.

$$\tilde{P}_{\mu} = P_{\mu} - S_{\mu} \tag{15}$$

with S_μ defined by (12). Note that if one intends to impose a projective transformation[15] together with a conformal one, then the scalar factor A must be constant. Projective mappings can be associated with Yang-Mills fields[16], and in this case a torsion field must be taken into account, since the connection becomes nonsymmetric.

The P-field of **(13)** acquires an intrinsic geometrical character if P_μ is related to quantities of V„ for instance if $P_\mu = \partial_\mu$ R or $P_\mu = R_{\mu\nu}\, \partial^\nu R$.

Conforma1copies that also deserve consideration are those entirely determined by vielbein fields. An example is

$$\bar{\Gamma}_{\alpha\beta}{}^{\gamma} = e_j^{\gamma}\partial_\alpha e_\beta^j + \frac{2}{n-1}(\delta_\alpha{}^{\gamma}\delta_\beta{}^{\tau} - g_\beta^{\gamma\tau} g_{\alpha\beta}) e_j{}^{\sigma}\partial_{[\tau} e^j{}_{\sigma]}, \tag{16}$$

where $\underline{j}$ is an internal index.

The first term on the right hand side of (16) is the affine connection of Weitzenbock space A„ obtained from the metricity condition imposed on the vielbein e_σ^j, namely

$$\nabla_\lambda e_\sigma^j = 0. \tag{17}$$

In *A*, the fields $e_{j\alpha}$ constitute a set of parallel vectors which fix the nonsymmetric connection of the manifold[15]. The existence of a copy like (16) poses the question of an additional ambiguity in the Hamiltonian dynamics of a theory based on vielbein vectors as canonical variables. In the vielbein formalism the internal connection (i.e., the affinity corresponding to an internal space) is related to a canonical variable called hypermomentum[17]. Thus the existence of a copy of the connection implies extra degrees of freedom in the propagation of the canonical variables. This fact also indicates a new motivation for the study of copies in gravitational theories.

5. Interaction with matter

The preceding sections dealt with conformal copies in space-time, without any consideration about coupling with matter. Although in W_n or V, theories geometry and matter remain segregated, gravitational fields are expected, of course, to interact with other fields and particles. Hence it is reasonable to investigate the influence of ICC of a given manifold on Lagrangians describing these interactions.

A matter field belongs to an internal space described by the internal indices of the vielbein $e_{j\mu}$. In a W_n manifold the full covariant derivative is given as

$$D_\mu e^j_\nu = \partial_\mu e^j_\nu - \Gamma_{\mu\nu}{}^\lambda e^j_\lambda + \Lambda^j{}_{k\nu} e^k_\mu, \tag{18}$$

with

$$\Gamma_{\mu\nu}{}^\lambda = \left\{ {\lambda \atop \mu\nu} \right\} + \frac{1}{2}(\delta_\mu{}^\lambda Q_\nu + \delta_\nu{}^\lambda Q_\mu - g_{\mu\nu} Q_\lambda) \tag{19}$$

being the space-time affinity of W_n, and $\Lambda^j{}_{k\nu}$ representing the internal connection.

The semi-metricity conditions yields

$$\Lambda^j{}_{k\sigma} = -e_j{}^\lambda \nabla_\sigma e^j_\lambda - \frac{1}{2}\delta^j_k Q_\sigma, \tag{20}$$

where ∇_λ is the covariant derivative associated with **(19).** The internal connection (20) is asymmetric and contains the affinity **(19)** through the derivative ∇_λ. One concludes that copies $\bar{\Gamma}^\lambda_{\mu\nu}$ of $\Gamma^\lambda_{\mu\nu}$ imply copies $\bar{\Lambda}^\lambda{}_{k\sigma}$ of $\Lambda^\lambda{}_{k\sigma}$, and consequently all the cases of ICT presented in Sec. 3 and Sec. 4 affect the gravitational interaction with matter. This will be clarified by the following examples.

The minimal coupling of a spin - 1/2 particle with gravitation in a 4-dimensional space, is expressed by the Lagrangian[18]

$$\mathcal{L}^{(1)}_m = \frac{i}{2} e^\lambda_k (\bar{\Psi}\gamma^k \delta_\lambda \psi - \psi \gamma^k \delta_\lambda \bar{\psi}) - m\psi\bar{\psi}, \tag{21}$$

with $\delta_\lambda = \partial_\lambda + \frac{1}{8}\Lambda_{jk\lambda}[\gamma^j, \gamma^k]$ in W_4. The copy $\bar{\Gamma}^\lambda_{\mu\nu}$ originates a $\bar{\Lambda}_{jk\lambda} \neq \Lambda_{jk\lambda}$, so that the modified Lagrangian $\mathcal{L}_m$ yields additional terms in the resulting field equations. In a non-minimal coupling instance fermions are influenced by ICT as well, as can be seen from

$$\mathcal{L}^{(2)}_m = ia\sqrt{-g}\bar{\psi}\gamma^k \psi e^\lambda{}_k e^\sigma{}_j \partial_{[\lambda} e^k_{\sigma]}, \tag{22}$$

a a constant, and $g = \det(e_{j\mu} e^j{}_\nu)$. The P-field defined by (16) with n = 4 produces $\bar{\mathcal{L}}^{(2)}_m \neq \mathcal{L}^{(2)}_m$. It is interesting to mention that in the context of scale transformations it can be proven that Weyl 1-form does not couple to any spin -1/2 particle, while in the case of ICT treated here the Q-vectors cannot be disregarded. This can also be illustrated by a model of interaction of the Weyl vector with a complex scalar field[ig]. The corresponding Lagrangian can be rewritten as

$$\mathcal{L}_m^{(3)} = \sqrt{-h}h^{\alpha\beta}d_\alpha(\phi^{-1}q)d_\beta(\phi^{-1}q)^*, \tag{23}$$

with

$$h^{\alpha\beta} = \phi^2 g^{\alpha\beta}, \tag{24}$$

$$d_\mu(\phi^{-1}q) = [\partial_\mu - ia(Q_\mu - 2\partial_\mu \log\ \phi)](\phi^{-1}q), \tag{25}$$

4 and q are scalar and matter fields, respectively. The action functions becomes independent of ϕ, and this makes the conformal invariance that supports the model devoid of physical content, as pointed out in Ref. 4. However, if instead of the scale mapping one takes a copy transformation for the eletromagnetic potencial $A_\mu \equiv Q_\mu - 2\partial_\mu \log$ 4, i.e. a copy of the type $\tilde{A}_\mu = A_\mu +$ M,(Qu), where M_μ satisfies $\nabla_{[\mu}M_{\nu]} = 0$, then a contribution related to the Weyl vector will appear in the equations of motion, ensuring a meaningful role for A_μ. In this case M_μ is considered as a function of the Weyl 1-form, for instance $M_\mu = Q_\mu Q^\sigma Q_\sigma$, so that it is not a gauge field in the usual sense.

It should be observed that if Q, is interpreted as a Dirac current[7] then conformal copies may provide a natural way to introduce interactions. A similar speculation can be made concerning the proposal of Ref. 8. where a vector meson related to Weyl 1-form absorbs the magnitude of the Higgs field in the Weinberg-Salam theory, leading to a model for the interaction of gravitational and electroweak fields.

6. Final remarks

The examples of ICC considered here show the existence of new symmetries in pure gravitational Lagrangians. When these copy symmetries involve the Weyl vector the interaction of gravitation with matter can be viewed as a copy effect, and Q_α may be conveniently chosen to represent matter fields. This aspects of ICC seems to deserve further investigation, since recent works give promissing indications about the possibility that the Weyl vector could shed some light on the search for quantum models of the gravitational field[7,8,17].

Another point is that these new "intrinsic" symrnetries introduce an additional ambiguity in the Hamiltonian analysis of gravitational models. Cauchy data at an initial hypersurface $t_0=$ constant may carry the propagation of geometrical objects associated with copies, even after a gauge fixing procedure. The hypermomentum[17], for instance, is a canonical variable which incorporates a copy effect via the P-field of the interna1 connection. This dynamical ambiguity has to be taken into account in the construction of observables.

I also wish to point out that certain quadratic combinations of the curvature tensor and/or its contractíons appearing in gravitational Lagrangians can arise from invariance under ICT. Indeed, the well-known Gauss-Bonnet theorem in an L, manifold shows the existence of a class of quadratic Lagrangians equivalent to Lagrangians depending on the Weyl tensor, which are as a consequence copy-invariant Lagrangians.

Finally I mention the effect of ICC on trajectories in L,. Autoparallel curves[20], along which a vector is transported parallel to itself according to the connection of the manifold, will differ in a copy space from similar curves considered in the original one. For P-fields related to the Weyl vector this difference will mean copy trajectories emerging from the dilation property of vectors in the manifold.

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Resumo

Cópias de campo associadas ao tensor de curvatura de Weyl são consideradas no contexto de variedades de Riemann e de Weyl. Discutem-se aspectos geométricos bem como implicações físicas dos exemplos aqui apresentados.