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Discrete Alfvén waves in a cylindrical plasma: arbitrary beta and magnetic twist

H. Shigueoka
Instituto de Física, Universidade Federal Fluminense, 24024, Niterói, RJ, Brasil

P. H. Sakanaka
Instituto de Física, Universidade Estaduai de Campinas, 13081, Campinas, SP, Brasil

C. A. de Azevedo
Instituto de Física, Universidade do Estado do Rio de Janeiro, 20550, Rio de Janeiro, RJ, Brasil

and

A. S. de Assis
Instituto de Matemática, Universidade Federal Fluminense, 24024, Niterói, RJ, Brasil

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Abstract Discrete Alfvén wave (DAW) modes in diffuse, current-carrying plasma are studied numerically, with a cylindrical model and toroidal correction. In this model, with WKB approximation, we have shown that there are three eigenmodes: the discrete Alfvén, the magnetosonic and the helical mode. We show an approximate expression for the discrete mode in terms of known wave parameters and of the toroidal effect. The discrete Alfvén frequencies, for a fixed azimuthal mode number m and an axial mode number k (= n/R, where n is the toroidal mode number and R is the major radius), are calculated as function of the plasma current using the complete equation, with no WKB approximation. The result is compared with that obtained from the WKB approximation. We present the results for *TCA* and NET tokamaks.

1. Introduction

The spectrum of the ideal magnetohydrodynamics for cylindrical plasma systems with diffuse profile has two continua, known as the SLOW WAVE and ALFVÉN WAVE continua, due to the singularities of the eigenvalue equation.

There are also two non-Sturmian regions, due to the zeroes of the coefficient of the highest order term, intercalated by two continua (Goedbloed 1975). Discrete eigenvalues exist in between these four regions, in particular, the discrete Alfvén modes are those found below the Alfvén continuum.

Experimental and theoretical results (De Chambrier et al. 1981), Ross et al. 1982, and Appert et al. 1982) have shown the existence of stable discrete Alfvén eigenmodes with frequencies below the lower edge of the Alfvén continuum for a given toroidal mode number n and a poloidal mode number m, i. e., $\omega^2_{DAW} < min[\omega_A(r)]$. We define $\omega^2_A = (m + nq)^2 v^2_A/(qR)^2$, where q is the safety factor, R is the major radius, and v_A is the Alfvén speed. These modes are like stable kink modes and appear only under certain well-defined conditions. These discrete modes are also called "global Alfvén eigenmodes" (Appert et al. 1982).

When a finite toroidicity is included, DAW modes with different poloidal mode numbers will become coupled. However, in this paper we study the discrete Alfvén waves still in cylindrical geometry, but with the inclusion of the toroidal effects.

In Section 2 we present the equations of the ideal magnetohydrodynamics with toroidal effects. In Section 3, we show the analytical results of the dispersion relation for the case of the large aspect ration expansion. Numerical solutions with TCA and NET tokamak parameters are shown in Section 4 and conclusion in Section 5.

2. Basic equations

The Euler equation with first-order toroidal corrections in a circular tokamak was derived by Copenhaver (1976) using the theory of the MHD spectrum in toroidal geometry developed by Goedbloed (1975). A simplified form of this equation has been used to study the unstable internal kink mode by Galvão, Sakanaka, and Shigueoka (1978). Copenhaver's equation for the radial component of the plasma displacement vector $\vec{\xi}$ is given by

$$\frac{d}{dr}\left[f(r)\frac{d}{dr}\left(r\xi_r\right)\right] + g(r)\left(r\xi_r\right) = 0, \qquad (1)$$

where

$$f(r)=\frac{N}{rD},\qquad N=(\rho\omega^2-F^2)q_B,\qquad q=\frac{rB_z}{\bar{R}B_\theta},$$
$$D=\rho^2\omega^4-Hq_B,\qquad F=\frac{m}{r}B_\theta+kB_z=(\vec{k}\cdot\vec{B}),$$
$$q_B=\rho\omega^2(\gamma p+B^2)-\gamma pF^2,\qquad H=\frac{m^2}{r^2}+\frac{n^2}{\bar{R}^2},$$
$$rg=\left(\rho\omega^2-F^2\right)-r\frac{d}{dr}\left(\frac{B_\theta}{r}\right)^2+2q^2\left(\frac{B_\theta}{r}\right)^2\left\{2+\frac{\varsigma'\bar{R}}{r}\right.$$
$$\left.+\Delta\left(r\frac{I'}{I}-1\right)\right\}-\frac{4n^2}{D\bar{R}}\left(\frac{B_\theta}{r}\right)^2\left(q_B-\rho\omega^2\gamma p\right)\left(1+q\frac{m}{n}\Delta\right)^2$$
$$+\qquad r\frac{d}{dr}\left[\frac{2n}{\bar{R}r}\left(\frac{B_\theta}{r}\right)\frac{Gq_B}{D}\left(1+q\frac{m}{n}\Delta\right)\right],$$
$$G=\frac{m}{r}B_z-\frac{n}{\bar{R}}B_\theta,$$

where primes indicate derivatives with respect to r. The quantities 7, p and p are the adiabatic constant, plasma pressure and mass density, respectively. B_θ is the poloidal and B, is the toroidal magnetic field. Here $\varsigma(r)$ is the displãcement of the axis of the magnetic surface (considered as of circular cross section with radius r) with respect to the geometric center of the plasma column. The axis of the magnetic surface is given, in terms of the toroidal coordinates, by $\bar{R}(r)=R_0+\varsigma(r)$. The outermost magnetic surface coincides with the wall and has the axis located at the geometric center, $\bar{R}(a)=R_0$ at $z=0$. The quantities I and **A** are defined as

$$I=\bar{R}(r)B_z(r),\qquad \Delta=2+\frac{R_0}{r}\int_0^r\frac{\varsigma'}{r}dr.$$

The Eq.(1) is strictly valid only for the weak toroidal mode coupling. Its use is restricted to the neighborhood of the magnetic axis because the boundary conditions for the m = 0, 1, and **2** modes are $\xi_{r0}(0)=0$, $\xi_{r1}(0)=1$, $\xi_{r2}(0)=0$, thus reducing mode coupling in this region. This uncoupled equation recovers the diffuse linear pinch equation, given by Hain and Lust 1958, in the large aspect ratio limit (with $n/\bar{R}$ maintained constant) when the toroidal correction terms disappear. The equation N = 0 defines the two continua, the shear Alfvén and the slow wave. The shear Alfvén wave frequency given by

$$\omega_A^2 = \frac{v_A^2}{\left[\bar{R}q(r)\right]^2}(m + nq)^2 \tag{2}$$

is dependent on the radial variation of Alfvén speed, v_A, on the position of the magnetic surface from the symmetric axis and on the safety factor $q(r)$. This value is minimum at or near the plasma center and maximum at the plasma edge, r = a.

The eigenfrequency ω_N (the subscript N denotes normalization) and the corresponding eigenvector $\xi_r(r)$, for a given k $= n/\bar{R}$ and m, can be obtained by solving equation (1)with the following boundary conditions: $r\xi_r = 0$ at r = 0 and r = a.

The profiles for p, B„ and B_θ obey the equilibrium equation

$$\frac{dp}{dr} + \frac{1}{2\bar{R}^2}\frac{dI^2}{dr} + \frac{1}{2}\frac{dB_\theta^2}{dr} + \frac{dB_\theta^2}{dr} = 0 \tag{3}$$

and

$$\frac{d\varsigma}{dr} = -\frac{1}{R_0 r B_\theta^2}\int_0^r \left(B_\theta^2 - 2s\frac{dp}{ds}\right) s\, ds. \tag{4}$$

One difference here, from the usual circular toroidal coordinate system, is that $d\varsigma/dr = -d\delta/dr$, where 6 wits first defined by Shafranov in **1964.** That Eq.(4) is reasonable follows directly, since $\varsigma + \delta =$ const for a given equilibrium. In the large aspect ratio limit (n $\longrightarrow \infty, q \longrightarrow 0, nq$ fixed) all toroidal correction terms disappear and we recover the well known Hain Lüst equation **(1958).**

3. Analytical study of the discrete eigenmodes

In this part of the paper, we study the discrete Alfvén modes by "WKB" analysis. Therefore, rather than introducing the most complete eigenvalue equation, here we adopt a simple model; yet, the essence of physics can be brought out. We will show that, within the frarnework of the MHD theory, when a bounded plasma is assumed, there will be discrete modes besides the helical and magnetosonic modes, with the angular frequency *w* a little below the shear Alfvén frequency. The spectrum of these modes depends on the plasma current, as a first order quantity.

We consider that the current density is relatively small so that $(B_\theta/B_z)^2 \sim \epsilon << 1$, that is, the twist is small, and that the plasma pressure is negligible (very small β). Thus, $B_z \cong B_{z0} = constant.$ We also assume that $R/a \sim \epsilon^{-\frac{1}{2}}$ and q ~ 1. Using the approximation $\frac{\partial}{\partial r} \approx ik,$ in equation (1), known as the "WKB" approximation, and expanding the dispersion relation up to the order one in ϵ, we derived the following dispersion relation:

$$\left[\omega^2 - \omega^2_{MSW}\right]\left[\omega^2 - \omega^2_{DAW}\right]\left[\omega^2 - \omega^2_{HLW}\right] = 0 \tag{5}$$

where

$$\omega^2_{MSW} = k_T^2 v_A^2 + s - e + f, \qquad \omega^2_{DAW} = k_\|^2 v_A^2 + e,$$

$$\omega^2_{HLW} = H v_A^2 - s, \qquad H = \frac{m^2}{r^2} + k^2 \cong \frac{m^2}{r^2},$$

$$B^2 = B_z^2 + B_\theta^2, \qquad v_A^2 = \frac{B^2}{\rho}, \qquad k_T^2 = H + k_r^2, \qquad b = \frac{B_\theta}{r},$$

$$k_\|^2 v_A^2 = \frac{F^2}{\rho} = \frac{[mb(r) + kB_z]^2}{\rho},$$

$$s = 4kbm\frac{B_z t_1}{\rho},$$

$$e = -\frac{\{4k^2b^2t_1^2 - 2k[(bt_1)'/r]mB_z - H[r(b^2)' - 2q^2b^2t_2]\}}{k_T^2\rho},$$

$$f = \frac{r(b^2)' - 2q^2b^2t_2}{\rho},$$

$$t_1 = 1 + q\frac{m}{n}\Delta,$$

$$t_2 = 2 + \frac{\varsigma' R}{r} + \Delta\left(r\frac{I'}{I} - 1\right)$$

and primes indicate the derivative with respect to r; $k_\|$ is the component of the wave vector parallel to the magnetic field and v_A is the Alfvén velocity. When we take the cylindrical approximation we have $t_1 \longrightarrow 1$, $t_1' \longrightarrow 0$, and $t_2 \rightarrow 0$. This dispersion relation is more complete than that reported by Appert et *al. (1982).* The r-dependence is eliminated by replacing r by an effective radius r_{eff}. This effective radius is estimated to be of order of $0.1a$ to $0.2a$, deduced from the results of a more complete calculation done with the scheme presented in Section 4.

There are three eigenmodes, the first, ω^2_{MSW}, representing the magnetosonic, the second, ω^2_{DAW}, the discrete Alfvén wave, and the last, ω^2_{HLW}, the helical wave. Taking the current profile as $J_z(r) = J_0(1 - r^2/a^2)^{\alpha_j}$ and using Ampère's law we obtain $b(r)$. Now, taking $k = n/\bar{R}$, we can express the discrete Alfvén frequency in terms of the known quantities at r = 0 :

$$\omega^2_{DAW} = k_\parallel^2 v_{A0}^2 - \frac{2}{(k_T a)^2}\left(\frac{v_{A0}}{q_0 R_0}\right)^2 \Big[nmq_0\alpha_j(m^2 + k_r^2 r^2) + m^2\alpha_j\left(1 + \frac{nq_0}{m}\right) + 2n^2\left(\frac{a}{R}\right)^2\left(1 + \frac{7}{4}nq_0\frac{m}{n^2}\right)^2 - \frac{7}{4}\frac{m^2}{4I_0R_0^2}\left(1 + I_0q_0^2\alpha_j^2\right)\Big], \quad (6)$$

where α_j represents the currerit profile shape.

We see from equation (6) that the eigenfrequency is shifted downward from the shear Alfvén frequency by a quantity of the order of ϵ. Considering that the last two terms of this equation are of higher order, we can say that the frequency shift is proportional to n, α_j, v_A^2, and $1/R_0^2$. Its dependence with m and q_0 is confirmed by numerical calculation of a more complete model in next section . The value of k, is unknown; however, a good guess is that it should be of the order of $1/r_{eff}$.

4. Numerical Results

In this section we present the numerical technique to solve Eqs.(1)-(4). These equations are solved using the shooting method. Two profiles can be chosen arbitrarily. We choose for pressure $p(r) = p_{0N}\, exp(\alpha_{p2}r^2 + \alpha_{p4}r^4)$, where p_{0N}, α_{p2} and α_{p4} are constant. The B_θ profile is obtained by solving the Ampère's law with a given J_z profile. We take for current density $J_z(r) = J_{0N}\,(1 - (r/a_N)^{\alpha_{j1}}\}^{\alpha_{j2}}$ where α_{j1} and α_{j2} are free parameters and the constant J_{0N} must satisfy the condition that the safety factor at the axis $q(0)$ is greater than 1. We choose a parabolic density profile, $\rho(r) = \rho_{0N}\left\{1 - 0.95(r/a_N)^2\right\}$. As a consequence of the inhomogeneous density, we have a continuous spectrum of the Alfvén wave, in addition to discrete espectra. The shooting method is used to obtain the eigenvalue ω_N which satisfies the boundary condition at r = 0 and a. We choose the following parameters: $B_{zN} = 1$, $\rho_{0N} = 1$, $\alpha_{p2} = -4/a_N^2$, $\alpha_{p4} = -6/a_N^4$, $a_N = 1$.

In figure 1 we plot the distance $\Delta\omega^2 = \omega^2_{DAW} - \omega^2_{A0}$, where $\omega^2_{A0} = k^2_{\parallel} v^2_{A0}$, as a function of constant ar, , for different parameters in the stable region. In this figure it the dependence analysed in the last paragraph of the previous section is evident. We can see that the discrete Alfvén wave only appears for α_j larger than 2. This means that the current profile should be very peaked to have a discrete Alfvén mode. For α_j smaller than 2 this mode enters the Alfvén continuum, that is, we do not have the discrete mode anymore.

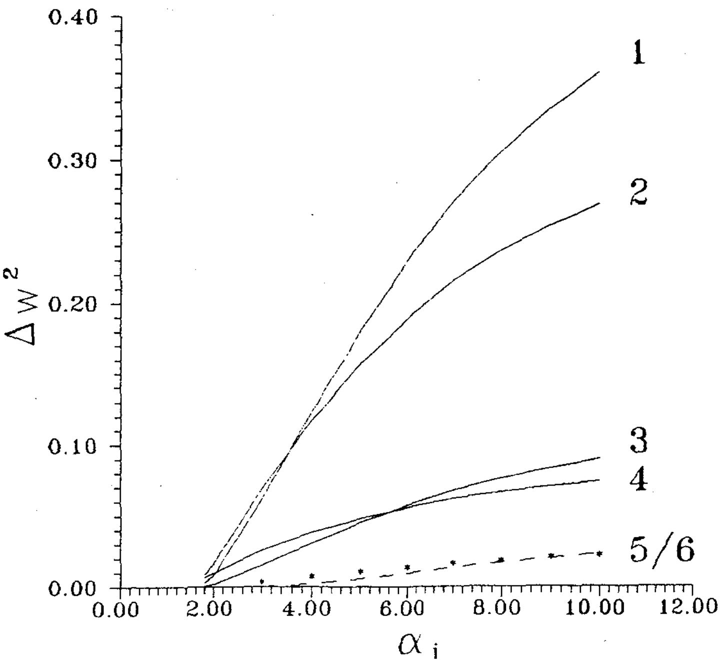

Fig. 1 – Distance between the discrete Alfvén frequency and the lower edge of the Alfvén continuum, $Aw^2 = \omega^2_{DAW} - \omega^2_{A0}$ versus α_j. The following input parameters are used for the curves:

(1) $R_0 = 5,\ m = 1,\ n = 2,\ q_0 = 1,\ B_{z0} = 2$;

(2) $R_0 = 5,\ m = 2,\ n = 2,\ q_0 = 1,\ B_{z0} = 1$;

(3) $R_0 = 5,\ m = 1,\ n = 2,\ q_0 = 1,\ B_{z0} = 1$;

(4) $R_0 = 5,\ m = 1,\ n = 1,\ q_0 = 1,\ B_{z0} = 1$;

(5) $R_0 = 5,\ m = 1,\ n = 2,\ q_0 = 2,\ B_{z0} = 1(*)$;

(6) $R_0 = 10,\ m = 1,\ n = 2,\ q_0 = 1,\ B_{z0} = 1(- - -)$

In figure 2 and 3 it is shown that equation (1) recovers the Hain and Lüst equation for large aspect ratio with fixed *k* value. In figure 2 the normalized eigenfrequency, ω_N, is plotted against R_0. As R_0 increases the frequency approaches, asymptotically, the cylindrical case. In figure 3 it is shown that the displacement of the magnetic axis approaches zero as R_0 increases.

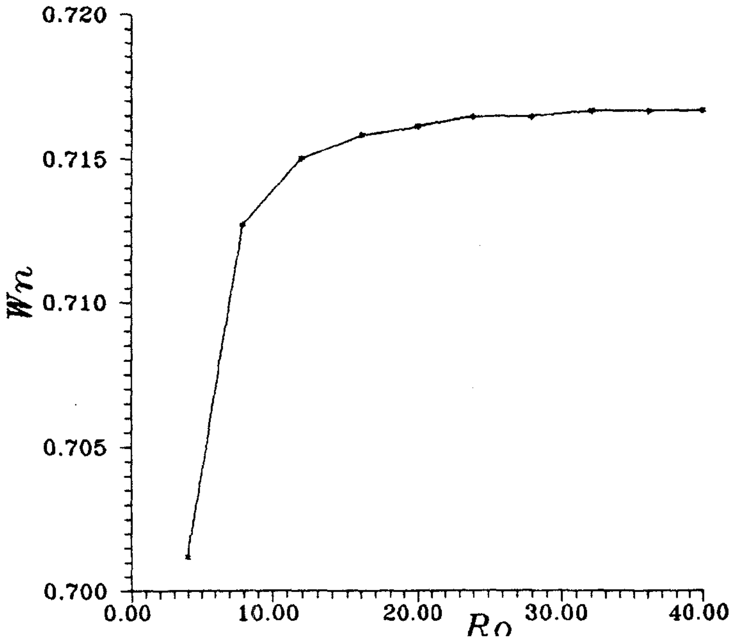

Fig. 2 – Normalized eigenfrequency, ω_N, versus R_0 for k constant (k $= n/R_0 =$ 0.5, $J_0 = 0.5$, $\beta = 0.01$, $\alpha_j = 4$. The ω_N for the cylindrical plasma column is 0.717.

We have applied this analysis for two tokamaks: TCA and NET.

We have taken parameters for TCA: $R_0 = 0.61m$, $a = 0.18m$, $B_{z0} = 1.6T$, $n_{e0} = 1.0 10^{19} m^{-3}$, and for NET: $R_0 = 5.18m$, a $= 2.17m, B_{z0} = 5.0T$, $n = 1.8 10^{19} m^{-3}$. The equation $\omega = [\rho_{0N}/(\mu_0\rho_0)^{1/2}][B_{z0}/B_{zN}][R_N/R_0]\omega_N$ recovers the frequency in s^{-1}. The dependence of the eigenfrequencies on current, in terms of the normalized current density, is shown in figure 4, for both TCA and NET. The distance which separates the discrete Alfvén frequency to the lower edge of the

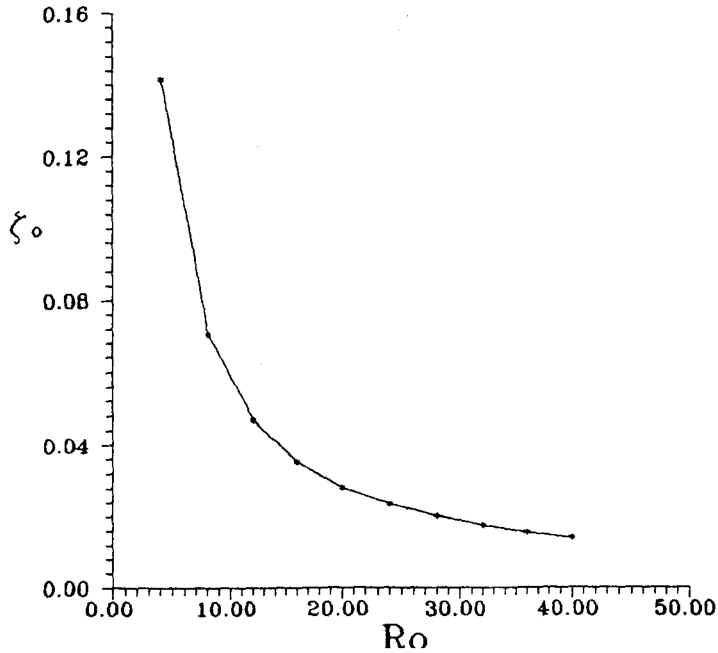

Fig. 3 - Displacement of the magnetic axis $\zeta_0 = \zeta(r = 0)$, from the geometric center of the cylinder, versus R_0, for k $= 0.5$, $J_0 = 0.5$, $\beta = 0.01$, a $= 4$.

Alfvén continuum, Aw $= \omega_{min} - \omega_N$, is plotted as a function of the normalized current density, in figure 5. The eigenfrequencies for TCA and NET are found to be 2.3 MHz and 0.8 MHz, respectively, as shown in Table I and II. This shows a good agreement with experimental values given by De Chambrier (1983) for TCA and those predicted for NET given by Borg et al. (1989).

Table I - Comparison between the cylindrical and toroidal eigenfrequencies for TCA; n $= 3, q_0 = 1, q_w = 5$.

m	$\omega_N(cyl.)$	$\omega_N(toro.)$	$\omega/2\pi(MHz)$
1	1.189	1.153	2.30
2	1.436	1.403	2.80

The main effect comes from the fact that the effective major radius $\bar{R} = R_0 + \zeta$, where, ζ is the displacement of the axis of the magnetic surface, affects the value

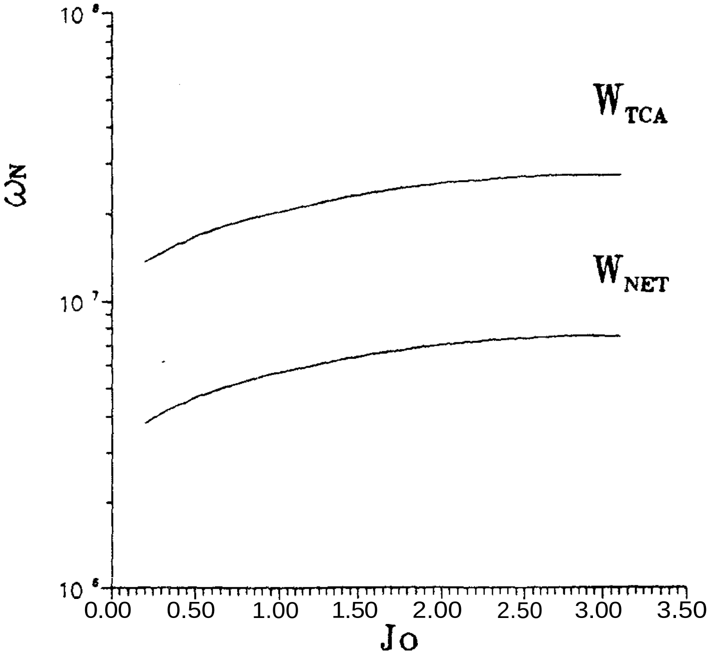

Fig. 4 – Normalized eigenfrequency, ω_N, for NET and TCA tokamaks, as function of the normalized current density ($\sim q_0^{-1}$). The following parameters were used:

NET: $R_0/a = 2.2,\ J_0 = 0.9,\ q_0 = 1,\ n = 3,\ \beta = 0.05$;

TCA: $R_0/a = 3.3,\ J_0 = 0.6,\ q_0 = 1,\ n = 3,\ \beta = 0.01$.

Table II - Comparison between the cylindrical and toroidal eigenfrequencies for $NET; n = 3, q_0 = 1, q_w = 3$.

m	$\omega_N(cyl.)$	$\omega_N(toro.)$	$\omega/2\pi(MHz)$
1	1.817	1.815	0.80
2	2.264	2.199	1.00

of k given by $n/$R.

5. Conclusions

We have shown that the discrete Alfvén wave exists in tokamaks when the current profile is very peaked, a, $\geq$ 2, where $J_z \sim (1 - r^2/a^2)^{\alpha_j}$. An approximate formula is derived to calculate the discrete Alfvén frequency for the case

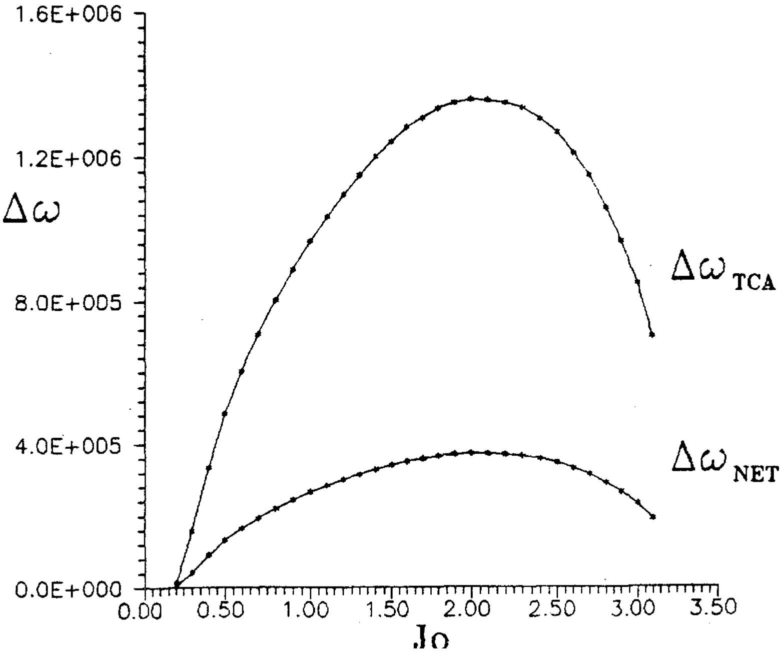

Fig. 5 – Distance for the discrete eigenfrequency to the lower edge of the Alfvén continuum, *Aw,* for NET and **TCA** as function of the normalized current density.

of small twist and very low β value. We have applied this analysis to TCA and NET tokamaks. The eigenfrequencies thus obtained show a good agreement with experimental results. We have shown that the WKB dispersion relation has three branches, the discrete or shear Alfvén, the magnetosonic and the helical wave. The helical wave is a new mode which has not appeared before in cylindrical modes. The most important result of this analysis is equation (6) which shows the dependence of the DAW mode with wave parameters and toroidal effect.

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Resumo

Os modos discretos da onda de Alfvén (DAW) em um plasma difuso e portador de corrente são estudados numericamente num modelo cilíndrico e com correções toroidais. Utilizando aproximação 'WKB', mostramos que há três automodos: discreto de Alfvén, magnetosônico e hélico. Uma expressão aproximada é deduzida para o modo discreto em termos dos parametros da onda conhecida e do efeito toroidal. As frequências discretas de Alfvén para os modos azimutal m e axial k ($= n/R$, onde n é o número de modo toroidal e R é o raio maior do toróide) são calculadas em função da corrente de plasma usando a equação completa, isto é, sem o uso da aproximação 'WKB'. O resultado é comparado com o que é obtido a partir da aproximação 'WKB'. Também são calculados usando os parâmetros dos tokamaks TCA e NET.