Revista Brasileira de Física, Vol. 21, no 3, 1991

On the quantum equivalence of antisymmetric tensor field theories

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Received June 11, 1991

Abstract We show, in a very simple way, the quantum equivalence of massless antisymmetric second rank tensor field theories with massless scalar and gauge field theories, in four and five dimensions, respectively. The technique can be straightforwardly extended to higher rank antisymmetric tensor fields.

Antisymmetric tensor field theories have recently become the subject of intense research, since they emerge in the low energy limit of string theories. They appear coupled to gravity or supergravity fields with higher curvature terms in four and ten dimensions[1] and a complete understanding of these couplings is needed in order to have anomaly cancelations. They also have been used to have a better understanding of the quantization of reducible field theories, either from the Lagrangian or Hamiltonian point of view[2], motivated by string field theory which is also a reducible theory[3].

The quantum equivalence of antisymmetric tensor field theories in four dimensions has been discussed by several authors*. Either by coupling to a background gravity field[6,7], or using a Lagrangian BRST approach with a careful

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† **Partially supported by CNPq.**

choice of ghosts for ghosts[8] the quantum equivalence has been established. Here we want to present a simple and straightforward proof in any dimension using the Hamiltonian **BRST** formalism for reducible theories[9], which brings in all ghosts for ghosts with the right counting of ghosts and ghost numbers. The application of the formalism is straightforward and we will not discuss its details. An exposition of this formalism can be found e.g. in ref.[10].

As is well known a second rank antisymmetric tensor field is equivalent to a scalar field in four dimensions. In general, if we denote the antisymmetric tensor field by $B_{\mu\nu}$ and its field strenght by $F_{\mu\nu\rho}$ then the first order action

$$S = -\frac{1}{12}\int d^Dx\Big(\frac{1}{2}F^{\mu\nu\rho}F_{\mu\nu\rho} - F^{\mu\nu\rho}\partial_{[\mu}B_{\nu\rho]}\Big) \tag{1}$$

implies for the $B_{\mu\nu}$ field equation that $\partial^\mu F_{\mu\nu\rho} = 0$. This equation can be solved in D dimensions by $F_{\mu_1\mu_2\mu_3} = \epsilon_{\mu_1\ldots\mu_D}\partial^{\mu_4}A^{\mu_5\cdots\mu_D}$. So for D = 4 A is a scalar field, for $D = 5$ A^μ is a vector field and for D $\geq$ 5 $A^{\mu_5\cdots\mu_D}$ is another antisymmetric tensor. If we now use the field equation for $F_{\mu\nu\rho}$, then, for $D = 4$ the scalar field A satisfies the Klein-Gordon equation and, for D = 5 the vector field A^μ obeys the Maxwell equations.

We now want to show that these classical equivalences remain valid at the quantum level, using the Hamiltonian BRST formalism for reducible systems[9] since it seems the most suitable technique to handle such systems. We start with the second order form of the action (1) to find out the canonical momenta of $B_{\mu\nu}$

$$\Pi_{0i} \equiv \frac{\partial L}{\partial \dot{B}^{0i}} = 0 \tag{2a}$$

$$\Pi_{ij} \equiv \frac{\partial L}{\partial \dot{B}^{ij}} = \frac{1}{2}\dot{B}_{ij} - \partial_{[i}B_{0j]} \tag{2b}$$

and the canonical Hamiltonian

$$H_c = \frac{1}{2}(\Pi_{ij})^2 - B^{0j}\partial^i\Pi_{ij} - \frac{1}{4}(\partial_i B_{ij})^2 + \frac{1}{8}(\partial_i B_{jk})^2 \tag{3}$$

The requirement that the primary constraint (2a) does not evolve in time furnishes the secondary constraint

$$\phi_i \equiv \{\Pi_{0i}, H_c\} = \partial^j\Pi_{ji} = 0 \tag{4}$$

and no more constraints are obtained in this way. The constraints Π_{0i} and ϕ_i are first class constraints and have an abelian Poisson bracket algebra. However the theory is reducible since we can find a linear combination of the constraints which vanishes without the use of the constraint equations. Consider the linear combination

$$\int d^{D-1}x' Z^i(x,x')\phi_i(x'). \tag{5}$$

It vanishes identically if we choose $Z^i(x,x') = \frac{\partial}{\partial x'}\delta(x-x')$. We now have to check whether there is a linear combination of Z^i

$$\int d^{D-1}x' Z_2^i(x,x') Z_i(x',x'') \tag{6}$$

which also vanishes. We find that this is not possible and the theory is said to be reducible at the first stage[9,10].

We now enlarge the phase space of the theory. We associate a Lagrange multiplier to each constraint as well as its associated canonical momentum: λ_{0i}, P_{0i} to Π_{0i} and λ_i, P_i to ϕ_i. Now the constraints are Π_{0i}, P_{0i}, ϕ_i and P_i and we associate to each of them a pair of fermionic canonical conjugated ghosts, respectively $\eta_{0i}, \bar{\wp}_{0i}$, $\wp_{0i}, \bar{\eta}_{0i}$; η_i, $\bar{\wp}_i$, and $\wp_i$, $\bar{\wp}_i$. To the function Z^i we associate a fermionic Lagrange multiplier ξ and its canonical momentum Π_ξ and two pairs of bosonic ghosts c, $\bar{b}$ and b, $\bar{c}$. We also need extra ghosts[9,10], a pair which is fermionic λ', Π' and a pair which is bosonic ξ', $\Pi'_{\xi'}$. They are also canonically conjugated.

We can then built the BRST charge

$$Q = \int d^{D-1}x(\eta^{0i}\Pi_{0i} + \eta^i\phi_i + c\partial^i\bar{\wp}_i + P^{0i}\wp_{0i} + P^i\wp_i + \Pi_\xi b + \Pi'\xi') \tag{7}$$

and we choose the gauge fixing fermion Ψ as

$$\Psi = \int d^{D-1}x\Big[\frac{1}{\epsilon}\bar{\eta}^{0i}(\lambda_i - B_{0i}) - \bar{\eta}^i\partial^j B_{ij} + \bar{\wp}^{0i}\lambda_{0i} \\ - \bar{\wp}^i\lambda_i + \bar{c}\partial^i\eta_i + \bar{b}\xi + \Pi'_{\xi'}\partial^i\lambda_i - \bar{\eta}^i\partial_i\lambda'\Big], \tag{8}$$

where ϵ is a parameter which will be set to zero later on. Then the effective action is given by

$$S_{eff} = S_k + S_c + S_\Psi, \tag{9}$$

where

$$S_k = \int d^D x(\Pi_{0i}\dot{B}^{0i} + \frac{1}{2}\Pi_{ij}\dot{B}^{ij} + P_{0i}\dot{\lambda}^{0i} + P_i\dot{\lambda}^i + \bar{\eta}_{0i}\dot{\wp}^{0i} + \bar{\wp}_{0i}\dot{\eta}^{0i} + \\ + \bar{\eta}_i\dot{\wp}^i + \bar{\wp}_i\dot{\eta}^i + \bar{c}\dot{b} + \bar{b}\dot{c} + \Pi_\xi\dot{\xi} + \Pi'\dot{\lambda}' + \Pi'_{\xi'}\dot{\xi}'), \tag{10}$$

$$S_c = \int d^D x\Big[-\frac{1}{2}(\Pi_{ij})^2 + \frac{1}{4}(\partial_i B_{ij})^2 - \frac{1}{8}(\partial_i B_{jk})^2\Big], \tag{11}$$

and

$$S_\Psi = -\int d^D x\{Q,\Psi\} = -\int d^D x\Big[\Pi^{0i}\lambda_{0i} + \Pi^{ij}\partial_i\lambda_j + \frac{1}{\epsilon}P^{0i}(\lambda_i - B_{0i}) \\ - P^i(\partial^j B_{ji} + \partial_i\lambda') + \frac{1}{\epsilon}\bar{\eta}^{0i}(\wp_i - \eta_{0i}) + \bar{\eta}^i(\partial^j\partial_j\eta_i - \partial^j\partial_i\eta_j - \partial_i\xi') \\ + \bar{\wp}^{0i}\wp_{0i} - \bar{\wp}^i(\wp_i + \partial_i\xi) - \bar{c}\partial^i\partial_i c + \bar{b}b + \Pi_\xi\partial^i\eta_i + \Pi'\partial^i\lambda_i + \Pi'_{\xi'}\partial^i\wp_i\Big] \tag{12}$$

We now perform a change of variables $P_{0i} \to \epsilon P_{0i}$, $\bar{\eta}_{0i} \to \epsilon\bar{\eta}_{0i}$, whose Jacobian is equal to one, and take the limit $\epsilon \to 0$. Then, the effective action (9) becomes

$$S_{eff} = \int d^D x\Big[\Pi^{0i}(\dot{B}_{0i} - \lambda^{0i}) - \frac{1}{2}\Big(\Pi^{ij} - \frac{1}{2}\dot{B}^{ij} + \partial^{[i}\lambda^{j]}\Big)^2 \\ + \frac{1}{8}(\dot{B}_{ij} - 2\partial_{[i}\lambda_{j]})^2 - P^{0i}(\lambda_i - B_{0i}) + P^i(\dot{\lambda}_i + \partial^j B_{ji} + \partial_i\lambda') \\ - \bar{\eta}^{0i}(\wp_i - \eta_{0i}) + \bar{\eta}^i(\dot{\wp}_i - \partial^j\partial_j\eta_i + \partial^j\partial_i\eta_j) + \bar{\wp}^{0i}(\dot{\eta}_{0i} - \wp_{0i}) + \bar{\wp}^i(\dot{\eta}_i + \wp_i + \partial_i\xi) \\ + \bar{c}(\dot{b} + \partial^i\partial_i c) + \bar{b}(\dot{c} - b) + \Pi_\xi(\dot{\xi} - \partial^i\eta_i) + \Pi'(\dot{\lambda}' - \partial^i\lambda_i) \\ - \Pi'_{\xi'}\partial^i\wp_i + \xi'(\Pi'_{\xi'} + \partial^i\bar{\eta}_i) + \frac{1}{4}(\partial_i B_{ij})^2 - \frac{1}{8}(\partial_i B_{jk})^2\Big] \tag{13}$$

The partition function is, then, given by

$$Z = \int D\mu \exp\, iS_{eff}, \tag{14}$$

where the measure D_μ involves all the fields and ghosts.

The functional integrals over Π_{0i}, Π_{ij}, P_{0i}, P_i, $\bar{\eta}_{0i}$, $\bar{\wp}_{0i}$, $\bar{\wp}_i$, $\bar{b}$, Π_ξ, Π' and ξ' are easily performed and we obtain

$$Z = \int D\mu'\delta(\dot{B}_{0i} - \lambda_{0i})\delta(\lambda_i - B_{0i})\delta(\dot{\lambda}_i + \partial^j B_{ji} + \partial_i\lambda')\delta(\wp_i - \eta_{0i}) \\ \delta(\dot{\eta}_{0i} - \wp_{0i})\delta(\dot{\eta}_i + \wp_i + \partial_i\xi)\delta(\dot{c} - b)\delta(\dot{\xi} - \partial^i\eta_i)\delta(\dot{\lambda}' - \partial^i\lambda_i)\delta(\dot{\Pi}'_{\xi'} + \partial^i\bar{\eta}_i) \\ \exp\ i\int d^D x\Big[\frac{1}{8}(\dot{B}_{ij} + \partial_{[i}\lambda_{j]})^2 + \bar{\eta}^i(\dot{\wp}_i - \partial^j\partial_j\eta_i + \partial^j\partial_i\eta_j) + \\ + \bar{c}(\dot{b} + \partial^i\partial_i c) - \Pi'_{\xi'}\partial^i\wp_i + \frac{1}{4}(\partial_i B_{ij})^2 - \frac{1}{8}(\partial_i B_{jk})^2\Big], \tag{15}$$

where $D\mu'$ involves only the remaining fields and ghosts. We can now perform the integrals over λ_i, $\wp_i$, b, λ_{0i}, η_{0i}, and $\wp_{0i}$ to obtain

$$\begin{aligned} Z = \int & DB_{0i}DB_{ij}D\eta_i D\bar{\eta}_i DcD\bar{c}D\xi D\lambda' D\Pi'_{\xi'} \\ & \delta(\dot{B}_{0i} + \partial^j B_{ji} + \partial_i\lambda')\delta(\dot{\xi} - \partial^i\eta_i)\delta(\dot{\lambda}' - \partial^i B_{0i})\delta(\dot{\Pi}'_{\xi'} + \partial^i\bar{\eta}_i) \\ & \exp\ i\int d^Dx\Big[\frac{1}{8}(\dot{B}_{ij} - \partial_{[i}B_{0j]})^2 + \bar{\eta}^i\partial^2\eta_i + \bar{c}\partial^2 c + \Pi'_{\xi'}(\ddot{\xi} + \partial^i\partial_i\xi) \\ & + \frac{1}{4}(\partial_i B_{ij})^2 - \frac{1}{8}(\partial_i B_{jk})^2\Big] \end{aligned} \tag{16}$$

Now using the delta functionals in the integrand of (16) we can rewrite some terms in the action in a covariant form

$$\frac{1}{2}(\dot{B}_{ij} - \partial_{[i}B_{0j]})^2 + \frac{1}{4}(\partial_i B_{ij})^2 - \frac{1}{8}(\partial_i B_{jk})^2 - \frac{1}{4}\lambda'\partial^2\lambda' = \frac{1}{8}\partial_\mu B_{\nu\rho}\partial^\mu B^{\nu\rho} \tag{17}$$

and redefiníng $\eta_\mu \equiv (\xi, \eta_i)$ and $\bar{\eta}_\mu \equiv (\Pi'_{\xi'}, \bar{\eta}_i)$ we can rewrite eq. (16) in a covariant way as

$$\begin{aligned} Z = \int & DB_{\mu\nu}D\eta_\mu D\bar{\eta}_\mu DcD\bar{c}D\lambda'\delta(\partial^\mu\eta_\mu)\delta(\partial^\mu\bar{\eta}_\mu)\delta(\partial^\nu B_{\mu\nu} - \partial_\mu\lambda') \\ & \exp\ i\int d^Dx\Big(-\frac{1}{8}B^{\mu\nu}\partial^2 B_{\mu\nu} + \bar{\eta}^\mu\partial^2\eta_\mu + \bar{c}\partial^2 c + \frac{1}{4}\lambda'\partial^2\lambda'\Big) \end{aligned} \tag{18}$$

Next we notice that due to the delta functional $\delta(\partial^\nu B_{\mu\nu} - \partial_\mu\lambda')$ we have $\partial^2\lambda' = 0$ and so the last term of the action can be dropped out. We then exponentiate all delta functionals

$$\begin{aligned} Z = \int & DB_{\mu\nu}D\eta_\mu D\bar{\eta}_\mu DcD\bar{c}D\lambda' D\Pi D\bar{\Pi} D\alpha_\mu \\ & \exp\ i\int d^Dx\Big[-\frac{1}{8}B^{\mu\nu}\partial^2 B_{\mu\nu} + \bar{\eta}^\mu\partial^2\eta_\mu + \bar{c}\partial^2 c \\ & + \Pi\partial^\mu\eta_\mu + \bar{\Pi}\partial^\mu\bar{\eta}_\mu + \alpha^\mu(\partial^\nu B_{\mu\nu} - \partial_\mu\lambda')\Big] \end{aligned} \tag{19}$$

and perform the functional integrals over $\bar{\eta}_\mu$, η_μ, Π and $\bar{\Pi}$ to obtain

$$\begin{aligned} Z = \int & DB_{\mu\nu}DcD\bar{c}D\lambda' D\alpha_\mu(\det\ \partial^2)^D \\ & \exp\ i\int d^Dx\Big(-\frac{1}{8}B^{\mu\nu}\partial^2 B_{\mu\nu} + \bar{c}\partial^2 c + \alpha^\mu\partial^\nu B_{\mu\nu} - \alpha^\mu\partial_\mu\lambda'\Big). \end{aligned} \tag{20}$$

Now, integrating over $c, \bar{c}$ and λ' we get

$$Z = \int DB_{\mu\nu} D\alpha_\mu (\det\, \partial^2)^{D-1} \delta(\partial_\mu \alpha_\mu)$$
$$\exp\, i \int d^D x \Big(-\frac{1}{8} B^{\mu\nu} \partial^2 B_{\mu\nu} + \frac{1}{2} B_{\mu\nu} - \partial_{[\mu}\alpha_{\nu]} \Big) \tag{21}$$

and, finally performing the integral over $B_{\mu\nu}$ we find a partition function depending only on one functional integral over α_μ

$$Z = \int D\alpha_\mu (\det\, \partial^2)^{(D-1)-\frac{D(D-1)}{4}} \delta(\partial^\mu \alpha_\mu) \exp\, i \int d^D x \alpha^\mu \alpha_\mu \tag{22}$$

This is our master functional integral from which we can prove the equivalence of the antisymmetric tensor field theory to other field theories. If we exponentiate the delta functional in eq. (22) and integrate over α_μ we obtain

$$Z = \int D\phi (\det\, \partial^2)^{(D-1)-\frac{D(D-1)}{4}} \exp\, i \int d^D x \frac{1}{4} \phi \partial^2 \phi. \tag{23}$$

For this to be the partition function of a scalar field, no det ∂^2 should appear in the integrand of (23). This leads to the condition $D - 1 - \frac{D(D-1)}{4} = 0$, from which we find $D = 1$ and 4. Since we must have $D \geq 3$ only in four dimensions the second rank antisymmetric tensor is equivalent to a scalar field at the quantum level.

Let us now perform the following change of variables in (22), i.e., $\alpha_\mu \rightarrow 2^{-D/2}(\partial^2)^{1/2}\alpha_\mu$. Then taking into account the Jacobian of the transformation we can rewrite eq. (22) as

$$= \int D\alpha_\mu (\det\, \partial^2)^{(D-1)-\frac{D(D-1)}{4}+\frac{D}{2}-\frac{1}{2}} \delta(\partial^\mu \alpha_\mu) \exp\, i \int d^D x (-\alpha^\mu \partial^2 \alpha_\mu). \tag{24}$$

To obtain the final form of the action we took a realization for $(\partial^2)^{1/2}$ in terms of Dirac gamma matrices such that $(\partial^2)^{1/2}\alpha^\mu (\partial^2)^{1/2}\alpha_\mu = \mathrm{Tr}(\partial\!\!\!/ \alpha^\mu\, \partial\!\!\!/ \alpha_\mu)$ and performed an integration by parts. In order that (24) be the partition function for a gauge field in the Lorentz gauge we demand that only a factor of det ∂^2 remains in it. This means that $D - 1 - \frac{D(D-1)}{4} + \frac{D}{2} - \frac{1}{2} = 1$ and we find $D = 2$ and 5. So the theory is equivalent to a gauge field theory only in five dimensions.

In conclusion we have shown that using the Hamiltonian BRST technique for reducible theories and by a convenient choice of integrations and change of variables it is possible to recast the partition function of the original theory into the partition function for other field theories at some definite dimensions. This implies then that the two quantum field theories are equivalent.

This technique can be straightforwardly extended to higher rank antisymmetric tensor theories. In this case the theory will be reducible at higher and higher levels so that itwill be needed the inclusion of more and more ghosts. But again an appropriate choice of integrations and change of variables can be used to rewrite the original partition function in determined dimensions showing that at these dimensions the theories are equivalent.

Acknowledgements

One of the authors (V.O.R.) would like to thank Prof. Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste and also to FAPESP for financia1 help L.S.Jr. would like to acknowledge CNPq for financial help.

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Resumo

Demonstramos, de maneira bastante simples, a equivalência quântica de teorias de campos tensoriais antissimétricos de segunda ordem com teorias de campos escalares de massa nula e teorias de gauge em quatro e cinco dimensões, respectivamente. Esta técnica pode se estendida diretamente a campos tensoriais antissimétricos de ordem mais alta.