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A functional integral solution for Random fluid motion

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Abstract We propose a functional integral. representation for the probability distribution over the statistical ensemble of Newtonian fluid velocity fields satisfying an ultra-local gaussian random initial condition.

1.'Introduction

The main task in the statistical approach to fluid turburlence (or in the theory of random fluid dynamics[1]) is to solve the set of infinite hierarchy equations for the random fluid velocity correlation functions. There are two schemes to solve these equations: the first scheme consists in applying ad hoc closure approximations without any control[2,3]; the second, pionered by Hopf[3], consists in writing the above infinite set of equations as a single functional differential equation satisfyíng some suitable initial-time condition and trying to solve it by means of a functional integral[4].

Our aim in this paper is to present a (formal) functional integral solution for the Navier-Stokes equation with a Gaussian (ultra-local) random initial condition. This study is the content of section 2.

2. The functional integral

Let us start this section by writing the Navier-Stokes equation for the velocity field of an incompressible fluid in the presence of a non-random externa1 force $F_i(x, \tau)$ with a Gaussian (ultra-local) Random initial condition

$$\frac{\partial}{\partial\tau}v_i - \nu\Delta_x v_i + \left(v_k\frac{\partial}{\partial x_k}v^i\right)^{\mathrm{Tr}} = F_i \qquad (1.a)$$

$$v_i(x,0) = \varphi_i(x) \tag{1.b}$$

$$< \varphi_{i_1}(x_1)\varphi_{i_2}(x_2) >= \lambda\delta^{(3)}(x_1 - x_2)\delta(\tau_1 - \tau_2) \tag{1.c}$$

Let us remark that we have eliminated the pressure term $-\frac{1}{\rho}\vec{\Delta}\cdot\vec{p}$ by using the incompressibility condition $(\vec{\Delta}\vec{v}) \equiv 0$ which, in turn, lead us to consider only the transverse part of the force and non-inertial field terms in Navier-Stokes equation. The transverse part of a generical vector field $\vec{W}(x, 7)$ is defined by the relationship

$$\vec{W}^{\mathrm{Tr}}(x,\xi) = \vec{W}(x,\xi) + \frac{1}{4\pi}\vec{\Delta}_x\left(\int_{R^3} dy\,\frac{(\vec{\Delta}_y\vec{W})(y,\xi)}{|x-y|}\right)$$
$$\vec{W}(x,\xi) = \{W_i(x,\xi) \quad ; \quad i = 1,2,3\} \tag{2}$$

Our task, now, is to compute the φ-average of the N-point fluid velocity fields eq. (1.a), for arbitrary space-time points, by means of a functional integral representation for the characteristic functional of the random fluid velocity fields $Z[J_i(x,\xi)]$, namely[3]

$$\left\langle V_{i_1}(x_1,\xi_1),[\varphi])\ldots V_{i_N}(x_N,\xi_N,[\varphi])\right\rangle_\varphi$$
$$= (-1)^N \frac{\delta^{(N)}}{\delta J_{i_1}(x_1,\xi_1)\ldots\delta J_{i_N}(x_N)} Z[J_i(x,\xi)]\Big|_{J_i(x,\xi)=0} \tag{3.a}$$

where

$$Z[J_i(x,\xi)] = \int_M d\mu[V_i]\times$$
$$\exp\left(-\int_{R^3} dx \int_0^\infty d\xi (V_i\cdot J_i)(x,\xi)\right) \tag{3.b}$$

The functional measure $d\mu[V_i]$ in eq. (4) is defined over the functional space M of all possible realizations of the random fluid motion defined by eq. (1). An explicit (formal) expression for the above functional measure should be given by the product of the usual Feynman measure weighted by a certain functional $S[V_i]$ to be determined[5].

$$d\mu[V_i] = D^F[V^i]\exp(-S[V^i]) \tag{5.a}$$

$$D^F[V^i] = \prod_{\substack{x\in R^3\\ 0<\xi<\infty\\ i=1,2,3}} (dV_i(x,\text{E})) \tag{5.b}$$

In order to determine the Weight Functional $S[V_i]$ we first rewrite the Navier-Stokes Equation as a pure integral equation which has an explicit term taking into account the initial condition[4]

$$A_i[\tilde{v}] = B_i[\varphi] \tag{6}$$

with

$$A_i[v] = V_i(x,\xi) - \int_0^\infty ds \int_{R^3} dy O_{ijk}(x-y,\xi-s) \times (V_j V_k)(y,s) - \int_0^\infty ds \int_{R^3} dy H_{(1)}(x-y,\xi-s) F_i(y,s) \tag{7.a}$$

$$B_i[\varphi] = \int_{R^3} dy H_{(0)}(x-y,\xi)\varphi_i(y) \tag{7.b}$$

Here, the Kernels $O_{ijk}, H_{(1)}, H_{(0)}$ are given respectively by

$$O_{ik\ell}(z,\xi) = -\frac{1}{2}\left(\frac{\partial}{\partial Z_\ell}\bar{O}_{ik} + \frac{\partial \bar{O}_{i\ell}}{\partial Z_k}\right)(z,\xi) \tag{8.a}$$

$$\bar{O}_{pq}(z,\xi) = \delta_{pq}\theta(\xi)H_{(0)}(|z|,\xi) + \frac{\partial^2}{\partial z_p \partial z_q}\left(\frac{2\nu\xi}{|z|}\int_0^{|z|} H_0(|z'|,\xi)dz'\right) \tag{8.b}$$

$$H_{(0)}(|z|,\xi) = \frac{1}{(4\pi\nu\xi)^{3/2}}\exp\left(-\frac{|z|^2}{4\pi\nu\xi}\right) \tag{8.c}$$

$$H_{(1)}(|z|,\xi) = \theta(\xi)H_{(0)}(|z|,\xi) \tag{8.d}$$

Let us now introduce the following functional representation for the generating functional $Z[J_i(x,\xi)]^{(os)}$

$$Z[J_i(x,\xi)] = \int D^F[V^i]\left\langle \delta^{(F)}(V^i - \tilde{V}^i[\varphi])\right\rangle_\varphi \times \exp\left(-\int_{R^3} dx \int_0^\infty d\xi (V^i . J^i)(x,\xi)\right) \tag{9}$$

where $\delta^{(F)}(\cdot)$ denotes the delta - functional integral representation defined by the rule

$$\int_M D^F[V_i]\delta^{(F)}(V_i - A_i)\Sigma(V_i) = \Sigma(A_i) \tag{10}$$

with $\Sigma(V_i)$ being an arbitrary functional defined on M.

By writing the (o-average in eq. **(9)** by means of a Gaussian functional integral in $\varphi(x,\xi)$, we obtain the following functional integral representation for the weight $S[V^i]$

$$\exp(-S[V^i]) = \int D^F[\varphi^i]\exp\left[-\frac{1}{2\lambda}\int_{R^3}dx\int_0^\infty d\xi(\varphi^i\cdot\varphi^i)(x,\xi)\right]$$
$$\times\left\{\int D^F[K^i]\exp i\int_{R^3}dx\int_0^\infty d\xi K_i\cdot(A^i[v]-B^i[\varphi])\right\} \tag{11}$$

where we have used the Fourier Functional Integral representation for the Delta - functional in eq.(9)[4]

$$\delta^{(F)}(V^i - V^i[\varphi]) = \delta^{(F)}(A_i[v]-B_i[\varphi]) = \mathrm{DET}\left(\frac{\delta}{\delta V_i}A_k[v]\right)\times$$
$$\int D^F[K^i]\exp\left(i\int_{R^3}dx\int_0^\infty d\xi K_i(A^i[v]-B^i[\varphi])(x,\xi)\right) \tag{12}$$

It is worth remarking that the functional determinant in eq. (12) is unity as a straightforward consequence of the fact that the Green function of the operator $\partial/\partial\xi$ is the step function (see eq. (40) in ref. 4).

We, then, face the problem of evaluating the (o and K functional integrals in eq. (12).

The φ-functional integral is of Gaussian type and easily evaluated

$$\int D^F[\varphi^i]\exp\Big(-\frac{1}{2\lambda}\int_{R^3}dx\int_0^\infty d\xi(\varphi_i\cdot\varphi_i)(x,\xi)\Big)\times$$
$$\exp\Big(i\int_{R^3}dx\int_0^\infty d\xi(K_i\cdot B^{i,*}[\varphi])(x,\xi)\Big)$$
$$=\exp\Big\{-\frac{\lambda}{2}\int_{R^3}dx_1\int_{R^3}dx_2\int_0^\infty d\xi_1\int_0^\infty d\xi_2 K^i(x_1,\xi_1)\times$$
$$C(x_1,\xi_1;x_2,\xi_2)K^i(x_2,\xi_2)\Big\} \tag{13}$$

where the Kernel $C(x_1,\xi_1;x_2,\xi_2)$ is given by

$$C(x_1,\xi_1;x_2,\xi_2) = \int_{R^3} dz H_{(0)}(x_1 - z,\xi_1)H_{(0)}(z - x_2,\xi_2) \tag{14}$$

and is the (formal) Green function of the self - adjoint extension of the square $B^{i*}B_i$ diffusion operator

$$\Big(-\frac{\partial}{\partial\xi_1}-\nu\Delta_{x_1}\Big)\Big(\frac{\partial}{\partial\xi_1}-\nu\Delta_{x_1}\Big)C(x_1,\xi_1;x_2,\xi_2)=0 \tag{15}$$

with the (well-posed) initial and boundary conditions

$$\lim_{\xi_1\to 0^+} C(x_1,\xi_1;x_2,\xi_2)=\delta^{(3)}(x_1-x_2) \qquad (16.a)$$

$$C(x_1,0,x_2,\xi_2)=C(x_1;\infty,x_2,\xi_2) \qquad (16.b)$$

Its explicit expression in K-momentum space is given by (see ref. 6).

$$\tilde{C}(k;\xi_1,\xi_2)=-\frac{1}{\nu k^2}[e^{-nuk^2|\xi_1-\xi_2|}-e^{-nuk^2(\xi_1+\xi_2)}] \qquad (17)$$

As a consequence of eq. (13) we have represented the weight $S[v^i]$ by a Gaussian functional integral in the $K_i(x,\xi)$ field

$$\exp(-S[v^i])=\int D^F[K^i]\exp\Big[-\frac{\lambda}{2}\int_{R^3}dx_1dx_2\int_0^\infty d\xi_1 d\xi_2\Big(K^i(x_1,\xi_1)C(x_1,\xi_1;x_2,\xi_2)K_i(x_2,\xi_2)\Big)\Big] \times \exp\Big(i\int_{R^3}dx\int_0^\infty d\xi (K^i\dot{A}_i[v])(x,\xi)\Big) \qquad (18)$$

By evaluating eq. (18) we, thus, obtain the result

$$\exp(-S[v^i])=\exp\Big(-\frac{1}{2\lambda}\int_{R^3}dx_1dx_2\int_0^\infty d\xi_1 d\xi_2\, A^i[v](x_1,\xi_1)C^{-1}(x_1,\xi_1;x_2,\xi_2)A_i[v](x_2,\xi_2)\Big) \qquad (19)$$

By noting that (see ref. 4).

$$B_i^{-1}[A[v]]=\Big(\frac{\partial}{\partial\xi}-\nu\Delta_x\Big)A^i[v] =\Big(\frac{\partial}{\partial\xi}-\nu\Delta_x\Big)v^i+\Big(v_k\frac{\partial}{\partial x_k}\Big)^{\mathrm{Tr}}-F_i^{\mathrm{Tr}} \qquad (20)$$

we finally obtain the expression for the weight $S[v^i]$

$$S[v^i]=\frac{1}{2\lambda}\int_{R^3}dx\int_0^\infty d\xi\Big[\Big(\frac{\partial}{\partial\xi}-\nu\Delta_x\Big)A^i[v]\Big]^*(x,\xi)\times \Big[\Big(\frac{\partial}{\partial\xi}-\nu\Delta_x\Big)A_i[v]\Big](x,\xi) =\frac{1}{2\lambda}\int_{R^3}dx\int_0^\infty d\xi\Big\{\Big(\frac{\partial}{\partial\xi}-\nu\Delta_x\Big)v^i +\Big(v_k\frac{\partial}{\partial x_k}v^i\Big)^{\mathrm{Tr}}-F_i^{\mathrm{Tr}}\Big\}^2(x,\xi) \qquad (21)$$

By substituting eq. **(22)** in eq. **(9)** we obtain our proposed functional integral representation for eq. **(1)**

$$Z[J_i(x,\xi)] = \int D^F[v^i]\exp(-S[v^i]$$
$$\exp\Big(-\int_{R^3} dx \int_0^\infty d\xi (J_i.V_i)(x,\boldsymbol{i})\Big) \qquad (22)$$

The above written functional integral is the main result of this paper.

A perturbative analysis for eq. **(22)** may be implemented by using the free propagator eq. (17) in the context of a background field decomposition V, = $\bar{V}_i + \beta V_i^q$ where $\bar{V}_i$ satisfies the non-Random **Navier** Stokes equation

$$\frac{\partial}{\partial \xi}\bar{V}_i = \nu\Delta_x \bar{V}^i - \left(\bar{V}_k \frac{\partial}{\partial x_k}\bar{V}^i\right)^{\mathrm{Tr}} + F^{\mathrm{Tr}} \qquad (23.a)$$

with

$$\bar{V}^i(x,0) \ \mathbf{r}\ \text{o}, \qquad (23.b)$$

with β being a coupling constant ($\beta << 1$). It is worth remaking that the cross term

$$\int_{R^3} dx \int_0^\infty d\xi (\partial_i v^i \Delta_x v_i)(x,\xi) \qquad (24.a)$$

vanishes in $S[v^i]$ as a result of the boundary condition

$$V_i(x,0) = v_i(x,\infty) = 0 \qquad (24.b)$$

for the "free propagator" (see eq. **(19)** and ref. **6.**

Finally, we point out that our proposed functional integral eq. (22) differs from that proposed in ref. **7.**

A complete perturbative analysis will be presented elsewhere.

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Resumo

Apresentamos uma solução tipo integral funcional para a Equação de Navier-Stokes tendo como condição inicial uma distribuição aleatória gaussiana ultra-local.