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Multifractality in Magnetic Models

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Abstract The connection between the multifractal properties and the critical behavior of the local magnetization of the ferromagnetic Ising model on hicrarchical lattices is established. A linear relation between the β-critica1 exponent of the local magnetization and the a-crowding index (Hõlder exponent) is obtained showing that a continuous infinite set of critical exponents $\{\beta_i\}$ is required to describe the critical behavior of the local magnetization of the system. Each β_i exponent corresponding to a class of sites of the system is related to a γ_i-local susceptibility exponent by a "hyperscaling" relation where the $f(\alpha_i)$ function value plays the role of the dimensionality of this class of sites.

Introduction

In this paper we discuss the multifractal properties and critical behavior of the local magnetization of the Ising model on hierarchical lattices. We review the main results of our previous work [see reference 1] and discuss new insights to the problem.

The study of spin models on hierarchical lattices is of special interest to statistical mechanics since the Migdal-Kadanoff renormalization group equations[2] were proved to be exact on the diamond hierarchical lattice (hereafter DHL)[3]. Several properties of the Ising model on these lattices were established. For instance, the thermodynamic limit of the free energy has been proved to be well defined[4,5]. The total spontaneous magnetization has been analytically obtained[5] and the total susceptibility has been found to be infinite for T > T_c [5,6] and finite for T < T_c [5,7]. In reference 1 we have shown that the local magnetization of the ferromagnetic Ising model on the diamond hierarchical lattice is a fractal measure, i.e., it has a multifractal structure. We found that the local magnetization of a given site can be obtained by an exact recursion relation between the local magnetization of two sites of previous, distinct, hierarchical levels. From this relation we were able to get the f (a)-function which characterizes the multifractality, the average magnetization and its critica1 exponent. Our approach allows us to calculate the critical exponent β_i associated with the local magnetization of every class of sites of the lattice. Now we search for the connections between the critical exponents describing the local behavior of the order parameter and those associated with its multifractal structure. The former are more related to the phase transition undergone by the order parameter, that is, to the thermodynamical behavior of the model hamiltonian close to the transition, while the latter are more related to the spatial distribution of the local order parameter over the lattice, that is, to the geometry and topology of the lattice. Actually we found a linear relation between the local β_i critica1 exponent associated with the local magnetization of a particular class of sites of the lattice and the corresponding a,-exponent (called Holder exponent or crowding index). Furthermore we obtain a relation between the γ_i-exponent of the local susceptibility, the v-exponent of the correlation length, β_i, and the f (α_i) value for this particular class of siites. The latter corresponding to a "hyperscaling relation" for each class of sites of the lattice.

In section 2 we describe the model Hamiltonian and review the multifractal properties of the local and average magnetization of the Ising model on the DHL. Section 3 is devoted to the study of the connection between the order parameter critica1 exponent, the α-exponent and f (a)-function. Finally the conclusions are summarized in section 4.

2. The Model Hamiltonian and the Local Magnetization

We consider here the ferromagnetic Ising model on the simplest hierarchical lattice, namely the diamond hierarchical lattice. Starting with a bond linking the two sites (called roots, vertices or surface sites), the DHL with N levels is constructed by replacing this bond by the basic cell (see figure 1-a). Then the new bonds of the cell are successively replaced by the basic cell it self. Each step introduces new sites which constitute the levels of the DHL (see figure 1-b). The reduced Hamiltonian of the spin-1/2 ferromagnetic Ising model with zero field in an N-level DHL is (the zero-level DHL is the initial bond):

$$-\beta \mathcal{H}_N = K_N \sum_{<ij>} \sigma_i \sigma_j \tag{1}$$

where K_N is the reduced coupling constant of the exchange interaction between all pairs $< ij >$ of nearest-neighbor spins of the N-level DHL, and the *a's* are the spin variables $(\sigma = \pm 1)$. To analyze the structure of the local magnetization it is sufficient to look at the sites of one of the shortest paths joining the two roots of the DHL. All the shortest paths are equivalent since each one contains (in a symmetrically arranged way with respect to the middle point) all kind of sites of the DHL with distinct coordination numbers and depths with respect to the roots. We can identify the sites on this path by a pair (s,ℓ) where ℓ is the level (ℓ=1,2,...,N) an s is the position (s=1,3,5,7,...,2^ℓ-1) of the site within the ℓ-th level with respect to one root (see figure i-c). If ℓ=N, then s represents the chemical distance from the considered site (s,N) to one of the roots. The local magnetization of a given site of the last hierarchical level (ℓ = N) and that of its nearest-neighbor sites can be evaluated as a function of the coupling constant K_N, the effective fields and the effective interaction acting upon these neighbors. These effective fields and coupling which enclose the effect of the remaining lattice can be formally obtained by tracing over spin variables of the remaining lattice, following the spirit of the decoration transformation[a]. However, by eliminating these effective fields and interaction we end up with a recurrence equation between the local magnetization of a given site of the N^{th}-level and the local magnetizations of two sites belonging to the $(N-1)^{th}$ and j^{th} levels (j=0,1,2,...N-2) respectively. By induction this equation can be generalized to each site of the lattice giving rise to

$$m_{s,\ell} = A_\ell(m_{s',\ell-1} + m_{s'',j}) \tag{2}$$

where $s' = (s \pm 1)/2$, $s'' = (s \mp 1)/2^{\ell-j}$ and

$$A_\ell = t_\ell/(1+t_\ell^2) \tag{3}$$

with $t_\ell = \tanh K_\ell$. For temperatures below the ferromagnetic critica1 temperature T_c, the local magnetization profile of the DHL can be obtained from eqs.(2) and (3), and with the help of the renormalization equation for the coupling constants of the DHL, given by

$$t_{\ell-1} = 2t_\ell^2/(1+t_\ell^4) \tag{4}$$

A. The Local Magnetization Profile at the Critical Point

The local magnetization profile at the critical point for one of the shortest paths joining the two vertices of the DHL can be straightforwardly given from eq.(2) by replacing A_ℓ for A, where

$$A_c = \frac{t_c}{1+t_c^2} = 0.419643...$$

t_c being the exact unstable fixed point solution of eq.(4), that is

$$t, = (a, - 2/a_c - 1)/3 = 0.543689...$$

with a, $= (3\sqrt{33} + 17)^{1/3}$. For an N-level DHL the sites of a given shortest path on the ℓ^{th}-level can be arranged over the interval [0,1] such that the pair (s, ℓ) corresponds to the point $s.2^{-\ell}$ (with $s = 1.3.5....2^\ell - 1$ and $\ell = 1.2.....N$).

In figure 2 we display the local magnetization profile of the DHL with N=10 levels at the critical point with fixed ferromagnetic boundary conditions which correspond to assuming both spins at the roots [0 and 1] with fixed values a = 1. Actually, to obtain the local magnization profile it is sufficient to fix just one spin of the lattice in order to break symmetry. However this leads to an asymmetric profile. We remark some properties of the profile shown in figure 2.

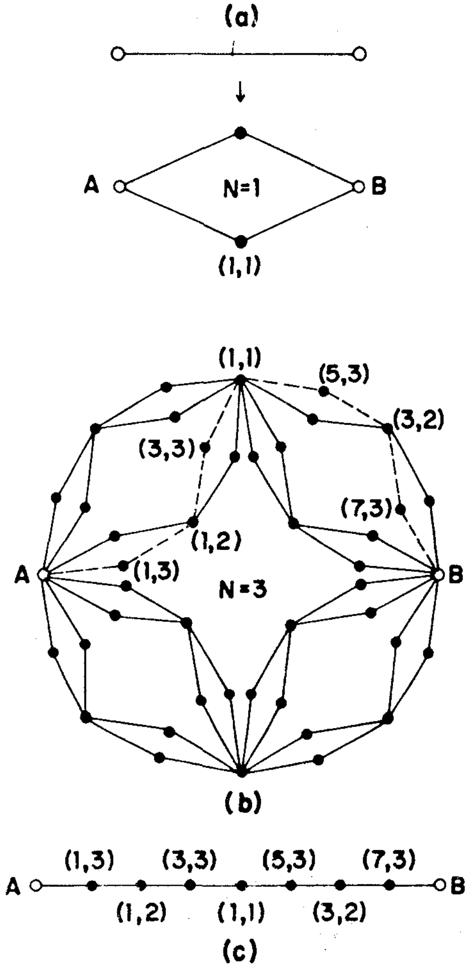

Fig. 1 – Diamond Hierarchical Lattice (DHL). a) One bond replaced by the basic cell. b) The DHL up to three levels. The open circles are the roots. The broken line indicates an arbitrary shortest path joining the roots. c) The sequence of sites (s, ℓ) appearing in a shortest path between the roots A and B for the 3-level DHL.

i) The local magnetization is symmetric by reflection with respect to the middle point, due to the present boundary conditions.

ii) The average magnetization vanishes in the $N \rightarrow \infty$ thermodynamic limit as it should be. This will be shown later.

iii) The magnetization profile has no smallest scale. Figure **3** shows the magnification of figure *2* between two deep sites.

iv) There is a discontinuity everywhere in the infinite level limit. For every value of m there is a $mA_c/(1-A_c) \approx 0.72m$ discontinuity both to the left and right side limits.

v) The fractal dimension of the profile is about 1.8791... (box counting).

vi) The local magnetization is a fractal measure. (See II.B).

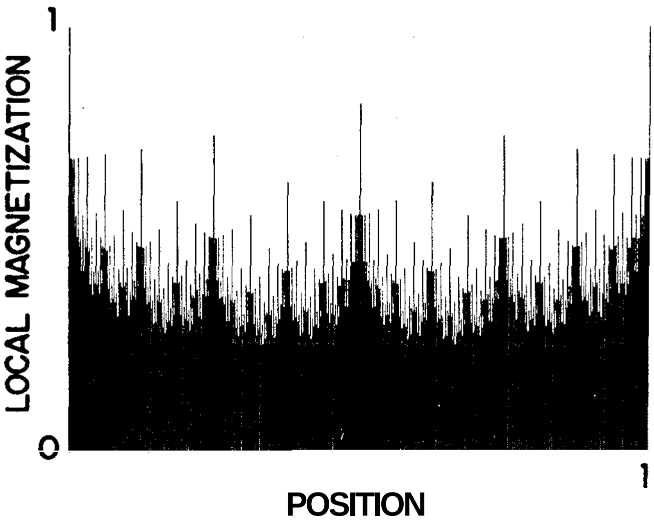

Fig. 2 - Local magnetization (at the critical point) against spin position (in an arbitrary scale) within a shortest path joining the two roots of a DHL with N=10 levels. Note: the average magnetization of this profile does vanish at the $N \Rightarrow \infty$ thermodynamic limit, which is not evident from this N=10 plot.

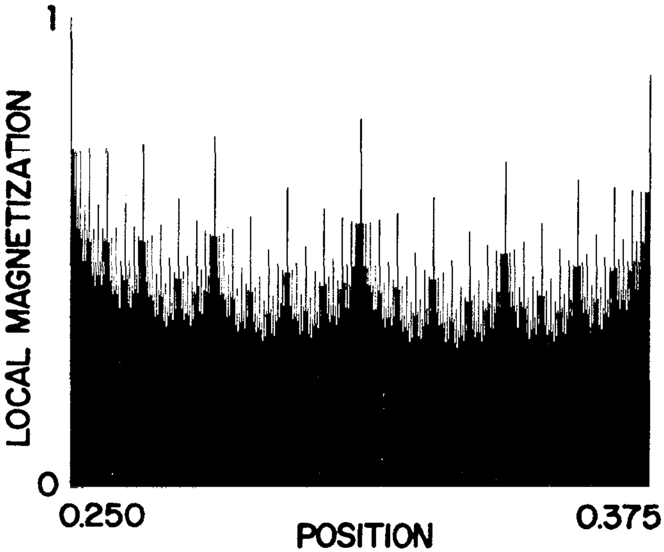

Fig. 3 – Magnification of figure 2 between 0.250 and 0.375, on a renormalized scale.

B. Multifractal properties. The f(a)-function

Consider a magnetization profile of an N-level DHL covered by a set of boxes $\{\varepsilon_i\}$ each one with the same box width ε. *A* local measure within the box ε_ρ can be defined by

$$\mu_\rho = lim_{\varepsilon\to 0}\frac{m_\rho}{\sum_\rho m_\rho} \tag{5}$$

where m_ρ is the average local magnetization within the box ε_ρ and $\sum_P m_\rho$ is the total magnetization of the profile. Let S_j be the set of boxes for which the measure vanishes with exponent α_j as the box width goes to zero, that is

$$\mu_\rho \sim \varepsilon^{\alpha_j} \qquad \forall \varepsilon_\rho \in S_j \qquad \varepsilon \to 0 \tag{6}$$

Let $\mathcal{N}_j$ be the number of elements of the set S_j, and f(α_j) the exponent such that $\mathcal{N}_j$ diverges as $\varepsilon \to 0$, that is

$$\mathcal{N}_j \sim \varepsilon^{-f(\alpha_j)} \qquad \varepsilon \to 0 \tag{7}$$

The dependence of $f(\alpha_j)$ with a, can be obtained by displaying the measure distribution in a reduced double logarithmic plot (with reducing factor equal to $\ln\varepsilon$). In the $\varepsilon \rightarrow 0$ limit this plot should converge to the f(a)-function, which reflects how densely the singularities of the measure are dístributed. In figure (4) we show the f(a)-function for the local magnetization of'the DHL with N=29 levels for temperatures close to the critica1 point. Despite the slow convergence, figure (4) contains most features of an f(a)-function, say: a concave curve with a single maximum equal to the Hausdorff dimension of the support (which is one in our case), and an infinite slope at the points in which f(a) = 0. In this figure (dashed line) we also show what we expect the true f(a)-function (N $\Rightarrow \infty$ limit) to look like. The values α_{min} and a,,,, which reflect, respectively, how the measures of the most concentrated and the most rarified intervals scale with the box width, are calculated exactly. We get

$$\alpha_{min} = -\ln t_c/\ln 2 \simeq 0.8791...$$
$$\alpha_{max} = 2 - \left\{\ln\left[(1-t_c)(1+(1+4t_c^{-1}+4t_c)^{\frac{1}{2}})\right]/\ln 2\right\} \simeq 1.0460... \qquad (8)$$

a,,, is in extremely good agreement with th numerical end point of the graph for N=29 levels. For α_{min} as well as the value $f_{max}(\alpha) = 1$, convergence is slow. In the inset of figure (4) we show the values of $f_{max}(\alpha)$ as a function of N^{-1}, which indicate this convergence.

C. The average magnetization

The average magnetization (per site) of the entire lattice is defined by

$$m = \lim_{N\Rightarrow\infty}\left[\left(2+\sum_{\ell=1}^{N}2^{\ell}\sum_{s=1,3...}^{2^{\ell}-1}m_{s,\ell}\right)\Big/\Phi(N)\right], \qquad (9)$$

where $\Phi(N) = \frac{2}{3}(2+2^{2N})$ is the number of sites in an N-level DHL. With help of eq.(2) we can show that the temperature dependent average magnetization is given by

$$m(T) = \prod_{i=1}^{\infty}\frac{1}{2}(1+2A_i) \qquad (10)$$

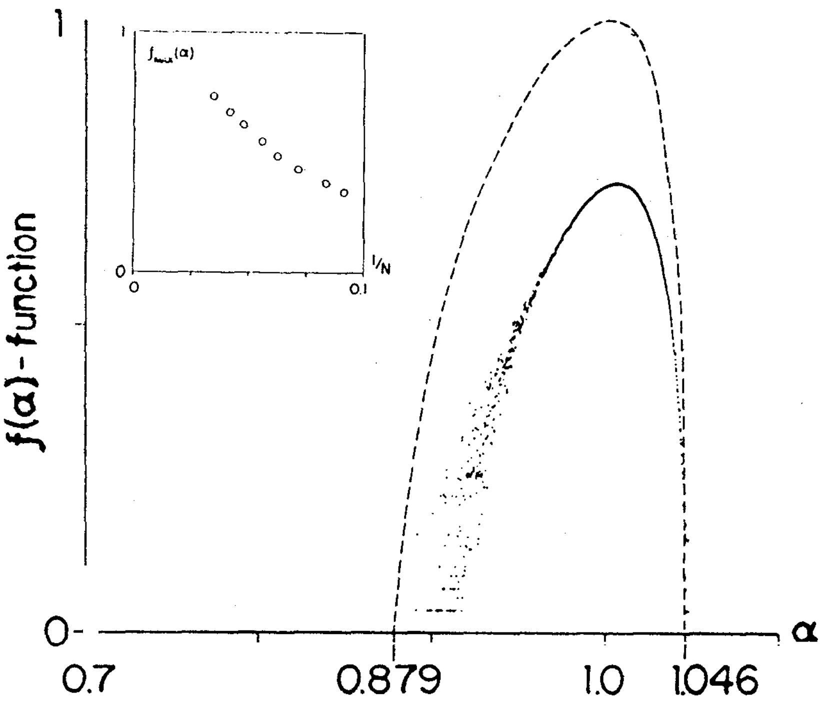

Fig. 4 – f(a)-function of the DHL local magnetization profile for N=29 levels. The dashed curve represents the expected N ⇒ ∞ limit. The inset gives the maximum values of f(a) as function of N^{-1}, showing the slow convergence towards one as N ⇒ ∞.

recovering a previous result[5].

At the critical point A_i = A, = 0.419643..., then with help of eq.(3) we get from eq.(10) that

$$m(T_c) = \lim_{n\to\infty}\left[\frac{1}{2}(1+2A_c)\right]^n = \lim_{n\to\infty}\left(\frac{1}{2t_c}\right)^n \to 0 \tag{11}$$

as should be expected. From eq.(2) and (9) one is able to show that the total magnetization also follows a recursion equation given by

$$M_N = \left[1 + 2(1+2A_{n-1})\frac{A_n}{A_{n-1}}\right] M_{N-1} - 2(1+2A_{n-1})\frac{A_n}{A_{n-1}} M_{N-2} \tag{12}$$

which at the critical point gives for the average magnetization the recursion equation

$$m_N = \frac{1}{4}(1+2t_c^{-1})m_{N-1} - \frac{1}{8}t_c^{-1}m_{N-2} \qquad (N \to \infty) \tag{13}$$

3. Critical Behavior

A. Average critical behavior

The critical exponent β associated with the average magnetization can be evaluated assuming that the total magnetization at the critical point shows a power law behavior with respect to the linear size of'the system L that is, $M(L) \sim L^D$ [9]. Close to T_c the average magnetization $m(L) = M(L)/L^d$ (d being the dimensionality of the system) behaves in the L $\to \infty$ limit as $(t)^\beta$ where $t = (T - T_c)/T_c$. On the other hand the correlation length, which at the critical point scales as the linear size of the system behaves like $\xi \sim$ L $\sim (t)^{-\nu}$ where ν is the correlation length exponent. Therefore one can write that the average magnetization has a $t^{-\nu(D-d)} \sim t^\beta$ dependence, which leads to

$$\beta = -\nu(D-d) \tag{14}$$

For our system we get from eq.(9) that

$$M_N(T_c) \sim (2/t_c)^N \qquad (N \to \infty) \tag{15}$$

giving that $D = 1 - \ln t_c/\ln 2 = 1.87914...$ On the other hand, v can be obtained from the coupling constant renormalization equation (see eq.(4)) giving $1/\nu = 2\ln(1+t_c^2)/\ln 2 = 0.747235....$ Since $d = \ln 4/\ln 2 = 2$ for the DHL, the result from eq.(14) is that

$$\beta = \ln(2t_c)/2\ln(1+t_c^2) = 0.161734... \tag{16}$$

This exact value of β for the average magnetization of the entire lattice, which was already obtained by other approaches[5,10], can also be calculated exactly from the recursíon equation for the average magnetization. Actually assuming that close to the transition temperature $m_N \sim (\delta t_N)^\beta$, where $\delta t_N \simeq r_c \delta t_{N-1}$ with

$r_c = [dt_N/dt_{N-1}]_{t_c} = (1+t_c^2)^2$, we get from eq.(13) the exact value of β given by eq.(16). The γ critical exponent associated to the susceptibility per spin can be also estimated by using the finite size scaling argument. The susceptibility per spin is equivalent to the average of the squared magnetization. Henee at the critical point the susceptibility per spin $\chi \sim L^{2D-d}$ and therefore χ has a $t^{\nu(2D-d)}$ dependence giving $\gamma = \nu(2D-d)$. For our particular system we end up with

$$\gamma = -\frac{\ln t_c}{\ln(1+t_c^2)} = 2.35306... \tag{17}$$

which is the γ exponent for the average susceptibility per spin of the Ising model on the DHL.

B. Local critical behavior

- Relation with the $f(\alpha)$-spectrum

In this section we focus our attention on the critical behavior of a particular subset of sites S_j represented by a point in the $f(\alpha)$-spectrum, that is, for which their measure behaves with exponents α_j and $f(\alpha_j)$ as defined by eqs.(6) and (7). Assuming that the average magnetization m_j of the set S_j behaves with β_j exponent at the critical point, that is, $m_j \sim (t)^{\beta_j}$ one can write that

$$\mu_j = \frac{m_j}{\sum_j m_j \mathcal{N}_j} \sim \frac{(t)^{\beta_j}}{L^D} \tag{18}$$

where the summation runs over the subsets with distinct values of α_j, i.e., $\sum_i m_j \mathcal{N}_j$ equals the total magnetization of the system. From eq.(6) we have that $\mu_j \sim L^{-\alpha_j}$. Therefore from eq.(18) we get $(L)^{-\beta_j/\nu - D} \sim L^{-\alpha_j}$ yielding in the $L \to \infty$ limit:

$$\beta_j = \nu(\alpha_j - D) \tag{19}$$

which relates the β_j critical exponent of the local magnetization to the α_j exponent. For the present model this relation can be evaluated explicitly. At the critical point with $N \to \infty$ we have $m_j \sim (\delta t_N)^{\beta_j}$, $\delta t_N \sim (r_c)^N \delta t_0$, $r_c = (1+t_c^2)^2$, $M_j = \sum_i m_j \sim (2/t_c)^N$ and $\varepsilon = 2^{-N}\varepsilon_0$. With help of eqs.(5) and (6) we get

$$\beta_j = \frac{\ln 2}{\ln r_c}\left[\alpha_j - 1 + \frac{\ln t_c}{\ln 2}\right] \tag{20}$$

Comparing eqs.(19) and (20) we recover our previous results (see 3.A), respectively $D = 1 - \ln t_c / \ln 2$ with $\nu = \ln 2 / \ln$ r,.

Now combining eqs.(19) and (14) to eliminate D we end up with

$$\alpha_j = d + \frac{1}{\nu}(\beta_j - \beta) \tag{21}$$

For the local magnetization profile (figure 2) the $f(\alpha)$-function can be rescaled with help of eq.(21) (with d=1) yielding a "$f(\beta)$-function" shown in figure (5). We remark that the set of sites represented by α_{min} and f(a,,,) = 0 for which the measure is most concentrated has $\beta_{min} = 0$. This correponds in our model to the set of sites with finite magnetization at T_c. We recall that this happens as a consequence of the fixed boundary conditions at the roots, which induce a finite magnetization at the "magnetically neighboring" surface sites. On the other hand the set of points represented by a,,, and f (a,,,) = 0 for which the measure is most rarified corresponds to

$$\beta_{max} = \frac{1}{2}\left[1 - \ln(t_c(1 + (1 + 4t_c^{-1} + 4t_c)^{1/2}))/2) / \ln(1 + t_c^2)\right] = 0.22335\ldots \tag{22}$$

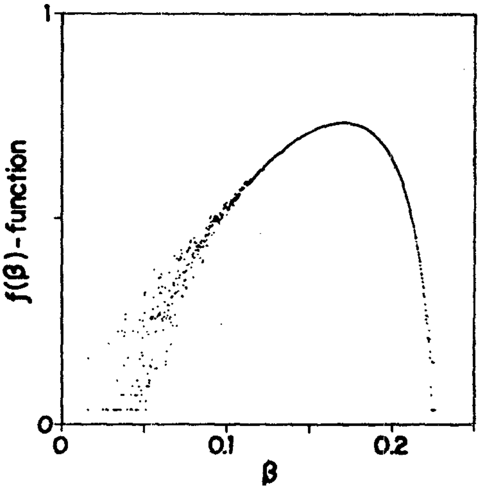

Fig. 5 – "$f(\beta)$–function" of the DHL local magnetization profile for N=29 levels.

Actually the set of sites described by β_{max} corresponds to the "magnetically deepest" sites of the DHL. One can show that the local magnetization of these sites at T_c can be reached by the recursion relation

$$m_\ell = A_c[m_{\ell-1} + m_{\ell-2}] \qquad (\ell \to \infty) \tag{23}$$

starting from every pair of neighboring sites at all levels. Taking $m_\ell \sim (t_\ell)^\beta$, and $t_\ell = r_c t_{\ell-1}$ we obtain from eq.(23) and the definition (3) the exact expression of β given by eq.(22). From eq.(23) is also easy to see that $m_\ell \to 0$ as $\ell \to \infty$.

From eq.(21) it is evident that the class of sites for which the local magnetization behaves at T_c with he same exponent of the whole lattice is the one characterized by $\alpha = d$. In the present model, for the local magnetization profile this corresponds to $\alpha = 1$ (or $\beta = 0.161...$, see eq.(16)).

- Local hyperscaling relation

The average magnetization m_j of the class of sites S_j behaves at T_c with a β_j critical exponent, that is $m_j \sim (t)^{\beta_j}$. On the other hand we can write that $m_j = M_j/\mathcal{N}_j$. Now assuming that at T_c the total magnetization of S_j behaves like $M_j \sim L^{D_j}$, D_j being a characteristic exponent and L is the linear size of the relevant cluster which corresponds to the correlation length at $T_c (L \sim \xi \sim t^{-\nu})$ we get that

$$\beta_j = -\nu[D_j - f(\alpha_j)] \tag{24}$$

The susceptibility per spin of this class S_j behaves at T_c with a critical exponent γ_j $(\chi_j \sim (t)^{-\gamma_j})$. Since the susceptibility per spin is equivalent to the average squared magnetization $(\chi_j \sim L^{2D_j - f(\alpha_j)})$ we get

$$\gamma_j = \nu[2D_j - f(\alpha_j)] \tag{25}$$

By eliminating D_j from eqs.(24) and (25) we end up with

$$\nu f(\alpha_j) = 2\beta_j + \gamma_j \tag{26}$$

which corresponds to the "hyperscaling relation" for each class of sites characterized by α_j and $f(\alpha_j)$. Note that $f(\alpha_j)$ plays the role of the dimensionality of S_j.

4. Conclusions

The multifractal properties of the local magnetization of the ferromagnetic Ising model on hierarchical lattices were reviewed and extended from ref. **1.** In that reference the f (a)-spectrum of the normalized local magnetization profile was numerically obtained by an exact recursion procedure, showing a multifractal behavior at the critica1 temperature. The boundary values α_{min} and α_{max} for which the f (a)-function vanishes were analitically obtained confirming the numerical results. The multifractal behavior described by the f (a)-function implies that there exists an infinite class of sites of the system such that the normalized local magnetization (measure) of each one scales with a particular exponent. Now we have shown that the local magnetization of each class of sites characterized by a point of the f (a)-function, say a, and $f(\alpha_i)$, has a distinct β_i critical exponent at T_c. A linear relation between α_i and β_i was established as shown by eq. (21). We note that the class of sites for which the local magnetization behaves like the average magnetization, that is with $\beta_i = \beta$, scales with the dimensionality of the system, that is with $\alpha_i = d$. We remark that eq. **(21)** which constitutes the main result of this paper should hold for a general system presenting a second order phase transition with a multifractal order parameter. With help of cq. **(21)** the f (a)-spectrum can be rescaled leading to an "$f(\beta)$-spectrum" as shown in figure (5) for the present model. Since an infinite number of exponents is required to describe the critica1 behavior of the order parameter we are driven to look the "f (β)-spectrum" of the lattice as a good candidate to express the "universality class" of the model. Moreover we have also shown that for every set of sites described by a β_i exponent there exists a γ_i exponent associated with the local susceptibility which related to β_i through a "hyperscaling" relation, where the corresponding $f(\alpha_i)$-function value plays the role of the "dimensionality" of the corresponding class of sites. This is given by eq. (26). We note that all sites characterized by critical exponents β_i and γ_i, are spread out over the whole system and undergo a phase transition at the same critical temperature. Therefore the correlation length should diverge at

T_c with the same exponent ν, otherwise a particular class of sites would dominate the behavior of the system at the transition. Regarding the behavior of the whole system, we have calculated recursion equations for the total and average magnetization which allow us to obtain the critical exponents of the whole system verifying that they obey the hyperscaling relation.

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References

1. W. A. M. Morgado, S. Coutinho and E. M. F. Curado, J. Stat. Phys. **61**, 913 (1990).
2. A. A. Migdal, JETP 69, 810; 1457 (1975) and L. P. Kadanoff, Ann. Phys. Phys. 100, 359 (1976).
3. P. M. Blěher and E. Zâlys, Commun. Math. Phys. 67, 17 (1979); A. N. Berker and S. Ostlund, J. Phys. C 12, 4961 (1979).
4. R. B. Griffiths and M. Kaufman, Phys. Rev. B 26, 5022 (1982).
5. P. M. Blěher and E. Zālys, Commun. Math. Phys. 120, 409 (1989).
6. M. Kaufman and R. B. Griffiths, J. Phys. **A15**, L239 (1982).
7. S. R. McKay and A. N. Berker, Phys. Rev. B 29, 1315 (1984).
8. M. Fisher, Phys. Rev. 113, 969 (1959).
9. N. Ito and M. Suzuki, Prog. Theor. Phys. 77, 1391 (1987); J. Phys. (Paris) 49; C8-1565 (1988).
10. A. O Caride and C. Tsallis, J. Phys. **A20,** L667 (1987) and J. R. Melrose, J. Phys. **A16**, 3077 (1983).

Resumo

Investigamos a relação entre o comportamento crítico e as propriedades multifractais da magnetização local do modelos de Ising ferromagnético em redes hierárquicas. Mostramos que existe uma relação linear entre o intervalo de variação

dos expoentes de Hõlder-a e o conjunto contínuo, infinito, de expoentes críticos $\{\beta_i\}$ necessário para descrever o comportamento-crítico da **magnetização** local do sistema. Mostramos ainda que para cada classe de sítios da rede hierárquica existe uma relação de "hiperescala" conectando β_i, o expoente crítico da susceptibilidade de correlação ν. Nesta relação $f(\alpha_i)$ desempenha o papel da **"dimensionalidade"** da classe de sítios correspondente.