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Stability of Parisi solutions for the clock spin glass

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Abstract The infinite-range pstate clock spin glass is studied within Parisi's replica-symrnetry-breaking scheme. A simplified stability analysis of these solutions is performed by taking into account longitudinal fluctuations only. It is shown that for the case *p=3,* the simple step-function solution is stable, whereas for all *p* # 3 the conventional Parisi solutions lead to a marginal stability. It is argued that such a picture should remain true in a more general analysis.

1. Introduction

The mean-field theory for the Ising spin glass[1] is well established nowadays through the understanding of the infinite-range-interaction model proposed by Sherrington and Kirkpatrick (SK)[2-6]. The one-parameter theory as introduced by SK[2], presented a negative entropy at zero temperature, and was shown to be unstable in the spin-glass phase[3]. The Parisi replica-symmetry-breaking scheme[4,5] is, at the moment, accepted as the correct solution for this problem: a continuous and monotonically increasing function defined in the interval [0,1] is used to describe the low-temperature phase, i.e., one has an infinite number of order parameters. At zero temperature, the free energy presents a highly non-trivial structure with many valleys separated by barriers which diverge in the thermodynamic limit[7]. Surprisingly, any three states chosen at random are restricted to an ultrametric condition[8]: by taking their respective distances in phase space, one can only construct triangles which are always equilateral or isosceles, in which case, the different size must be the smaller one[6].

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Although a big controversy remains as to whether such features are present in real systems[9], recent applications of this mean-field theory in the areas of neural networks and optimization problems give a lot of encouragement in pursuing the study of infinite-range spin-glass models; one is readily tempted to generalize the SK model in order to include spin variables other than the Ising ones.

The generalization to the infinite-range m-vector spin glasses, in what concerns replica-symmetry breaking, gives results which are very similar to the SK model, i.e., the Parisi functions are continuous and monotonically increasing[10]. All models studied so far, in which the spin variables are symmetric under reflection $(-\vec{S}_i \in \{\vec{S}_i\})$, showed such *conoentional* behaviour in their order-parameter functions.

However, the same procedure when applied to systems in which the spins do not present symmetry under reflection, like quadrupolar[11] and Potts glasses[12–14], turned out to provide rather surprising behaviour. Discontinuities in the Parisi functions as well as first-order phase transitions were observed.

In order to investigate whether such *unconventional* effects are peculiar to Potts and quadrupolar glasses only, or if they also happen on other systems, Nobre and Sherrington[15] studied the infinite-range pstate clock spin-glass model. In this problem, the spins present (do not present) reflection symmetry for every even (odd) value of *p*. It was shown that the case $p = 3$ is very special; only for this value of *p* one finds the peculiar behaviour already observed for Potts and quadrupolar glasses. The absence of refiection symmetry was qualitatively irrelevant for all other odd-state clock glasses, which presented the usual continuous monotonically increasing order function. One expects the conventional Parisi solutions to pass the stability tests for all $p \neq 3$, at least marginally as in the case of the SK model[16,17], whereas the step-function as proposed for $p = 3$ [14], deserves further investigation.

In this paper we examine the stability of the Parisi solutions for the *p*-state clock spin glass, restricting our analysis to the longitudinal sector as done by Thouless et al.[16] for the SK model. In section 2 we define the model and apply the replica method to it. In section 3 we consider the first stage towards Parisi's replica-symmetry-breaking scheme by means of a step-function. We show that such a solution is unstable for all $p \neq 3$, but surprisingly, for $p = 3$, it leads, within this simplified analysis, to a full stability for temperatures just below the spin-glass critical temperature. In section 4 we consider the full Parisi scheme which leads to conventional solutions for all $p \geq 4$; in particular, for $p = 4$, two possible, but similar solutions, in what concern replica-symmetry breaking, are discussed. Whenever the conventional Parisi function is applicable, one obtains a marginal stability only, like in the SK case[16,17]. Finally, in section 5 we present our conclusions.

2. The model and the replica formalism

Let us consider the pstate clock spin glass as defined by the hamiltonian

$$H = -\sum_{(ij)} J_{ij} \vec{S}_i \cdot \vec{S}_j, \tag{2.1}$$

where the $\vec{S}_i$ are unit vectors restricted to p orientations in a plane, with components given by

$$S_{ix} = \cos\theta_i \quad , \quad S_{iy} = \sin\theta_i, \tag{2.2a}$$

$$\theta_i = \frac{2\pi}{p} k_i \quad (k_i = 0, 1, ..., (p-1)). \tag{2.2b}$$

Similarly to the SK model[2], one has infinite-range interactions, i.e., the summation in (2.1) is over all pairs (ij), and the $\{J_{ij}\}$ are quenched random couplings following a gaussian probability distribution,

$$P(J_{ij}) = (N/2\pi J^2)^{1/2} \exp(-N J_{ij}^2 / 2J^2). \tag{2.3}$$

As is well known for the quenched case, one should average the free energy over the disorder; this is usually done by means of the replica trick[18],

$$-\beta[F]_{av} = [\ln Z]_{av} = \lim_{n\to 0} \frac{1}{n}([Z^n]_{av} - 1), \tag{2.4}$$

where Z^n is defined, for integer n, as the partition function of n independent and equivalent replicas of the original system defined in the hamiltonian (2.1), and $[\;]_{av}$ stands for an average over the disorder. The analytic continuation from integer n to $n = 0$ is one of the main difficulties found in infinite-range spin-glass problems;

for the *SK* model this was only solved satisfactorily by Parisi's replica-symmetry-breaking scheme[4].

It is important to remember, at this point, that one is interested in the thermodynamic limit, $N \to \infty$, and it is very convenient in this case to use the steepest descent method in order to evaluate $[Z^n]_{av}$. Strictly speaking, the $n \to 0$ limit must be taken before the $N \to \infty$ and although no rigorous proof exists, it is usually assumed that these two limits can be freely interchanged. For some years, it was suspected that this interchange of limits was responsible for the failure of the SK solution at low temperatures[19], but it is now believed that this does not really cause trouble. Then, interchanging the limits, $[Z^n]_{av}$ can be evaluated by steepest descent[2], and the free energy per spin, $f = [F]_{av}/N$, will be given, in the thermodynamic limit, by

$$\beta f = \lim_{n\to 0} \frac{1}{n} min[g(R^\alpha, \{Q^{\alpha\beta}_{\mu\nu}\})]. \tag{2.5}$$

In the equation above, the functional $g(R^a, \{Q^{\alpha\beta}_{\mu\nu}\})$ is given by

$$g(R^\alpha, \{Q^{\alpha\beta}_{\mu\nu}\}) = -\frac{n}{8}(\beta J)^2 + \frac{(\beta J)^2}{2}\sum_\alpha (R^\alpha)^2 + \frac{(\beta J)^2}{2}\sum_{(\alpha\beta)}\sum_{\mu\nu}(Q^{\alpha\beta}_{\mu\nu})^2$$
$$- \ln \mathrm{Tr} \exp\{H_{eff}\}, \tag{2.6a}$$

where

$$H_{eff} = (\beta J)^2 \sum_\alpha R^\alpha[(S^\alpha_x)^2 - 1/2] + (\beta J)^2 \sum_{(\alpha\beta)}\sum_{\mu\nu} Q^{\alpha\beta}_{\mu\nu} S^\alpha_\mu S^\beta_\nu . \tag{2.6b}$$

As is well known for infinite-ranged models, the problem is reduced to a single-site one; therefore, we discarded the site index for simplicity; μ, ν denote cartesian components (x, y), whereas α and β are replica labels; $\alpha, \beta = 1, ..., n$. The summations $\sum_{(\alpha\beta)}$ stand for sums over distinct pairs of replicas, $\alpha \neq \beta$. The quadrupolar parameter, R^α, which is a measure of anisotropy in the replicated spin space, as well as the spin-glass ones, $Q^{\alpha\beta}_{\mu\nu}$, are determined by extremizing the functional $g(R^\alpha, \{Q^{\alpha\beta}_{\mu\nu}\})$. They are given respectively by

$$R^\alpha = < (S^\alpha_x)^2 > -1/2, \tag{2.7a}$$

$$Q^{\alpha\beta}_{\mu\nu} = < S^{\alpha}_{\mu} S^{\beta}_{\nu} > \quad ; \quad \alpha \neq \beta \qquad (2.7b)$$

where the < > brackets denote thermal averagings with respect to the effective hamiltonian H_{eff}. For p = 2, the above model reduces to the SK spin glass whose solution is well understood nowadays, and R^a is a trivial constant. For the remainder of this paper we shall restrict ourselves to p > 2.

The next step now is to find the correct solutions of equations (2.7) in the limit n → 0. It is obvious that H_{eff} is invariant under permutations of replica indices, as long as n is a positive integer; however, what is not obvious is that this symmetry is preserved when one takes n → 0. This leads to the point of finding a particular parametrization for R^a and $Q^{\alpha\beta}_{\mu\nu}$ which gives sensible physics after n → 0.

If no external fields are present, one expects in general that, on the average, the system will be isotropic in spin space, and the solutions of (2.7) simplify a lot by assuming the isotropic conditions,

$$R^{\alpha} = 0, \qquad (2.8a)$$

$$Q^{\alpha\beta}_{\mu\nu} = Q^{\alpha\beta} \delta_{\mu\nu}, \qquad (2.8b)$$

which means that all directions in spin space are equivalent*. Within this space of solutions, the free-energy functional in equations (2.6) may be written as

$$g(Q^{\alpha\beta}) = -\frac{n}{8}(\beta J)^2 + (\beta J)^2 \sum_{(\alpha\beta)} (Q^{\alpha\beta})^2 - \ln \mathrm{Tr} \exp\{\mathrm{H}_{\text{eff}}\}, \qquad (2.9a)$$

$$H_{\text{eff}} = (\beta J)^2 \sum_{(\alpha\beta)} Q^{\alpha\beta} (S^{\alpha}_{x} S^{\beta}_{x} + S^{\alpha}_{y} S^{\beta}_{y}), \qquad (2.9b)$$

where

$$Q^{\alpha\beta} = \frac{1}{2} < S^{\alpha}_{x} S^{\beta}_{x} + S^{\alpha}_{y} S^{\beta}_{y} > . \qquad (2.10)$$

Since all replicas are equivalent, what appears naturally as a first solution to be tried, is the Replica-Symmetric (RS) one[2],

$$Q^{\alpha\beta} = \mathrm{Q} \quad \text{for all} \quad \alpha \neq \beta. \qquad (2.11)$$

* In section 4 we wil discuss a highly anisotropic solution which is also possible for the case p=4 .

Such a solution leads to a phase transition from a paramagnetic ($T > T_g$, $Q = 0$) to a spin-glass state ($T < T_g$, $Q \neq 0$) at a critical temperature[15],

$$T_g = \frac{J}{2} \quad ; \quad (p > 2). \tag{2.12}$$

As usual for spin glasses, one gets that f is a ***maximum*** with respect to Q in both states such that for $T < T_g$, the spin-glass solution ($Q \neq 0$) presents a ***higher*** free energy than the $Q = 0$ solution, contrary to what normally happens in other systems. The explanation for this, comes from the fact that the number of parameters $Q^{\alpha\beta}$, $n(n-1)/2$, becomes *negative* in the limit $n \to 0$. This is responsible for changing the minirnum in equation (2.5) into a maximum condition. Unfortunately, the solution (2.11) is unstable below T_g, but the Parisi solution, which is believed to be the correct one, presents an even higher free energy than the RS solution. The minimum condition in (2.5) only makes sense when seen as a local stability condition, that is, minimum with respect to each one of the $Q^{\alpha\beta}$ parameters. This is done by requiring the stability matrix $\underline{\Sigma}$, with elements[3],

$$\Sigma^{\alpha\beta\gamma\delta} = \frac{\partial^2 g}{\partial Q^{\alpha\beta} \partial Q^{\gamma\delta}}, \tag{2.13}$$

to be positive definite, i.e. all its eigenvalues should be positive for stability.

The Parisi *ansatz*[4] consists in a hierarchical process in which the diagonal blocks of the matrix $\underline{Q}$ defined by the elements (2.10), are broken into subblocks; the procedure is repeated for the diagonal subblocks and so on. At each step a different parameter is introduced; that gives in the limit $n \to 0$, a function $Q(x)$ defined in the interval $[0,1]$, i.e., an infinite number of order parameters. The free energy becomes a functional $f[Q]$, and in order to find the shape of $Q(x)$, one needs to solve the extremal equation,

$$\frac{\delta(\beta f[Q])}{\delta Q(t)} = 0. \tag{2.14}$$

Since the order parameter is itself a function, the above equation presents a dependente on its argument; one can now take the derivative of (2.14) with respect to this argument,

$$\frac{d}{dt} \frac{\delta(\beta f[Q])}{\delta Q(t)} = Q'(t) \Phi[Q] = 0. \tag{2.15}$$

Equation (2.15) has two types of solutions,

$$1) \quad Q'(t) \neq 0; \quad \Phi[Q] = 0 \quad , \tag{2.16a}$$

$$2) \quad Q'(t) = 0; \quad \Phi[Q] \neq 0. \tag{2.16b}$$

Solutions of type **2** are the replica-symmetric ones which are known to be unstable; breaking the replica symmetry implies searching for type-1 solutions. For such a solution to be accepted, a maximum of the free-energy functional will be required, i.e.,

$$\Sigma[Q] = \frac{\delta^2(\beta f[Q])}{\delta Q(s)\delta Q(t)}, \tag{2.17}$$

which may be expressed in general, as

$$\Sigma[Q] = \delta(t-s)\Phi[Q] + \Omega[Q], \tag{2.18}$$

must be negative definite. That means the eigenvalue equation,

$$\int_0^1 ds\ f(s)\Sigma[Q] = \lambda f(t), \tag{2.19}$$

should present no positive eigenvalues for stability.

In what follows, we shall discuss the application of Parisi's replica-symmetry-breaking scheme to the p-state clock spin glass, restricting ourselves to temperatures $T < T_g$ and small τ,

$$\tau = 1 - \frac{T}{T_g} = 1 - \frac{2T}{J}. \tag{2.20}$$

The relavant power-series expansions for small τ are shown explicitly in the Appendix. We start in the simplest level, proposing a step-function defined in the interval $[0,1]$ as a solution. We show that this is stable for $p = 3$, but unstable for all other values of p. Next, we carry the full Parisi's scheme, showing that for all $p \neq 3$ a marginal stability is obtained for the conventional solution.

3. The step-function solution

In this section we discuss the step function as proposed initially for the Potts spin glass[14], as a possible solution of (2.14),

$$Q(t) = Q_m\theta(t - \bar{t}), \tag{3.1}$$

where

$$\theta(u)=\begin{cases}0, & u<0\\ 1, & u>0\end{cases}. \tag{3.2}$$

Except for the discontinuity at the breaking point, this is a type–2 solution in equations (2.16); it represents the first step towards Parisi's replica-symmetry-breaking scheme.

Near the spin-glass phase transition, one can substitute (3.1) into (A.1) to get the free energy as

$$\begin{aligned}\beta f(Q_m,\bar{t}) = &-\bar{g}_0+\frac{1}{2}\bar{A}_2Q_m^2(1-\bar{t})+\frac{1}{3}\bar{A}_3Q_m^3(1-\bar{t})-\frac{1}{3}\bar{B}_3Q_m^3(1-\bar{t})(2-\bar{t})\\ &+\frac{1}{12}\bar{A}_4Q_m^4(1-\bar{t})+\frac{1}{12}\bar{C}_4Q_m^4(1-\bar{t})^2-\frac{1}{12}\bar{D}_4Q_m^4(1-\bar{t})(2-\bar{t})\\ &+\frac{1}{12}\bar{B}_4Q_m^4(1-\bar{t})+\{3-3\bar{t}+(\bar{t})^2\}+O(\tau^5).\end{aligned} \tag{3.3}$$

The equilibrium conditions,

$$\frac{\partial(\beta f(Q_m,\bar{t}))}{\partial Q_m}=0 \quad ; \quad \frac{\partial(\beta f(Q_m,\bar{t}))}{\partial \bar{t}}=0\,, \tag{3.4}$$

gives respectively,

$$\begin{aligned}\bar{A}_2+\bar{A}_3Q_m-\bar{B}_3Q_m(2-\bar{t})+\frac{1}{3}\bar{A}_4Q_m^2+\frac{1}{3}\bar{C}_4Q_m^2(1-\bar{t})-\frac{1}{3}\bar{D}_4Q_m^2(2-\bar{t})\\ +\frac{1}{3}\bar{B}_4Q_m^2\{3-3\bar{t}+(\bar{t})^2\}+O(\tau^3)=0,\end{aligned} \tag{3.5a}$$

$$\begin{aligned}&-\frac{1}{2}\bar{A}_2-\frac{1}{3}\bar{A}_3Q_m-\frac{1}{3}\bar{B}_3Q_m(-3+2\bar{t})-\frac{1}{12}\bar{A}_4Q_m^2+\frac{1}{12}\bar{C}_4Q_m^2(-2+2\bar{t})\\ &-\frac{1}{2}\bar{D}_4Q_m^2(-3+2\bar{t})+\frac{1}{12}\bar{B}_4Q_m^2\{-6+8\bar{t}-3(\bar{t})^2\}+O(\tau^3)=0.\end{aligned} \tag{3.5b}$$

By solving equations (3.5), comes

$$\bar{t}=\frac{\bar{A}_3}{\bar{B}_3}+O(\tau)=\frac{1}{2}\delta_{3,p}+O(\tau), \tag{3.6a}$$

$$Q_m=\frac{\bar{A}_2}{2(\bar{B}_3-\bar{A}_3)}+O(\tau^2)=\frac{\tau}{(2-\delta_{3,p})}+O(\tau^2), \tag{3.6b}$$

which gives a continuous phase transition at T_g for all p, i.e., Q_m goes continuously to zero as $T\to T_g$.

The next question one should address concerns the stability of the present solution. In order to do this, let us substitute *(3.1)* into the stability functional $\Sigma[Q]$ as defined in the previous section; in doing this, one gets,

$$\begin{aligned}\Phi(Q_m,\bar{t}) = {} & \bar{A}_2 + 2\bar{A}_3 Q_m \theta(t-\bar{t}) - 2\bar{B}_3 Q_m\{(1-\bar{t}) + \bar{t}\theta(t-\bar{t})\} + \bar{A}_4 Q_m^2 \theta(t-\bar{t}) \\ & + \frac{1}{3}\bar{C}_4 Q_m^2 (1-\bar{t}) - \frac{1}{6}\bar{D}_4 Q_m^2 \{(1-\bar{t}) + 6\theta(t-\bar{t})\} \\ & + \bar{B}_4 Q_m^2 \{(1-\bar{t})^2 + \bar{t}(2-\bar{t})\theta(t-\bar{t})\} + O(\tau^3),\end{aligned} \tag{3.7a}$$

$$\Omega(Q_m,\bar{t}) = -2\bar{B}_3 Q_m \theta(t-\bar{t})\theta(s-\bar{t}) + O(\tau^2). \tag{3.7b}$$

The quantity $\Phi(Q_m,\bar{t})$ can be decomposed in two parts, **i.e.**, for $t > \bar{t}$ and $t < \bar{t}$, respectively. Making use of equations (3.5), these may be expressed as

$$\Phi_{\pm} = \frac{1}{6}\bar{A}_4 Q_m^2 - \frac{1}{6}\bar{D}_4 Q_m^2 \bar{t} + \frac{1}{6}\bar{B}_4 Q_m^2 (\bar{t})^2 + O(\tau^3), \tag{3.8}$$

where the + (-) sign refers to $t > \bar{t}$ $(t < \bar{t})$. The stability functional becomes,

$$\Sigma[Q] = \{\Phi_+ \theta(t-\bar{t}) + \Phi_- \theta(\bar{t}-t)\}\delta(t-s) - 2\bar{B}_3 Q_m \theta(t-\bar{t})\theta(s-\bar{t}). \tag{3.9}$$

To find the eigenvalues associated to *(3.9)* one needs to solve equation *(2.19)*. In doing this, one gets the following eigenvalues and their corresponding eigenfunctions,

$$\begin{array}{ll} \Phi_+ & f_+(t), \\ \Phi_- & f_-(t)\ ;\ \kappa_-\theta(\bar{t}-t), \\ \Phi_+ - 2\bar{B}_3 Q_m (1-\bar{t}) & \kappa_+\theta(t-\bar{t}), \end{array}$$

where κ_+, κ_- are constants and $f_+(t)$ $(f_-(t))$ vanishes for $t < \bar{t}$ $(t > \bar{t})$, non-zero otherwise, restricted to

$$\int_0^1 dt f_+(t) = \int_0^1 dt f_-(t) = 0. \tag{3.10}$$

It is clear that the above eigenfunctions do form a complete set, since they are orthogonal to each other and any arbitrary function $f(t)$, $0 \le t \le 1$, may be expressed as a linear combination,

$$f(t) = f_-(t) + f_+(t) + \kappa_-\theta(\bar{t}-t) + \kappa_+\theta(t-\bar{t}). \tag{3.11}$$

Hence, one has the eigenvalues,

$$\lambda_1 = \frac{1}{6}\{\bar{A}_4 - \bar{D}_4\bar{t} + \bar{B}_4(\bar{t})^2\}Q_m^2 + O(\tau^3), \tag{3.12a}$$

$$\lambda_2 = -2\bar{B}_3 Q_m(1-\bar{t}) + O(\tau^2). \tag{3.12b}$$

It is interesting to note that for any $p \neq 3$, equation (3.6a) gives $\bar{t} = O(\tau)$ and then,

$$\lambda_1 = \frac{1}{6}\bar{A}_4 Q_m^2 + O(\tau^3) \quad ; \quad p \neq 3, \tag{3.13}$$

which is positive, signaling the instability of the step-function solution. However, for $p = 3$, one has $\bar{t} = 1/2 + O(\tau)$, and using the coefficients given in the Appendix (cf. equations (A.2)),

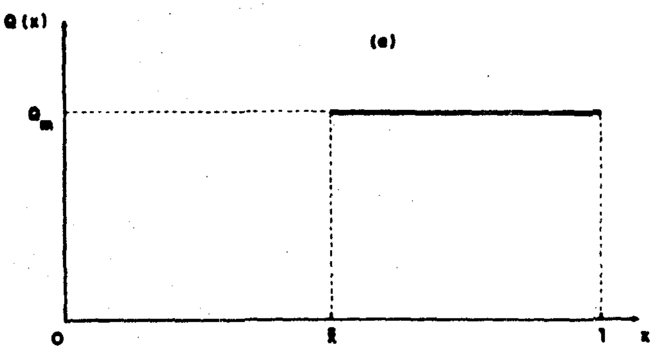
$$\lambda_1 = -\frac{(\beta J)^8}{128}Q_m^2 + O(\tau^3) \quad ; \quad \lambda_2 = -\frac{(\beta J)^6}{8}Q_m + O(\tau^2), \tag{3.14}$$

providing stability. Notice that there are no zero eigenvalues. This is to be contrasted with the marginal stability obtained from conventional Parisi solutions for the cases $p \neq 3$, as we discuss in the next section. One sees that the case $p = 3$ is very special in the sense that the first step in Parisi's replica-symmetry-breaking scheme, i.e., solution (3.1) (see Fig. 1(a)), is enough for stability.

4. The full replica-symmetry-breaking solutions

For $p = 3$, if one continues with the usual replica-symmetry-breaking process, searching for a conventional order-parameter function, as the one known for the SK model, one will find a negative slope for the region over which (2.16a) is valid[13,15]. According to the physical interpretation of the Parisi solution[5,21], this leads to a negative probability, being clearly incorrect. The correct solution for this problem is therefore, the step-function as discussed in the previous section.

For $p \geq 4$, the full replica-symmetry-breaking scheme is applicable as we discuss below; the absence of reflection symmetry in the spin variable for odd values of p is irrelevant and one finds the conventional solution in such cases. For $p = 4$, however, besides the isotropic solution (2.8), a highly anisotropic one[20], but qualitatively similar in what concerns replica-symmetry-breaking, is also possible, as considered next.



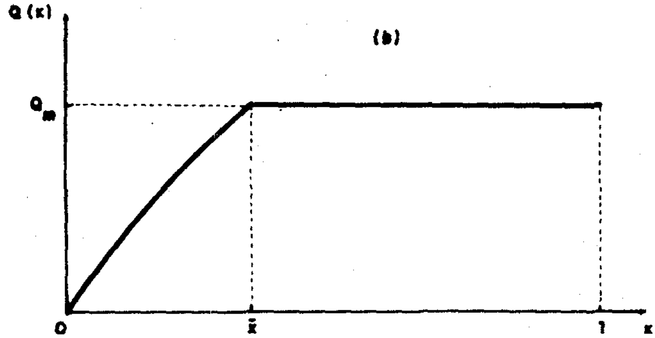


Fig. 1 – The Parisi functions for the pstate clock spin glasses just below the the freering temperature T_g: (a) case p = 3; (b) cases p $\neq$ 3. The height of the plateau (Q,), the breaking point ($\bar{x}$) and the slope ($Q'(x)$) (cases $p \neq 3$ only) are specified in Table 1 for each value of p.

4.a. p $\geq$ 4: the isotropic solution

The free-energy functional within the isotropic subspace of solutions (cf. equations (2.9)), may be expanded perturbatively near the spin-glass transition and the Parisi parametrization irnplemented; the resulting free-energy $\beta f[Q]$ can be seen in the Appendix (equation (A.l)). The shape of the order-parameter function can be obtained by successive differentiations of the extremal equation (2.14)[15]. That gives the function shown in Figure 1(b) and quantified in Table 1.

The stability of such salutions may be considered by solving the eigenvalue equation (2.19)[16,22], where the stability functional $\Sigma[Q]$ is given by (2.18), $\Phi[Q]$

in the Appendix (cf. equation (A.4)), and

$$\Omega[Q] = -2\bar{B}_3\{Q(s)\theta(t-s) + Q(t)\theta(s-t)\} + O(\tau^2), \tag{4.1}$$

with $\theta(u)$ being the usual Heaviside step function defined in (3.2).

Therefore, solutions of type 1 in (2.16) give

$$\Sigma[Q] = \Omega[Q], \tag{4.2}$$

which is clearly a non-positive quantity for the function $Q(x)$, as shown in Figure 1(b). In order to find the longitudinal eigenvalues of $\Sigma[Q]$, one substitutes (4.2) into (2.19),

$$-2\bar{B}_3 \int_0^t dsQ(s)f(s) - 2\bar{B}_3Q(t)\int_t^1 dsf(t) = \lambda f(s), \tag{4.3}$$

and differentiating the equation above one gets,

$$-2\bar{B}_3Q'(t)\int_t^1 dsf(s) = \lambda f'(t). \tag{4.4}$$

That gives for the plateau in Figure 1(b) ($Q'(t) = 0$), either $\lambda = 0$ or $f(t)$ a constant; therefore, making fluctuations which disturb this flat part, one gets a zero eigenvalue. For the case where $f(t) =$ constant for $t \geq \bar{t}$, one gets by evaluating (4.4) at $t = \bar{t}$,

$$\omega^2 f(\bar{t})(1-\bar{t}) = f'(\bar{t}) \quad ; \quad \omega^2 = -\frac{2\bar{B}_3Q'(t)}{\lambda}. \tag{4.5}$$

In the region where $Q'(t)$ is a positive constant, one can differentiate (4.4) to obtain,

$$f''(t) + \omega^2 f(t) = 0 \quad , \quad \text{i.e.,} \quad f(t) = \sin(\omega t). \tag{4.6}$$

Substituting the result (4.6) into (4.5) one obtains,

$$\text{cotan}(\omega\bar{t}) = \omega(1-\bar{t}), \tag{4.7}$$

which can be solved to give the eigenvalues,

$$\lambda = -2\bar{B}_3[Q'(t)]\bar{t} + O(\tau^2)\ , \tag{4.8a}$$

$$\lambda = -\frac{2\bar{B}_3[Q'(t)]}{m^2\pi^2}(\bar{t})^2 + O(\tau^3) \ ; \ (m = 1,2,3,...), \tag{4.8b}$$

Table 1 – Characteristics of the Parisi functions for the pstate clock spin glasses as shown in Figure 1; $Q'(x)$, Q_m and $\bar{x}$ are given to leading order in r ($\tau = (T_g - T)/T_g$; $T_g = J/2$).

pstate clock spin glass	$Q'(x)$	Q_m	$\bar{x}$	Type of Solution	Stability
p = 2	$\frac{1}{2}$	r	27	Conventional Fig. 1(b)	Marginal
$p = 3$	-	τ	$\frac{1}{2}$	Step-Function Fig. 1(a)	Stable
p = 4 (Isotropic)	$\frac{1}{4}$	$\frac{\tau}{2}$	2τ	Conventional Fig. 1(b)	Marginal
p = 4 (Collinear)	1	τ	r	Conventional Fig. 1(b)	Marginal
P ≥ 5	$\frac{1}{3}$	$\frac{\tau}{2}$	$\frac{3}{2}\tau$	Conventional Fig. 1(b)	Marginal

being all negative as required for stability. However, due to fluctuations around its plateau, which lead to a zero eigenvalue, one gets that the conventional Parisi solution as shown in Figure 1(b), is only marginally stable for all p ≥ 4.

4.b. $p = 4$: the anisotropic solution

In the preceding sections we treated the pstate clock spin glass, in the absence of externa1 fields, within the isotropic subspace of solutions (equations (2.8)), assuming all directions in spin space to be equivalent. However, by considering the 4-state clock glass in terms of two independent Ising models, Nobre, Sherrington and Young[20] have argued that besides (2.8), a spontaneously-anisotropic, i.e., a *collinear* solution, is also possible. That state may be induced by applying a small symmetry-breaking field which is taken to zero after the limit N → ∞. Such a solution may be represented in replica space as,

$$R^{\alpha} \neq 0 \tag{4.9a}$$

$$Q^{\alpha\beta}_{\mu\nu} = Q^{\alpha\beta}\delta_{\mu\nu}\delta_{\mu x} \quad \text{or} \quad Q^{\alpha\beta}\delta_{\mu\nu}\delta_{\mu y}, \tag{4.9b}$$

depending on whether the small symmetry-breaking field is chosen to be initially applied in the x− or y−direction, respectively. By choosing a collinear solution in the x-direction, one gets after substituting (4.9) into (2.6),

$$g(R^\alpha, Q^{\alpha\beta}) = -\frac{n}{8}(\beta J)^2 + \frac{(\beta J)^2}{2}\sum_\alpha (R^\alpha)^2 + \frac{(\beta J)^2}{2}\sum_{(\alpha\beta)}(Q^{\alpha\beta})^2 - \ln \mathrm{Tr}\exp\{H_{\text{eff}}\}, \tag{4.10a}$$

where

$$H_{\text{eff}} = (\beta J)^2 \sum_\alpha R^\alpha[(S_x^\alpha)^2 - 1/2] + (\beta J)^2 \sum_{(\alpha\beta)} Q^{\alpha\beta} S_x^\alpha S_x^\beta. \tag{4.10b}$$

As before, the free-energy functional in (4.10) may be expanded perturbatively for temperatures just below T_g; one can readily see a simultaneous ordering of the quadrupolar parameters R^α, together with the spin-glass ones, $Q^{\alpha\beta}$, suggesting a spontaneous anisotropy. As is well known, parameters depending on a single-replica index can be taken in the replica symmetry approximation, and so, the parameter R^a does not cause any trouble in what concerns replica-symmetry breaking. The Parisi scheme can be implemented for the spin-glass parameters $Q^{\alpha\beta}$ as usual, to give the free-energy functional $f[R,Q]$ as shown in the Appendix (equation (A.5)). The extremal equations,

$$\frac{\delta(\beta f[R,Q])}{\delta R} = 0 \quad ; \quad \frac{\delta(\beta f[R,Q])}{\delta Q(t)} = 0, \tag{4.11}$$

may be solved so as to eliminate the parameter R; that gives an equilibrium equation which depends only on $Q(t)$, as seen in (A.7). As before, the same procedure may be followed in order to get a function as shown in Figure 1(b) and described quantitatively in Table 1. Also, a stability analysis may be carried out to give, besides a zero eigenvalue,

$$\lambda = -6D_3[Q'(t)]\bar{t} + O(\tau^2) = -8\tau + O(\tau^2), \tag{4.12a}$$

$$\lambda = -\frac{6D_3[Q'(t)]}{m^2\pi^2}(\bar{t})^2 + O(\tau^3) = -\frac{8}{m^2\pi^2}\tau^2 + O(\tau^3) \;;\; (m = 1,2,3,...) \tag{4.12b}$$

leading to a marginal stability like in the case of the isotropic solution.

This marginality for the conventional Parisi solutions ($p \geq 4$), comes as a direct consequence of fluctuations around the flat part for $x \geq \bar{x}$ (see Figure 1(b)). The zero eigenvalue, responsible for that, should also be present in a more general analysis, similarly to what happens for the SK model[17].

5. Conclusion

We have studied the *p*-state clock spin glass in Parisi's replica-symmetry-breaking formalism. A stability analysis of such solutions was carried restricted to the longitudinal sector, as the one done by Thouless et al.[16] for the SK model. It was shown that the simple step-function solution is stable for the case $p = 3$, whereas for all other values of p the conventional Parisi solutions lead only to a marginal stabiiity. We believe this picture, revealed by such a simple analysis, remains true even in a more general situation, in which one takes into account transversal fluctuations, as done by de Dorninicis and Kondor[17] for the SK model. For $p \geq 4$, the zero eigenvalue should also appear as a consequence of a disturbance of the flat portion of the conventional Parisi solution, similarly to the Ising case, and the marginality should persist. For $p = 3$ the full stability is ensured by means of a *p*-state Potts spin-glass general analysis[23] which states the step-function to be stable just below the freezing temperature, for any value of p in the range $2.82 < p < 4$ (through the well-known isomorphism between the 3-state Potts and clock models).

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Appendix: Series expansions for the free-energy functionals

In this Appendix we show explicitly the development in power series of the free-energy functionals within Parisi's replica-symmetry-breaking scheme, restricting our analysis of the ordered phase to τ small ($\tau = (T_g - T)/T_g$; $T_g = J/2$).

First, we deal with the isotropic solution (2.8) for all p. The anisotropic solution for $p = 4$ will, then, be considered later. Near the phase transition ($Q^{\alpha\beta}$ small) the free-energy functional within the isotropic conditions (eq.(2.9)), can be written as a power series[15]. The Parisi parametrization can, then, be implemented; in doing that, one gets the free-energy functional,

$$\begin{aligned}
\beta f[Q] &= -\bar{g}_0 + \frac{1}{2}\bar{A}_2 < Q^2 > + \frac{1}{3}\bar{A}_3 < Q^3 > \\
&- \frac{1}{3}\bar{B}_3 \int_0^1 dx \Big[xQ^3(x) + 3Q(x) \int_0^x dy Q^2(y) \Big] + \frac{1}{12}\bar{A}_4 < Q^4 > \\
&- \frac{1}{12}\bar{C}_4 \Big\{ < Q^4 > -2 < Q^2 >^2 - \int_0^1 dx \int_0^x dy [Q^2(x) - Q^2(y)]^2 \Big\} \\
&- \frac{1}{12}\bar{D}_4 \Big\{ 2 < Q >< Q^3 > + \int_0^1 dx Q^2(x) \int_0^x dy [Q(x) - Q(y)]^2 \Big\} \\
&- \frac{1}{12}\bar{B}_4 \Big\{ < Q^2 >^2 -4 < Q >^2 < Q^2 > -4 < Q > \int_0^1 dx Q(x) \int_0^x dy [Q(x) - Q(y)]^2 \\
&- \int_0^1 dx \int_0^x dy \int_0^x dz [Q(x) - Q(y)]^2 [Q(x) - Q(z)]^2 \Big\} + \ldots, \qquad (A.1)
\end{aligned}$$

where

$$\begin{aligned}
\bar{g}_0 &= \ln p + \frac{(\beta J)^2}{8} \quad ; \quad \bar{A}_2 = (\beta J)^2 \Big\{ \frac{(\beta J)^2}{4} - 1 \Big\}, \\
\bar{A}_3 &= \frac{(\beta J)^6}{16} \delta_{3,p} \quad ; \quad \bar{B}_3 = \frac{(\beta J)^6}{8} , \\
\bar{A}_4 &= \frac{(\beta J)^8}{32} (3 + \delta_{4,p}) \quad ; \quad \bar{B}_4 = \frac{3}{16} (\beta J)^8 , \\
\bar{C}_4 &= \frac{3}{8} (\beta J)^8 \quad ; \quad \bar{D}_4 = \frac{3}{8} (\beta J)^8 \delta_{3,p} , \qquad (A.2)
\end{aligned}$$

and

$$< Q^m > = \int_0^1 dz Q^m(x). \qquad (A.3)$$

In equation (2.15) the functional $\Phi[Q]$ is given by,

$$\begin{aligned}
\Phi[Q] &= \bar{A}_2 + 2\bar{A}_3 Q(t) - 2\bar{B}_3 \Big\{ tQ(t) + \int_t^1 dy Q(y) \Big\} + \bar{A}_4 Q^2(t) + \frac{1}{3}\bar{C}_4 < Q^2 > \\
&- \frac{1}{6}\bar{D}_4 \Big\{ 6 < Q > Q(t) + 6tQ^2(t) - 6Q(t) \int_0^t dy Q(y) + < Q^2 > \Big\} \\
&- \frac{1}{3}\bar{B}_4 \Big\{ < Q^2 > -4 < Q >^2 -6t < Q > Q(t) - 3t^2 Q^2(t) + 6 < Q > \int_0^t dy Q(y)
\end{aligned}$$

$$+ 6tQ(t)\int_0^t dyQ(y) - t\int_0^t dyQ^2(y) - \int_t^1 dyyQ^2(y)$$
$$- 2\int_0^t dyQ(y)\int_0^t dzQ(z) + 2\int_t^1 dyQ(y)\int_0^y dzQ(z)$$
$$- \int_t^1 dy\int_0^y dzQ^2(z)\Big\} + \dots \,. \tag{A.4}$$

Now, we turn to the anisotropic solution (equations (4.9)) for p = 4. As before, we expand the free-energy functional for small τ (both R^α and $Q^{\alpha\beta}$ are small). It is a well-known fact that all parameters depending on a single replica index do not bring trouble in the replica symmetry approximation; therefore, we shall take $R^\alpha = R$ (all a), whereas for the spin-glass parameters, $Q^{\alpha\beta}$, the Parisi replica-symmetry-breaking scheme will be applied. One gets,

$$\beta f[R,Q] = -g_0 - A_2R^2 + B_2 < Q^2 > -D_3\int_0^1 dx\Big[xQ^3(x) + 3Q(x)\int_0^x dyQ^2(y)\Big]$$
$$+ G_3R < Q^2 > -A_4R^4 + B_4 < Q^4 > -R_4R\int_0^1 dx\Big[xQ^3(x) + 3Q(x)\int_0^x dyQ^2(y)\Big]$$
$$- C_4\Big\{- < Q^4 > +2 < Q^2 >^2 + \int_0^1 dx\int_0^x dy\Big[Q^2(x) - Q^2(y)\Big]^2\Big\} + O_4R^2 < Q^2 >$$
$$- E_4\Big\{ < Q^2 >^2 -4 < Q >^2 < Q^2 > -4 < Q > \int_0^1 dxQ(x)\int_0^x dy[Q(x) - Q(y)]^2$$
$$- \int_0^1 dx\int_0^x dy\int_0^x dz[Q(x) - Q(y)]^2[Q(x) - Q(z)]^2\Big\} + \dots, \tag{A.5}$$

where

$$g_0 = \ln 4 + \frac{(\beta J)^2}{8},$$
$$A_2 = \frac{(\beta J)^2}{2}\Big[\frac{(\beta J)^2}{4} - 1\Big] \;;\; B_2 = \frac{(\beta J)^2}{4}\Big[\frac{(\beta J)^2}{4} - 1\Big],$$
$$D_3 = \frac{(\beta J)^6}{48} \;;\; G_3 = \frac{(\beta J)^6}{16},$$
$$A_4 = -\frac{(\beta J)^8}{192} \;;\; B_4 = \frac{(\beta J)^8}{768} \;;\; C_4 = -\frac{(\beta J)^8}{128};$$
$$E_4 = \frac{(\beta J)^8}{128} \;;\; O_4 = \frac{(\beta J)^8}{64} \;;\; R_4 = \frac{(\beta J)^8}{32}. \tag{A.6}$$

At the extrema, the replica-symmetric parameter R can be eliminated to give

$$\frac{\delta(\beta f)}{\delta Q(t)} = 2B_2Q(t) - D_3\Big\{3tQ^2(t) + 3\int_0^t dyQ^2(y) + 6Q(t)\int_t^1 dyQ(y)\Big\}$$

$$\frac{G_3}{A_2}Q(t)\Big\{G_3 < Q^2 > -\frac{G_3^3 A_4}{2A_2^3} < Q^2 >^3 + \frac{G_3 O_4}{A_2} < Q^2 >^2$$
$$- R_4 \int_0^1 dx\Big[xQ^3(x) + 3Q(x)\int_0^x dyQ^2(y)\Big]\Big\}$$
$$+ 4B_4 Q^3(t) - 4C_4 < Q^2 > Q(t) + \frac{G_3^2 O_4}{2A_2^2} < Q^2 > Q(t)$$
$$- \frac{G_3 R_4}{2A_2} < Q^2 > \Big\{3tQ^2(t) + 3\int_0^t dyQ^2(y) + 6Q(t)\int_t^1 dyQ(y)\Big\}$$
$$- E_4\Big\{4 < Q^2 > Q(t) - 16 < Q >^2 Q(t) - 12 < Q^2 >< Q >$$
$$- 12t < Q > Q^2(t) - 4t^2 Q^3(t)$$
$$- 4\int_0^1 dyQ(y)\int_0^y dz[Q(y) - Q(z)]^2 + 12 < Q > \int_t^1 dyQ^2(y)$$
$$+ 24 < Q > Q(t)\int_0^t dyQ(y) + 12tQ^2(t)\int_0^t dyQ(y) + 4\int_t^1 dyyQ^3(y)$$
$$- 4tQ(t)\int_0^t dyQ^2(y) - 4Q(t)\int_t^1 dyyQ^2(y) - 8Q(t)\int_0^t dyQ(y)\int_0^t dzQ(z)$$
$$- 8\int_t^1 dyQ^2(y)\int_0^y dzQ(z) + 4\int_0^t dyQ(y)\int_0^t dzQ^2(z)$$
$$+ 4\int_t^1 dyQ(y)\int_0^y dzQ^2(z) + 8Q(t)\int_t^1 dyQ(y)\int_0^y dzQ(z)$$
$$- 4Q(t)\int_t^1 dy\int_0^y dzQ^2(z)\Big\} + \ldots = 0\,. \qquad (A.7)$$

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Resumo

O modelo de vidro de spins do tipo "clock" (relógio) como p estados e **interações** de alcance infinito é estudado no esquema de quebra da simetria entre réplicas de **Parisi.** Uma análise de estabilidade simplificada destas soluções é realizada, **levando-se** em conta somente **flutuações** longitudinais. Mostra-se que para o caso $p = 3$, a solução simples do tipo função degrau é estável, enquanto que para todo $p \neq 3$ as soluções convencionais de **Parisi** levam a uma estabilidade marginal. Argumenta-se que este quadro deve permanecer como verdadeiro em uma análise mais geral.