Revista Brasileira de Física, Vol. 21, nº 3, 1991

On Enskog's Dense Gas Theory. II. The Linearized Burnett Equations for Monatomic Gases

W. Marques Jr. and G. M. Kremer

Departamento de Física Universidade Federal do Paraná Caixa Postal 19091, Curitiba, 81531, PR, Brasil

Received June 6, 1991; in final form August 8, 1991

Abstract The third approximation of the distribution function for a moderately dense monatomic gas of hard spherical particles is determined from Enskog's dense gas theory through the methods of Chapman-Enskog and Grad. The linearized Burnett equations, obtained from the third approximation of the distribution function, indicate that the stress tensor has two terms which do not appear in the case of a rarefied gas. They represent a thermal (or temperature) pressure and a density pressure and are associated with the Laplacians of temperature and density, respectively. The phase speed and the attenuation coefficient of plane harmonic waves of small amplitudes are also determined in the low frequency limit.

1. Introduction

In a previous work,[1] henceforth denoted by I, a 13-field and a five-field theory were developed for monatomic dense gases of hard spherical particles, based on Enskog's dense gas theory[2] and on Grad's method of moments.[3] The constitutive relations for the pressure tensor and for the heat flux, corresponding to the second approximation of the distribution function (laws of Navier-Stokes and Fourier), were obtained from the transition of the 13-field theory to the five-field theory through an iteration method akin to the Maxwellian procedure.[4] Moreover, the problem concerning the propagation of plane harmonic waves of small amplitudes was analyzed for both cases.

In 1879 Maxwell[5] obtained an expression for the stress tensor from the kinetic theory of gases going beyond the Navier-Stokes equations. He showed the existence of a ***thermal (or temperature) stress by*** relating the stress tensor to second gradients of temperature. In the kinetic theory of gases this relationship is attained from the third approximation of the distribution function. The complete expression for the stress tensor which follows from the third approximation of the distribution function was given later by Burnett[6], and that corresponding to the heat flux by Chapman and Cowling.[7] These equations are known nowadays as the Burnett equations.

The aim of this paper is the determination of the linearized Burnett equations for a moderately dense gas of hard spherical particles. Based on the theory of Enskog for a dense gas we determine the third approximation of the distribution function through the method of Grad, following the methodology given in I. The same distribution function is obtained through the use of a different method, namely, the method of Chapman-Enskog.[7] In this method the distribution function is determined as a solution of the integro-differential equation of the modified Boltzmann equation.[8]

With the knowledge of the distribution function we obtain the linearized Burnett equations. The expression for the pressure (or stress) tensor of a moderately dense gas has three terms which vanish in the rarefied gas limit. One is the well known term proportional to the divergence of the velocity and whose coefficient is the volume viscosity. The two others are respectively proportional to the Laplacian of the temperature and to the Laplacian of the density. The first refers to a thermal (or temperature) pressure, and the second to a density pressure.

Like in the previous work, we analyze the problem concerning the propagation of plane harmonic waves of small amplitude and determine explicitly the phase speed and the attenuation coefficient of the wave in the low frequency limit. For the notation and the definition of the terms not given here, we refer the reader to 1.

2. The equation of transfer

The theory of Enskog for a moderately dense gas of hard spherical particles is based on the so-called Enskog equation, which is a generalization of the Boltzmann equation for the single-particle distribution function $f(\mathbf{x}, \mathbf{c}, t)$. This theory considers only two-body collisions but takes into account the difference in position of the colliding particles and the increase in frequency of collisions by a factor χ, which is identified with the local equilibrium radial distribution function.

Here we shall use the following approximation

$$\mathbf{D}f = \frac{\partial f}{\partial t} + c_i\frac{\partial f}{\partial x_i} = J^0(ff) + J^I(ff) + J^{II}(ff) + J^{III}(ff), \tag{1}$$

which is obtained from the Enskog equation by expanding the functions χ, f_1 and f_1' in Taylor series near $\mathbf{x}$ and neglecting the fourth- and higher-order terms (for more details one is refered to [1,2,7]). In Eq. (1) externa1 body forces were neglected, $J^0(ff)$, $J^I(ff)$ and $J^{II}(ff)$ are defined in I. by Eqs. (2.5) and $J^{III}(ff)$ stands for

$$J^{III}(ff) = \frac{a^3}{6}\int\Big\{\chi\Big[f'\frac{\partial^3 f_1'}{\partial x_i\partial x_j\partial x_k} + f\frac{\partial^3 f_1}{\partial x_i\partial x_j\partial x_k}\Big] + \frac{3}{2}\frac{\partial\chi}{\partial x_k}\Big[f'\frac{\partial^2 f_1'}{\partial x_i\partial x_j} + f\frac{\partial^2 f_1}{\partial x_i\partial x_j}\Big]$$
$$+\frac{3}{4}\frac{\partial^2\chi}{\partial x_i\partial x_j}\Big[f'\frac{\partial f_1'}{\partial x_k} + f\frac{\partial f_1}{\partial x_k}\Big] + \frac{1}{8}\frac{\partial^3\chi}{\partial x_i\partial x_j\partial x_k}[f'f_1' + ff_1]\Big\}k_ik_jk_ka^2(\mathbf{g}\cdot\mathbf{k})d\mathbf{k}d\mathbf{c}_1. \tag{2}$$

In Eq. (2) a is the diameter of a spherical particle and the prime and the index 1 in f refer to the velocities $(\mathbf{c},\mathbf{c}_1)$ and $(\mathbf{c}',\mathbf{c}_1')$ of two particles' before and after the collision, respectively. These velocities are connected by the relations

$$\mathbf{c}' = \mathbf{c} + \mathbf{k}\,(\mathbf{k}\cdot\mathbf{g}), \qquad \mathbf{c}_1' = \mathbf{c}_1 - \mathbf{k}\,(\mathbf{k}\cdot\mathbf{g}), \tag{3}$$

where $\mathbf{g} = \mathbf{c}_1 - \mathbf{c}$ is the relative linear velocity and $\mathbf{k}$ the unit vector in the direction of the line which joins the two particles centers at collision, pointing from the particle labeled by 1 to the other. Further, $d\mathbf{k} = \sin\theta\, d\theta\, d\epsilon$ is an element of solid angle with $0 \le \theta \le \frac{\pi}{2}$ representing the angle between $\mathbf{k}$ and $\mathbf{g}$ and $0 \le \epsilon \le 2\pi$ the angle containing $\mathbf{k}$ and $\mathbf{g}$ and a reference plane through $\mathbf{g}$.

For the approximation given by Eq. (1) we are interested only in the balance equations for mass, linear momentum and energy. Hence the multiplication of Eq. (1) by a summational invariant $\psi(\mathbf{c})$, and integration over all values of $\mathbf{c}$, lead to an equation of transfer which can be written as

$$\frac{\partial\Psi}{\partial t} + \frac{\partial}{\partial x_i}(\phi_i^0 + \phi_i^I + \phi_i^{II} + \phi_i^{III}) = 0, \tag{4}$$

where Ψ, ϕ_i^0, ϕ_i^I and ϕ_i^{II} are defined in I. by Eqs. (2.7) and

$$\phi_i^{III} = \frac{a^3}{16}\int \chi(\psi' - \psi)\Big[f\frac{\partial^2 f_1}{\partial x_j \partial x_k} + f_1\frac{\partial^2 f}{\partial x_j \partial x_k} - 2\frac{\partial f}{\partial x_j}\frac{\partial f_1}{\partial x_k}\Big]k_i k_j k_k d\Gamma$$
$$+\frac{a^3}{48}\frac{\partial^2}{\partial x_j \partial x_k}\int \chi(\psi' - \psi) f f_1 k_i k_j k_k d\Gamma, \tag{5}$$

where $d\Gamma = a^2(g\cdot \mathbf{k})d\mathbf{k}\, d\mathbf{c}_1\, d\mathbf{c}$.

3. The five-field theory

We recall that a macroscopic state of the gas in the five-field theory is characterized by the fields of

$$\begin{cases} & \text{- mass density,} \\ , & \text{- velocity, and} \\ T = \frac{m}{3k\varrho}\int mC^2 f d\mathbf{c} & \text{- temperature.} \end{cases} \tag{6}$$

m is the mass of a fluid particle, $C_i = c_i - v_i$ the peculiar velocity and k the Boltzmann constant.

By choosing respectively ψ equal to m, me, and $mc^2/2$ in the transfer equation (4), we get the balance equations for the five scalar fields, which read

$$\frac{\partial \varrho}{\partial t} + \frac{\partial \varrho v_i}{\partial x_i} = 0, \tag{7}$$

$$\frac{\partial \varrho v_i}{\partial t} + \frac{\partial}{\partial x_j}(\varrho v_i v_j + p_{ij}^\star) = 0, \tag{8}$$

$$\frac{\partial}{\partial t}\Big(\frac{3}{2}\varrho\frac{k}{m}T + \frac{1}{2}\varrho v^2\Big) + \frac{\partial}{\partial x_i}\Big[\Big(\frac{3}{2}\varrho\frac{k}{m}T + \frac{1}{2}\varrho v^2\Big)v_i + p_{ji}^\star v_j + q_i^\star\Big] = 0. \tag{9}$$

In the above equations $p_{ij}^\star$ and $q_i^\star$ are the total pressure tensor and the total heat flux, respectively. Here they are defined by

$$p_{ij}^\star = p_{ij} + p_{ij}^I + p_{ij}^{II} + p_{ij}^{III}, \qquad q_i^\star = q_i + q_i^I + q_i^{II} + q_i^{III}. \tag{10}$$

p_{ij}, p_{ij}^I, p_{ij}^{II}, q_i q_i^I and q_i^{II} are given in I through Eqs. (3.1), (3.6) and (4.4) and

$$p_{ij}^{III} = \frac{a^3}{16}\int \chi m(c_i' - c_i)\Big[f\frac{\partial^2 f_1}{\partial x_k \partial x_l} + f_1\frac{\partial^2 f}{\partial x_k \partial x_l} - 2\frac{\partial f}{\partial x_k}\frac{\partial f_1}{\partial x_l}\Big]k_j k_k k_l d\Gamma$$
$$+\frac{a^3}{48}\frac{\partial^2}{\partial x_k \partial x_l}\int \chi m(c_i' - c_i) f f_1 k_j k_k k_l d\Gamma, \tag{11}$$

$$q_i^{III} = \frac{a^3}{16}\int \chi m\Big(\frac{c'^2}{2} - \frac{c^2}{2}\Big)\Big[f\frac{\partial^2 f_1}{\partial x_j \partial x_k} + f_1\frac{\partial^2 f}{\partial x_j \partial x_k} - 2\frac{\partial f}{\partial x_j}\frac{\partial f_1}{\partial x_k}\Big]k_i k_j k_k d\Gamma$$
$$+\frac{a^3}{48}\frac{\partial^2}{\partial x_j \partial x_k}\int \chi m\Big(\frac{c'^2}{2} - \frac{c^2}{2}\Big) f f_1 k_i k_j k_k d\Gamma. \tag{12}$$

The aim of this paper is the knowledge of the total pressure tensor and of the total heat flux as a functions of the fields of density, velocity and temperature and their gradients. In order to determine them we need to know the distribution $f(\mathbf{x},\mathbf{c},t)$ as function of the above mentioned fields. In the next section we shall determine the distribution function through two methods, i. e., the method of moments of Grad and the method of Chapman-Enskog.

4. The determination of the distribution function

A. The method of moments of Grad

Here we shall characterize a macroscopic state of the gas by the 13 scalar fields of

$$\begin{cases} \varrho = \int mf d\mathbf{c} & \text{- mass density,} \\ v_i = \frac{1}{\varrho}\int mc_i f d\mathbf{c} & \text{- velocity,} \\ p_{ij} = \int mC_iC_j f d\mathbf{c} & \text{- kinetic pressure tensor, and} \\ q_i = \int \frac{1}{2}mC^2C_i f d\mathbf{c} & \text{- kinetic heat flux.} \end{cases} \tag{13}$$

The balance equations for these fields are obtained from the equation of transfer (2.6) given in I, by choosing ψ equal to m, mc_i, mC_iC_j and $mC^2C_i/2$, respectively.

$$\frac{\partial\varrho}{\partial t}+\frac{\partial\varrho v_i}{\partial x_i}=0, \tag{14}$$

$$\frac{\partial\varrho v_i}{\partial t}+\frac{\partial}{\partial x_j}(\varrho v_iv_j+p_{ij}+p_{ij}^I+p_{ij}^{II})=0, \tag{15}$$

$$\frac{\partial p_{ij}}{\partial t}+\frac{\partial}{\partial x_k}(p_{ij}v_k+p_{ijk}+p_{ijk}^I+p_{ijk}^{II})+(p_{ik}+p_{ik}^I+p_{ik}^{II})\frac{\partial v_j}{\partial x_k}$$
$$+(p_{jk}+p_{jk}^I+p_{jk}^{II})\frac{\partial v_i}{\partial x_k}=P_{ij}+P_{ij}^I+P_{ij}^{II}, \tag{16}$$

$$\frac{\partial q_i}{\partial t}+\frac{\partial}{\partial x_j}(q_iv_j+q_{ij}+q_{ij}^I+q_{ij}^{II})+(q_j+q_j^I+q_j^{II})\frac{\partial v_i}{\partial x_j}+(p_{ijk}+p_{ijk}^I+p_{ijk}^{II})\frac{\partial v_j}{\partial x_k}$$
$$-\frac{p_{ij}}{\varrho}\frac{\partial}{\partial x_k}(p_{jk}+p_{jk}^I+p_{jk}^{II})-\frac{p_{rr}}{2\varrho}\frac{\partial}{\partial x_k}(p_{ik}+p_{ik}^I+p_{ik}^{II})=Q_i+Q_i^I+Q_i^{II}, \tag{17}$$

where p_{ijk}, p_{ijk}^I, q_{ij}, q_{ij}^I, P_{ij}, P_{ij}^I, Q_i and Q_i^I are defined in I by Eqs. (3.6) and

$$p_{ijk}^{II}=\frac{a^2}{4}\int\chi m(C_i'C_j'-C_iC_j)ff_1\frac{\partial}{\partial x_l}\Big(ln\frac{f}{f_1}\Big)k_kk_l d\Gamma-\frac{a^2}{4}\int\chi m\Big[\frac{\partial v_i}{\partial x_l}(c_j'-c_j)$$

$$+\frac{\partial v_j}{\partial x_l}(c_i'-c_i)\Big] f f_1 k_k k_l d\Gamma - \frac{a^2}{8}\frac{\partial}{\partial x_l}\int \chi m(C_i'C_j'-C_iC_j) f f_1 k_k k_l d\Gamma, \tag{18}$$

$$q_{ij}^{II} = \frac{a^2}{4}\int \chi m\Big(\frac{C'^2}{2}C_i'-\frac{C^2}{2}C_i\Big) f f_1 \frac{\partial}{\partial x_k}\Big(ln\frac{f}{f_1}\Big)k_j k_k d\Gamma$$
$$-\frac{a^2}{4}\int \chi m\Big[\frac{\partial v_i}{\partial x_k}\Big(\frac{C'^2}{2}-\frac{C^2}{2}\Big)+\frac{\partial v_r}{\partial x_k}(C_r'C_i'-C_rC_i)\Big] f f_1 k_j k_k d\Gamma$$
$$-\frac{a^2}{8}\frac{\partial}{\partial x_k}\int \chi m\Big(\frac{C'^2}{2}C_i'-\frac{C^2}{2}C_i\Big) f f_1 k_j k_k d\Gamma, \tag{19}$$

$$P_{ij}^{II} = -\frac{a^2}{8}\int \chi m\Big[\frac{\partial^2 v_i}{\partial x_k\partial x_l}(c_j'-c_j)+\frac{\partial^2 v_j}{\partial x_k\partial x_l}(c_i'-c_i)\Big] f f_1 k_k k_l d\Gamma$$
$$+\frac{a^2}{8}\int \chi m(C_i'C_j'-C_iC_j)\Big[\frac{\partial^2 f}{\partial x_k\partial x_l}f_1 + f\frac{\partial^2 f_1}{\partial x_k\partial x_l} - 2\frac{\partial f}{\partial x_k}\frac{\partial f_1}{\partial x_l}\Big]k_k k_l d\Gamma, \tag{20}$$

$$Q_i^{II} = -\frac{a^2}{8}\int \chi m\Big[\frac{\partial^2 v_i}{\partial x_j\partial x_k}\Big(\frac{C'^2}{2}-\frac{C^2}{2}\Big)+\frac{\partial^2 v_r}{\partial x_j\partial x_k}(C_i'C_r'-C_iC_r)\Big] f f_1 k_j k_k d\Gamma$$
$$+\frac{a^2}{8}\int \chi m\Big(\frac{C'^2}{2}C_i'-\frac{C^2}{2}C_i\Big)\Big[\frac{\partial^2 f}{\partial x_j\partial x_k}f_1 + f\frac{\partial^2 f_1}{\partial x_j\partial x_k} - 2\frac{\partial f}{\partial x_j}\frac{\partial f_1}{\partial x_k}\Big]k_j k_k d\Gamma. \tag{21}$$

The fluxes p_{ijk}, p_{ijk}^I, p_{ijk}^{II}, q_{ij}, q_{ij}^I, q_{ij}^{II}, p_{ij}^I, p_{ij}^{II}, q_i^I and q_i^{II} and the production terms P_{ij}, P_{ij}^I, P_{ij}^{II}, Q_i, Q_i^I and Q_i^{II} are evaluated by insertion of Grad's distribution function

$$f = \frac{\varrho}{m}\Big(\frac{m}{2\pi kT}\Big)^{\frac{3}{2}}\exp\Big(-\frac{mC^2}{2kT}\Big)\Big\{1+\Big(\frac{m}{kT}\Big)^2\frac{1}{2\varrho}\Big[p_{\langle ij\rangle}C_iC_j+2q_i\Big(\frac{mC^2}{5kT}-1\Big)C_i\Big]\Big\}, \tag{22}$$

into their definitions and integration. By neglecting the gradients of $p_{\langle ij\rangle}$ and q_i in the fluxes and all non-linear terms we get the results (3.10), (3.11) and (4.5) of I and

$$p_{ijk}^{II} = -\frac{48}{25\pi}\frac{k}{m}\frac{\mu^{id}}{\chi}\chi^2 b^2\varrho^2\Big(\frac{\partial T}{\partial x_k}\delta_{ij}+\frac{\partial T}{\partial x_j}\delta_{ik}+\frac{\partial T}{\partial x_i}\delta_{jk}\Big), \tag{23}$$

$$q_{ij}^{II} = -\frac{264}{25\pi}\frac{k}{m}T\frac{\mu^{id}}{\chi}\chi^2 b^2\varrho^2\Big(\frac{\partial v_{\langle i}}{\partial x_{j\rangle}}+\frac{5}{6}\frac{\partial v_r}{\partial x_r}\delta_{ij}\Big), \tag{24}$$

$$P_{ij}^{II}=0, \qquad Q_i^{II}=0, \tag{25}$$

where

$$\mu^{id} = \frac{5}{16a^2}\Big(\frac{kTm}{\pi}\Big)^{\frac{1}{2}}, \quad b=\frac{2\pi}{3}\frac{a^3}{m}. \tag{26}$$

μ^{id} is the coefficient of shear viscosity for an ideal gas of hard spherical particles.

Now we insert the above results into the balance equations (14)-(17) and get the following system of linearized field equations for ϱ, v_i, p_{ij} and q_i, viz.,

$$\frac{\partial \varrho}{\partial t}+\varrho\frac{\partial v_i}{\partial x_i}=0, \tag{27}$$

$$\varrho\frac{\partial v_i}{\partial t}+\frac{k}{m}\varrho(1+\chi b\varrho)\frac{\partial T}{\partial x_i}+\frac{k}{m}T\left(1+2\chi b\varrho+\frac{\partial \chi}{\partial \varrho}b\varrho^2\right)\frac{\partial \varrho}{\partial x_i}$$
$$+\left(1+\frac{2}{5}\chi b\varrho\right)\frac{\partial p_{\langle ij\rangle}}{\partial x_j}-\frac{96}{25\pi}\frac{\mu^{id}}{\chi}\chi^2b^2\varrho^2\left(\frac{\partial^2 v_{\langle i}}{\partial x_j\partial x_{j\rangle}}+\frac{5}{6}\frac{\partial^2 v_r}{\partial x_i\partial x_r}\right)=0, \tag{28}$$

$$\varrho\frac{k}{m}\frac{\partial T}{\partial t}+\frac{2}{3}\varrho\frac{k}{m}T(1+\chi b\varrho)\frac{\partial v_i}{\partial x_i}+\frac{2}{3}\left(1+\frac{3}{5}\chi b\varrho\right)\frac{\partial q_i}{\partial x_i}-\frac{16}{5\pi}\frac{k}{m}\frac{\mu^{id}}{\chi}\chi^2b^2\varrho^2\frac{\partial^2 T}{\partial x_i\partial x_i}=0, \tag{29}$$

$$\frac{\partial p_{\langle ij\rangle}}{\partial t}+2\varrho\frac{k}{m}T\left(1+\frac{2}{5}\chi b\varrho\right)\frac{\partial v_{\langle i}}{\partial x_{j\rangle}}+\frac{4}{5}\left(1+\frac{3}{5}\chi b\varrho\right)\frac{\partial q_{\langle i}}{\partial x_{j\rangle}}$$
$$-\frac{96}{25\pi}\frac{k}{m}\frac{\mu^{id}}{\chi}\chi^2b^2\varrho^2\frac{\partial^2 T}{\partial x_{\langle i}\partial x_{j\rangle}}=-\frac{\chi}{\mu^{id}}\varrho\frac{k}{m}Tp_{\langle ij\rangle}, \tag{30}$$

$$\frac{\partial q_i}{\partial t}+\frac{5}{2}\varrho\left(\frac{k}{m}\right)^2T\left(1+\frac{3}{5}\chi b\varrho\right)\frac{\partial T}{\partial x_i}+\frac{k}{m}T\left(1+\frac{3}{5}\chi b\varrho\right)\frac{\partial p_{\langle ij\rangle}}{\partial x_j}$$
$$-\frac{24}{25\pi}\frac{k}{m}T\frac{\mu^{id}}{\chi}\chi^2b^2\varrho^2\left(\frac{\partial^2 v_{\langle i}}{\partial x_j\partial x_{j\rangle}}+\frac{5}{6}\frac{\partial^2 v_r}{\partial x_i\partial x_r}\right)=-\frac{2}{3}\frac{\chi}{\mu^{id}}\varrho\frac{k}{m}Tq_i. \tag{31}$$

Equations (29) and (30) are the trace and the traceless part of Eq. (16), respectively.

Following the same scheme as in I, we shall use Eqs. (30) and (31) and an iterative method akin to the Maxwellían procedure in order to express $p_{\langle ij\rangle}$ and q_i as functions of the five scalar fields ϱ , v_i and T. For the first iteration step we insert the equilibrium values $p^{(0)}_{\langle ij\rangle}=0$ and $q^{(0)}_i=0$ into the left-hand side of Eqs. (30) and (31) and obtain the first iterated values $p^{(1)}_{\langle ij\rangle}$ and $q^{(1)}_i$ on the right-hand side, i. e.,

$$p^{(1)}_{\langle ij\rangle}=-2\frac{\mu^{id}}{\chi}\left(1+\frac{2}{5}\chi b\varrho\right)\frac{\partial v_{\langle i}}{\partial x_{j\rangle}}+\frac{96}{25\pi}\left(\frac{\mu^{id}}{\chi}\right)^2\frac{1}{\varrho T}\chi^2b^2\varrho^2\frac{\partial^2 T}{\partial x_{\langle i}\partial x_{j\rangle}}, \tag{32}$$

$$q^{(1)}_i=-\frac{15}{4}\frac{k}{m}\frac{\mu^{id}}{\chi}\left(1+\frac{3}{5}\chi b\varrho\right)\frac{\partial T}{\partial x_i}+\frac{36}{25\pi}\frac{1}{\varrho}\left(\frac{\mu^{id}}{\chi}\right)^2\chi^2b^2\varrho^2\left(\frac{\partial^2 v_{\langle i}}{\partial x_j\partial x_{j\rangle}}+\frac{5}{6}\frac{\partial^2 v_r}{\partial x_i\partial x_r}\right). \tag{33}$$

Now by inserting the first iterated values $p^{(1)}_{\langle ij\rangle}$ and $q^{(1)}_i$ into the left-hand side of Eqs. (30) and (31), the second iterated values follow on the right-hand side

$$p^{(2)}_{\langle ij\rangle}=-2\frac{\mu^{id}}{\chi}\left(1+\frac{2}{5}\chi b\varrho\right)\frac{\partial v_{\langle i}}{\partial x_{j\rangle}}+\frac{1}{\varrho T}\left(\frac{\mu^{id}}{\chi}\right)^2\left[1+\frac{4}{5}\chi b\varrho+\left(\frac{7}{25}+\frac{96}{25\pi}\right)\chi^2b^2\varrho^2\right]\frac{\partial^2 T}{\partial x_{\langle i}\partial x_{j\rangle}}$$

$$-\frac{2}{\varrho^2}\left(\frac{\mu^{id}}{\chi}\right)^2\left[1+\frac{12}{5}\chi b\varrho+\frac{4}{5}\chi^2 b^2\varrho^2+\left(1+\frac{2}{5}\chi b\varrho\right)\frac{\partial\chi}{\partial\varrho}b\varrho^2\right]\frac{\partial^2\varrho}{\partial x_{\langle i}\partial x_{j\rangle}}, \tag{34}$$

$$q_i^{(2)}=-\frac{15}{4}\frac{k}{m}\frac{\mu^{id}}{\chi}\left(1+\frac{3}{5}\chi b\varrho\right)\frac{\partial T}{\partial x_i}+\frac{3}{\varrho}\left(\frac{\mu^{id}}{\chi}\right)^2\left[1+\chi b\varrho+\left(\frac{6}{25}+\frac{12}{25\pi}\right)\chi^2 b^2\varrho^2\right]\frac{\partial^2 v_{\langle i}}{\partial x_j\partial x_{j\rangle}}$$
$$-\frac{15}{4}\frac{1}{\varrho}\left(\frac{\mu^{id}}{\chi}\right)^2\left[1+\frac{8}{5}\chi b\varrho+\left(\frac{3}{5}-\frac{8}{25\pi}\right)\chi^2 b^2\varrho^2\right]\frac{\partial^2 v_r}{\partial x_i\partial x_r}. \tag{35}$$

In order to get Eqs. (34) and (35) we have eliminated the time derivatives of v_i and T *by* the use of Eqs. (28) and (29). Moreover, in the former equations we have not considered gradients of order higher than two.

With the above results we can express Grad's distribution function in terms of the five scalar fields ϱ, v, and T. Indeed, insertion of Eqs. (34) and (35) into Eq. (22), leads to

$$f=\frac{\varrho}{m}\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}}\exp\left(-\frac{mC^2}{2kT}\right)\left\{1+\left(\frac{m}{kT}\right)^2\frac{1}{2\varrho}C_iC_j\left\{-2\frac{\mu^{id}}{\chi}\left(1+\frac{2}{5}\chi b\varrho\right)\frac{\partial v_{\langle i}}{\partial x_{j\rangle}}\right.\right.$$
$$+\frac{1}{\varrho T}\left(\frac{\mu^{id}}{\chi}\right)^2\left[1+\frac{4}{5}\chi b\varrho+\left(\frac{7}{25}+\frac{96}{25\pi}\right)\chi^2 b^2\varrho^2\right]\frac{\partial^2 T}{\partial x_{\langle i}\partial x_{j\rangle}}$$
$$\left.-\frac{2}{\varrho^2}\left(\frac{\mu^{id}}{\chi}\right)^2\left[1+\frac{12}{5}\chi b\varrho+\frac{4}{5}\chi^2 b^2\varrho^2+\left(1+\frac{2}{5}\chi b\varrho\right)\frac{\partial\chi}{\partial\varrho}b\varrho^2\right]\frac{\partial^2\varrho}{\partial x_{\langle i}\partial x_{j\rangle}}\right\}$$
$$+\left(\frac{m}{kT}\right)^2\frac{1}{\varrho}\left(\frac{mC^2}{5kT}-1\right)C_i\left\{-\frac{15}{4}\frac{k}{m}\frac{\mu^{id}}{\chi}\left(1+\frac{3}{5}\chi b\varrho\right)\frac{\partial T}{\partial x_i}-\frac{15}{4}\frac{1}{\varrho}\left(\frac{\mu^{id}}{\chi}\right)^2\left[1+\frac{8}{5}\chi b\varrho\right.\right.$$
$$\left.\left.\left.+\left(\frac{3}{5}-\frac{8}{25\pi}\right)\chi^2 b^2\varrho^2\right]\frac{\partial^2 v_r}{\partial x_i\partial x_r}+\frac{3}{\varrho}\left(\frac{\mu^{id}}{\chi}\right)^2\left[1+\chi b\varrho+\left(\frac{6}{25}+\frac{12}{25\pi}\right)\chi^2 b^2\varrho^2\right]\frac{\partial^2 v_{\langle i}}{\partial x_j\partial x_{j\rangle}}\right\}\right\}. \tag{36}$$

B. The method of Chapman-Enskog

In this section we shall look for the solution of the approximate Enskog equation

$$\mathbf{D}f=J^0(ff)+J^I(ff)+J^{II}(ff), \tag{37}$$

which corresponds to the third approximation of the single-particle distribution function

$$f(\mathbf{x},\mathbf{c},t)=f^{(0)}+f^{(1)}+f^{(2)}. \tag{38}$$

$f^{(0)}$ is the Maxwellian distribution function

$$f^{(0)} = \frac{\varrho}{m}\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \exp\left(-\frac{mC^2}{2kT}\right). \tag{39}$$

$f^{(1)}$ is the second approximation of $f(x,c,t)$ which is the solution of the following integral equation

$$f^{(0)}\Big\{-\frac{1}{\chi}\Big(1+\frac{3}{5}\chi b\varrho\Big)S^{(1)}_{\frac{3}{2}}(C^2)C_i\frac{\partial T}{\partial x_i} + \frac{1}{\chi}\Big(1+\frac{2}{5}\chi b\varrho\Big)\frac{m}{kT}S^{(0)}_{\frac{5}{2}}(C^2)C_iC_j\frac{\partial v_{\langle i}}{\partial x_{j\rangle}}\Big\} = J^0(f^{(0)}f^{(1)}), \tag{40}$$

where $S^{(n)}_m(C^2)$ denote Sonine polynomials. $f^{(1)}$ was first determined by Enskog[2] and its first approximation in the infinite series of Sonine polynomials reads

$$f^{(1)} = f^{(0)}\Big\{\frac{3}{2}\frac{\mu^{id}}{\chi}\frac{m}{\varrho kT^2}\Big(1+\frac{3}{5}\chi b\varrho\Big)S^{(1)}_{\frac{3}{2}}(C^2)C_i\frac{\partial T}{\partial x_i} - \frac{\mu^{id}}{\chi}\frac{m^2}{\varrho k^2T^2}\Big(1+\frac{2}{5}\chi b\varrho\Big)S^{(0)}_{\frac{5}{2}}(C^2)C_iC_j\frac{\partial v_{\langle i}}{\partial x_{j\rangle}}\Big\}. \tag{41}$$

The determination of $f^{(2)}$ proceeds by insertion of Eq. (38) into Eq. (37) and keeping only the highest terms on each side of the latter equation. Hence it follows

$$\mathbf{D}(f^{(0)}+f^{(1)}) = J^0(f^{(0)}f^{(1)}) + J^0(f^{(0)}f^{(2)}) + J^I(f^{(0)}f^{(0)}) + J^I(f^{(0)}f^{(1)}) + J^{II}(f^{(0)}f^{(0)}). \tag{42}$$

In Eq. (42) we have neglected all non-linear terms and all terms that lead to gradients of order higher than two.

The third approximation $f^{(2)}$ is constrained by the relation

$$\int \psi f^{(2)} d\mathbf{c} = 0. \tag{43}$$

Moreover, in order that $f^{(2)}$ be a solution of Eq. (42) it is necessary that

$$\int \psi \mathbf{D}(f^{(0)}+f^{(1)}) d\mathbf{c} = 0. \tag{44}$$

In Eqs. (43) and (44) ψ represents a summational invariant.

By choosing ψ equal to m, mc_i and $mc^2/2$ in Eq. (44) it follows, after rearrangement and linearization

$$\frac{\partial \varrho}{\partial t} + \frac{\partial \varrho v_i}{\partial x_i} = 0, \tag{45}$$

$$\varrho\Big(\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j}\Big) + \frac{\partial p^\star}{\partial x_i} - \eta\frac{\partial^2 v_j}{\partial x_i \partial x_j} - 2\mu\frac{\partial^2 v_{\langle i}}{\partial x_j \partial x_{j\rangle}} = 0, \tag{46}$$

$$\varrho c_v\Big(\frac{\partial T}{\partial t} + v_i\frac{\partial T}{\partial x_i}\Big) + p^\star\frac{\partial v_i}{\partial x_i} - \lambda\frac{\partial^2 T}{\partial x_i \partial x_i} = 0. \tag{47}$$

Equations (45)-(47) represent the linearized field equations of a Navier- Stokes and Fourier fluid. In these equations p^*, μ, η and λ denote, respectively, the pressure and the coefficients of shear viscosity, volume viscosity and thermal conductivity of a moderately dense gas of hard spherical particles. They are given by

$$p^\star = \varrho\frac{k}{m}T(1+\chi\varrho^\star), \tag{48}$$

$$\mu = \frac{\mu^{id}}{\chi}\Big[1 + \frac{4}{5}\chi\varrho^\star + \Big(\frac{4}{25} + \frac{48}{25\pi}\Big)(\chi\varrho^\star)^2\Big], \tag{49}$$

$$\eta = \frac{16}{5\pi}\frac{\mu^{id}}{\chi}(\chi\varrho^\star)^2, \tag{50}$$

$$\lambda = \frac{5}{2}c_v\frac{\mu^{id}}{\chi}\Big[1 + \frac{6}{5}\chi\varrho^\star + \Big(\frac{9}{25} + \frac{32}{25\pi}\Big)(\chi\varrho^\star)^2\Big]. \tag{51}$$

In the above equations $c_v = 3k/2m$ is the specific heat at constant volume and $\varrho^\star = b\varrho$ the reduced density.

Equations (40) and (45)-(47) are used to eliminate the $J^0(f^{(0)}f^{(1)})$ term and the time derivatives of ϱ, v_i and T from Eq. (42) resulting in a linearized integral equation, which reads

$$f^{(0)}\Big\{-\frac{2}{\varrho T}\frac{\mu^{id}}{\chi}\Big[\Big(1+\frac{3}{5}\chi b\varrho\Big)^2 S^{(2)}_{\frac{1}{2}}(C^2) + \frac{8}{5\pi}\chi^2 b^2\varrho^2 S^{(1)}_{\frac{1}{2}}(C^2)\Big]\frac{\partial^2 T}{\partial x_i \partial x_i} - \frac{1}{2}\frac{m}{\varrho k T^2}\frac{\mu^{id}}{\chi}\Big[\Big(1$$
$$+\frac{4}{5}\chi b\varrho + \frac{7}{25}\chi^2 b^2\varrho^2\Big)S^{(0)}_{\frac{5}{2}}(C^2) - 3\Big(1+\frac{33}{35}\chi b\varrho + \frac{36}{175}\chi^2 b^2\varrho^2\Big)S^{(1)}_{\frac{5}{2}}(C^2)\Big]C_iC_j\frac{\partial^2 T}{\partial x_{\langle i}\partial x_{j\rangle}}$$
$$+\frac{m}{\varrho^2 kT}\frac{\mu^{id}}{\chi}\Big[1+\frac{12}{5}\chi b\varrho + \frac{4}{5}\chi^2 b^2\varrho^2 + \Big(1+\frac{2}{5}\chi b\varrho\Big)\frac{\partial\chi}{\partial\varrho}b\varrho^2\Big]S^{(0)}_{\frac{5}{2}}(C^2)C_iC_j\frac{\partial^2\varrho}{\partial x_{\langle i}\partial x_{j\rangle}}$$
$$+\frac{4}{5}\frac{m}{\varrho kT}\frac{\mu^{id}}{\chi}\Big[\Big(1+\chi b\varrho + \frac{6}{25}\chi^2 b^2\varrho^2\Big)S^{(1)}_{\frac{3}{2}}(C^2) + \frac{24}{5\pi}\chi^2 b^2\varrho^2 S^{(0)}_{\frac{3}{2}}(C^2)\Big]C_i\frac{\partial^2 v_{\langle i}}{\partial x_j \partial x_{j\rangle}}$$

$$-\frac{m}{\varrho kT}\frac{\mu^{id}}{\chi}\left[\left(1+\frac{8}{5}\chi b\varrho+\frac{3}{5}\chi^2b^2\varrho^2\right)S_{\frac{3}{2}}^{(1)}(C^2)-\frac{16}{5\pi}\chi^2b^2\varrho^2S_{\frac{3}{2}}^{(0)}(C^2)\right]C_i\frac{\partial^2 v_r}{\partial x_i\partial x_r}$$
$$-\frac{m^2}{\varrho k^2T^2}\frac{\mu^{id}}{\chi}\left[1+\frac{4}{7}\chi b\varrho+\frac{12}{175}\chi^2b^2\varrho^2\right]S_{\frac{7}{2}}^{(0)}(C^2)C_{\langle i}C_jC_{k\rangle}\frac{\partial^2 v_i}{\partial x_j\partial x_k}\Big\}$$
$$-\frac{a^2}{2}\int\chi f^{(0)}f_1^{(0)}\Big\{\frac{m}{2kT^2}[(C^{1\prime})^2-(C^1)^2]\frac{\partial^2 T}{\partial x_i\partial x_j}+\frac{m}{kT}[C_r^{1\prime}$$
$$-C_r^1]\frac{\partial^2 v_r}{\partial x_i\partial x_j}\Big\}k_ik_ja^2(\mathbf{g}\cdot\mathbf{k})d\mathbf{k}d\mathbf{c}_1=J^0(f^{(0)}f^{(2)}). \tag{52}$$

The solution of the linearized integral equation (52) which satisfies the constraints (43) and corresponds to the first approximation in the infinite series of Sonine polynomials, can be written as

$$f^{(2)}=f^{(0)}\Big\{\underline{a_1S_{\frac{1}{2}}^{(2)}(C^2)\frac{\partial^2 T}{\partial x_i\partial x_i}}+a_2S_{\frac{5}{2}}^{(0)}(C^2)C_iC_j\frac{\partial^2 T}{\partial x_{\langle i}\partial x_{j\rangle}}+a_3S_{\frac{5}{2}}^{(0)}(C^2)C_iC_j\frac{\partial^2 \varrho}{\partial x_{\langle i}\partial x_{j\rangle}}$$
$$+a_4S_{\frac{3}{2}}^{(1)}(C^2)C_i\frac{\partial^2 v_r}{\partial x_i\partial x_r}+a_5S_{\frac{3}{2}}^{(1)}(C^2)C_i\frac{\partial^2 v_{\langle i}}{\partial x_j\partial x_{j\rangle}}+\underline{a_6S_{\frac{7}{2}}^{(0)}(C^2)C_{\langle i}C_jC_{k\rangle}\frac{\partial^2 v_i}{\partial x_j\partial x_k}}\Big\}. \tag{53}$$

The scalar coefficients a_1 through a_6 do not depend on $\mathbf{C}$ and are determined from the integral equation (52). Since the two underlined terms contribute neither to the total pressure tensor $p_{ij}^\star$ nor to the total heat flux $q_i^\star$, we shall delete them from the expression given by Eq. (53). The coefficients a_2 and a_3 can be found by insertion of Eq. (53) into Eq. (52), multiplication by $S_{\frac{5}{2}}^{(0)}(C^2)C_lC_m$ and integration over all values of $\mathbf{c}$. Hence it follows

$$a_2=\frac{1}{2}\frac{m^2}{\varrho^2k^2T^3}\left(\frac{\mu^{id}}{\chi}\right)^2\left[1+\frac{4}{5}\chi b\varrho+\left(\frac{7}{25}+\frac{96}{25\pi}\right)\chi^2b^2\varrho^2\right], \tag{54}$$
$$a_3=-\frac{m^2}{\varrho^3k^2T^2}\left(\frac{\mu^{id}}{\chi}\right)^2\left[1+\frac{12}{5}\chi b\varrho+\frac{4}{5}\chi^2b^2\varrho^2+\left(1+\frac{2}{5}\chi b\varrho\right)\frac{\partial\chi}{\partial\varrho}b\varrho^2\right], \tag{55}$$

Following the same procedure but multiplying the Eq. (52) by $S_{\frac{3}{5}}^{(1)}(C^2)C_l$ one can get the coefficients a_4 and a_5, viz.

$$a_4=\frac{3}{2}\frac{m^2}{\varrho^2k^2T^2}\left(\frac{\mu^{id}}{\chi}\right)^2\left[1+\frac{8}{5}\chi b\varrho+\left(\frac{3}{5}-\frac{8}{25\pi}\right)\chi^2b^2\varrho^2\right], \tag{56}$$
$$a_5=-\frac{6}{5}\frac{m^2}{\varrho^2k^2T^2}\left(\frac{\mu^{id}}{\chi}\right)^2\left[1+\chi b\varrho+\left(\frac{6}{25}+\frac{12}{25\pi}\right)\chi^2b^2\varrho^2\right]. \tag{57}$$

Now the insertion of Eqs. (54)-(57) into Eq. (53) and the substituition of $f^{(0)}$, $f^{(1)}$ and $f^{(2)}$ into Eq. (38) leads, after some rearrangement, to the same distribution function given by Eq. (36).

5. The determination of $p^\star_{ij}$ and $q^\star_i$

Once the distribution function is known, one can determine the total pressure tensor $p^\star_{ij}$ and the total heat flux $q^\star_i$. If we insert Eq. (36) into the definitions of p_{ij} through p^{III}_{ij} and q_i through q^{III}_i, integrate and leave out all non-linear terms and all gradients of order higher than two, we get

$$p^\star_{ij} = \left(p^\star - \eta\frac{\partial v_r}{\partial x_r} + \alpha_1\frac{\partial^2 \varrho}{\partial x_r\partial x_r} + \alpha_2\frac{\partial^2 T}{\partial x_r\partial x_r}\right)\delta_{ij} - 2\mu\frac{\partial v_{\langle i}}{\partial x_{j\rangle}} - 2\alpha_3\frac{\partial^2 \varrho}{\partial x_{\langle i}\partial x_{j\rangle}} + \alpha_4\frac{\partial^2 T}{\partial x_{\langle i}\partial x_{j\rangle}} \quad (58)$$

$$q^\star_i = -\lambda\frac{\partial T}{\partial x_i} + \beta_1\frac{\partial^2 v_{\langle i}}{\partial x_r\partial x_{r\rangle}} - \beta_2\frac{\partial^2 v_r}{\partial x_i\partial x_r}, \quad (59)$$

where

$$\alpha_1 = \frac{64}{25\pi}\frac{1}{\varrho^2}\left(\frac{\mu^{id}}{\chi}\right)^2\left[1 + \frac{1}{8}\frac{\varrho^\star}{\chi}\frac{\partial\chi}{\partial\varrho^\star}\right](\chi\varrho^\star)^3, \quad (60)$$

$$\alpha_2 = \frac{6}{5\pi}\frac{1}{\varrho T}\left(\frac{\mu^{id}}{\chi}\right)^2\left[1 + \frac{5}{3}\chi\varrho^\star\right](\chi\varrho^\star)^2, \quad (61)$$

$$\alpha_3 = \frac{1}{\varrho^2}\left(\frac{\mu^{id}}{\chi}\right)^2\left\{1 + \frac{14}{5}\chi\varrho^\star + \frac{44}{25}(\chi\varrho^\star)^2 + \left(\frac{8}{25} - \frac{192}{125\pi}\right)(\chi\varrho^\star)^3 + \left[1 + \frac{4}{5}\chi\varrho^\star + \left(\frac{4}{25} - \frac{24}{125\pi}\right)(\chi\varrho^\star)^2\right]\frac{\partial\chi}{\partial\varrho^\star}\varrho^{\star 2}\right\}, \quad (62)$$

$$\alpha_4 = \frac{1}{\varrho T}\left(\frac{\mu^{id}}{\chi}\right)^2\left[1 + \frac{6}{5}\chi\varrho^\star + \left(\frac{3}{5} + \frac{132}{25\pi}\right)(\chi\varrho^\star)^2 + \left(\frac{14}{125} + \frac{492}{125\pi}\right)(\chi\varrho^\star)^3\right], \quad (63)$$

$$\beta_1 = \frac{3}{\varrho}\left(\frac{\mu^{id}}{\chi}\right)^2\left[1 + \frac{8}{5}\chi\varrho^\star + \left(\frac{21}{25} + \frac{44}{25\pi}\right)(\chi\varrho^\star)^2 + \left(\frac{18}{125} + \frac{124}{125\pi}\right)(\chi\varrho^\star)^3\right], \quad (64)$$

$$\beta_2 = \frac{15}{4}\frac{1}{\varrho}\left(\frac{\mu^{id}}{\chi}\right)^2\left[1 + \frac{11}{5}\chi\varrho^\star + \left(\frac{39}{25} - \frac{8}{25\pi}\right)(\chi\varrho^\star)^2 + \left(\frac{9}{25} - \frac{8}{25\pi}\right)(\chi\varrho^\star)^3\right]. \quad (65)$$

By inspecting Eqs. (48)-(51)and (60)-(65), we note that the linearized Burnett equations for a rarefied gas[6] are recovered, if we put $\varrho^\star$ equal to zero in these equations. In this limit, the volume viscosity vanishes as well as the coefficients

α_1 and α_2. The two latter coefficients lead to effects that do not appear in the case of a rarefied gas. Indeed, by taking the trace of Eq. (58), one obtains for the total pressure $p^t = p^\star_{rr}/3$ the following expression

$$p^t = p^\star - \eta\frac{\partial v_r}{\partial x_r} + \alpha_1\frac{\partial^2\varrho}{\partial x_r\partial x_r} + \alpha_2\frac{\partial^2 T}{\partial x_r\partial x_r}. \tag{66}$$

On the other hand, Eqs. (60) and (61) show that the coefficients α_1 and α_2 are positive. Hence the pressure increases for a gas at rest ($v_r = 0$) and with a uniform density providing the Laplacian of the temperature is positive. The same occurs if the temperature is uniform and the Laplacian of the density is positive. The term corresponding to the second gradient of temperature will be called *thermal* (or *temperature) pressure*, and the other *density pressure.*

6. The propagation of plane harmonic waves

In this section we shall analyze the problem concerning the propagation of longitudinal plane harmonic waves of small amplitudes. We begin with the substitution of Eqs. (58) and (59) into the balance equations (7)-(9) and get a system of linearized field equations for ϱ, v, and T, which reads

$$\frac{\partial\varrho}{\partial t} + \varrho_0\frac{\partial v_i}{\partial x_i} = 0, \tag{67}$$

$$\varrho_0\frac{\partial v_i}{\partial t} + \left(\frac{\partial p^\star}{\partial\varrho}\right)_0\frac{\partial\varrho}{\partial x_i} + \left(\frac{\partial p^\star}{\partial T}\right)_0\frac{\partial T}{\partial x_i} - \left(\eta_0 + \frac{1}{3}\mu_0\right)\frac{\partial^2 v_j}{\partial x_i\partial x_j} - \mu_0\frac{\partial^2 v_i}{\partial x_j\partial x_j}$$
$$-\left(\frac{4}{3}\alpha_3^0 - \alpha_1^0\right)\frac{\partial^3\varrho}{\partial x_i\partial x_j\partial x_j} + \left(\frac{2}{3}\alpha_4^0 + \alpha_2^0\right)\frac{\partial^3 T}{\partial x_i\partial x_j\partial x_j} = 0, \tag{68}$$

$$\varrho_0 c_v\frac{\partial T}{\partial t} + p_0^\star\frac{\partial v_i}{\partial x_i} - \lambda\frac{\partial^2 T}{\partial x_i\partial x_i} + \left(\frac{2}{3}\beta_1^0 - \beta_2^0\right)\frac{\partial^3 v_i}{\partial x_i\partial x_j\partial x_j} = 0. \tag{69}$$

The index zero corresponds to a reference state of constant ϱ and T and vanishing v_r.

We look for plane wave solutions of the system of partial differential equations (67)-(69) which have the form

$$\varrho = \varrho_0 + \tilde\varrho\,\exp[i(\omega t - k^c x)],\quad T = T_0 + \tilde T\,\exp[i(\omega t - k^c x)],\quad v_x = \tilde v\,\exp[i(\omega t - k^c x)]. \tag{70}$$

Without loss of generality we treat only the one-dimensional case and take the x-axis as the direction of propagation. In Eq. (70) $w > 0$ is the circular frequency of the forced wave, $k^c = k^r + ik^i (k^r > 0)$ is the complex wave number and $\tilde{\varrho}, \tilde{T}$ and $\tilde{v}$ are complex amplitudes. The amplitudes are considered to be small such that their products can be neglected.

The dispersion relation is a relationship between the circular frequency ω and the complex wave number k^c of a plane harmonic wave. It is obtained by insertion of Eqs. (70) into the linearized field equations (67)-(69) and taking the determinant of the system of linear homogeneous equations for the amplitudes of the wave equal to zero.

$$\left(\frac{k^c}{\omega}v_0^\star\right)^6\left[\frac{\gamma}{\gamma-1}\left(\frac{D_A D_B}{D_V^4}\right)\frac{1}{Re^4}-i\left(\frac{D_C D_T}{D_V^3}\right)\frac{1}{Re^3}\right]+\left(\frac{k^c}{\omega}v_0^\star\right)^4\left[\left(\frac{D_T}{D_V}-\frac{D_A+D_B-D_C}{D_V^2}\right)\frac{1}{Re^2}-i\frac{1}{\gamma}\frac{D_T}{D_V}\frac{1}{Re}\right]+\left(\frac{k^c}{\omega}v_0^\star\right)^2\left[1+i\left(1+\frac{D_T}{D_V}\frac{1}{Re}\right)\right]-1=0, \tag{71}$$

where

$$v_0^\star=\left(\frac{5}{3}\frac{k}{m}T_0\right)^{\frac{1}{2}}\left[1+2\chi_0\varrho_0^\star+\frac{2}{5}(\chi_0\varrho_0^\star)^2+\frac{3}{5}\frac{\partial\chi_0}{\partial\varrho_0^\star}\varrho_0^{\star 2}\right]^{\frac{1}{2}}, \tag{72}$$

$$\gamma=1+\frac{2}{3}\frac{1+2\chi_0\varrho_0^\star+(\chi_0\varrho_0^\star)^2}{1+2\chi_0\varrho_0^\star+\frac{\partial\chi_0}{\partial\varrho_0^\star}\varrho_0^{\star 2}}, \tag{73}$$

$$D_A=\frac{p_0^\star}{\varrho_0^2 c_v}\left(\frac{2}{3}\alpha_4^0+\alpha_2^0\right),\qquad D_B=\frac{p_0^\star}{T_0\varrho_0^2 c_v}\left(\frac{2}{3}\beta_1^0-\beta_2^0\right),\qquad D_C=\left(\frac{4}{3}\alpha_3^0-\alpha_1^0\right), \tag{74}$$

$$D_T=\frac{\lambda_0}{\varrho_0 c_v},\qquad D_V=\frac{1}{\varrho_0}\left(\frac{4}{3}\mu_0+\eta_0\right),\qquad Re=\frac{v_0^{\star 2}}{D_V\omega}. \tag{75}$$

In the above equations $v_0^\star$ is the adiabatic speed of sound, 7 the specific heat ratio, D_V the longitudinal kinematic viscosity, D_T the thermal diffusivity and Re the Reynolds number. The coefficients D_A, D_B and D_C do not have proper names.

From the dispersion relation (71) one can get the phase speed $v = \omega/k^r$ and the attenuation coefficient a: $= -k^i$ of the wave. Here we shall give the solution of the dispersion relation which corresponds to the low frequency limit. Hence by expanding k^c/w in powers of $1/Re$ and retaining terms up to order two, it follows

$$v=v_0^\star\left\{1+\frac{1}{v_0^{\star 4}}\left[\frac{(3\gamma-7)(\gamma-1)}{8\gamma^2}D_T^2+\frac{5(\gamma-1)}{4\gamma}D_T D_V+\frac{3}{8}D_V^2-\frac{D_A+D_B-D_C}{2}\right]\omega^2\right\}, \tag{76}$$

$$\alpha = \frac{1}{2v_0^{\star 3}}\left(\frac{\gamma-1}{\gamma}D_T + D_V\right)\omega^2. \tag{77}$$

If we put $D_A = D_B = D_C = 0$ in the above equations, we recover the results of the Navier-Stokes and Fourier theory.

As was remarked in Ref. 9 the speed of forced waves given by Eq. (76) together with the speed of free sound waves and with the speed that follows from light scattering experiments (Eqs. (36) and (55) of Ref. 9) lead to three different ways of measuring the dispersion of sound waves in fluids.

7. Final remarks

We have determined the linearized Burnett equations for a gas of hard spherical particles from Enskog's dense gas theory.

The expressions for the pressure tensor $p_{ij}^\star$ and for the heat flux $q_i^\star$ given by Eqs. (58) and (59) have the same form as those of a phenomenological theory based on an extended thermodynamic theory of dense gases.[10] On the other hand, the transport coefficients given by Eqs. (60)-(65) could differ from those of the so-called modified Enskog theory.[11] This was pointed out in Ref. 11, and it was confirmed in Ref. 12 that the two theories lead to different expressions for the transport coefficients of mixtures.

The determination of the linearized Burnett equations from the modified Enskog equation and the comparison of the transport coefficients of the two theories will the subject of a forthcoming paper.

Acknowledgment

One of the authors (G.M. K.) gratefully acknowledges the support by the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq).

Note added in proof

Recently it was proved[13] that the transport coefficients for the linearized Burnett equations that follow from the Enskog equation are the same as those that follow from the so-called modified Enskog theory.

References

1. G. M. Kremer and E. Rosa Jr., J. Chem. Phys. 89, 3240 (1988).
2. D. Enskog, translation in ***Kinetie Theory*** edited by S. G. Brush (Pergamon, New York, 1972), Vol. 3, pp. 226-259.
3. H. Grad, Commun. Pure Appl. Math. 2, 331 (1949).
4. E. Ikenberry and C. Truesdell, J. Rational Mech. Anal. 5, 1 (1956).
5. J. C. Maxwell, Phil. Trans. Roy. Soc. 170, 331 (1879).
6. D. Burnett, Proc. Lond. Math. Soc. 40, 382 (1935).
7. S. Chapman and T. G. Cowling, ***The Mathematical Theory of Non-Uniform*** Gases (Cambridge University Press, Cambridge, 1953).
8. Recently it was proved that the methods of Chapman-Enskog and Grad are equivalent with respect to the successive approximations to the transport coefficients for rarefied gases in the second approximation to the distribution function (see S. Reinecke and G. M. Kremer, Phys. Rev. A 42, 815 (1990), and S. Reinecke and G. M. Kremer, A ***Generalization of the Chapman-Enskog and Grad Methods***, Continuum Mech. Thermodyn. (in press)).
9. S. R. Alves and G. M. Kremer, Physica A 164,759 (1990).
10. I-Shih Liu and G. M. Kremer, Mat. Aplic. Comp. 9, 123 (1990).
11. H. van Beijeren and M. H. Ernst, Physica 68, 437 (1973).
12. M. López de Haro, E. G. D. Cohen and J. M. Kincaid, J. Chem. Phys. 78, 2746 (1983).
13. M. López de Haro, private communication.

Resumo

A partir da teoria de Enskog para gases densos e dos métodos de Chapman-Enskog e Grad, determina-se a terceira aproximação para a função de distribuição de um gás monoatômico moderadamente denso constituído de partículas esféricas rígidas. As equações linearizadas de Burnett, obtidas através da terceira aproximação para a função de distribuição, indicam que o tensor tensão contém dois termos que não aparecem no caso de um gás rarefeito. Estes estão relacionados aos Laplacianos de temperatura e densidade e são denominados de pressão térmica e pressão de densidade. A velocidade de fase e o coeficiente de atenuação para ondas harmônicas planas com pequenas amplitudes são determinadas no limite de baixas freqüências.