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Noether symmetries and integrable two-dimensional systems

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Abstract In this paper, the general form of the Noether theorem is used as a systematic procedure for the identification of integrable two-dimensional systems. We give some applications for polynomial potentials, including the generalized Hénon-Heiles case.

1. Introduction

The analysis of the regular or chaotic behavior of general nonlinear systems is an important problem in applied mathematics. In particular, the identification of integrable systems and the study of the relation between integrability and the symmetry structure of the system has been considered, in the last years, by several authors[1-6]. In this work, we use the general form of the Noether theorem in the analysis of two-dimensional hamiltonian systems. Let us start from the two-dimensional system described by the lagrangian

$$L = \dot{x}^2/2 + \dot{y}^2/2 - V(x,y) \tag{1}$$

and, therefore, with the following equations of motion

$$\begin{aligned} \ddot{x} &= -\frac{\partial V}{\partial x} \\ \ddot{y} &= -\frac{\partial V}{\partial y}. \end{aligned} \tag{2}$$

This system will be integrable, in the sense of Liouville[1], if, in addition to the energy, it admits a second isolated conserved quantity $I(\dot{x},\dot{y},x,y)$. In this case, no chaos will appear and the behavior of the system will be regular. By using the Noether theorem, we establish a systematic procedure for the search of two-dimensional systems with a nontrivial symmetry transformation and, consequently, with a second invariant.

There is no general method for determining whether a system of differential equations is integrable or not. However, in recent years, partial and important results in this direction were obtained by Ziglin and Yoshida[7], for two-dimensional homogeneous hamiltonian systems. We will summarize, now, some procedures used in the identification of integrable systems and for obtaining of the second invariant:

1 - Direct Method

This method was introduced by Laplace and developed y Bertrand and Whittaker[8]. Here, we assume the existence of a second invariant with a polynomial form on the velocities

$$I=\sum_{n,m} f_{n,m}(x,y)\dot{x}^n\dot{y}^m \tag{3}$$

and we impose

$$\frac{dI}{dt}=0,$$

for the eq. (2). We obtain a system of partial differential equations that, if solved, leads to integrable systems which have an invariant with the form (3). This procedure gives us quite general results and applies also in the case of non-hamiltonian systems[9]. A limitations of this method has to do with the computational difficulties, which restrict it to the obtaining low order polynomial invariants; this problem can be attenuated with the utilization of algebraic computation[10].

2 - Lie Symmetries

The method, introduced by Lie[11], consists of the determination of the symmetry transformations of the equations of motion and identification of the second invariant[4]. If we assume only geometrical symmetries, we obtain, at most, invariants with a quadratic dependence on the velocities. Due to this fact, many integrable systems can not be found by this procedure[4,12]. The method is applied also for nonhamiltonian systems.

3 - Painlevé test

The singular point analysis, introduced by S. Kowalewski and P. Painlevé, can serve, in a certain sense, to decide between integrable and nonintegrable dynamical systems. A system of ordinary differential equations in the complex domain is said to be of Painlevé type if the only movable singularities of all its solutions are poles. This means that there are no movable branch points, nor movable essential singularities[13]. The Painlevé test is used as a criterion for the integrability of the system, but there is no general proof of this conjecture. It is convenient, in many cases to extend the concept of Painlevé property to the, so-called, weak Painlevé property[14]. There are interesting connections between the analytical properties of the system and its integrability[15].

4 - Noether symmetries

The Noether symmetries are infinitesimal point transformations which maintain the invariance (up to a constant) of the action functional. By this theorem, for each symmetry there is an associated invariant. If we consider only geometrical transformations, the Noether symmetries constitute a subgroup of the Lie symmetry group for the corresponding equations of motion. In this case the, procedure of identification of integrable systems is very limited. However, the utilization of the generalized form of the Noether theorem, by assuming symmetry transformations with a dependence on the velocities, gives us a more general method for obtaining of invariants and generalised symmetries[16]. In this work, we explore this possibility for two-dimensional systems. We show how a general procedure for the identification of invariants and their associated symmetries can be obtained by considering the generalized symmetries. In this way, we get an over determined set of partial differential equations, which permits the identification of integrable potentials. These results are applied to the generalized Hénon-Heiles potential and we find the three integrable cases. We consider also the Kepler potential, perturbed with polynomial terms in x and y, with the form

$$V = -k/r + ax^n + by^n, \tag{4}$$

n being an integer, and the potential $V = x^n y^m$.

2. Noether theorem and integrability conditions

We take the general formulation of the Noether theorem as reviewed by Cantrijn and Sarlet[17]. If an infinitesimal transformation

$$\begin{aligned} t' &= t + \epsilon\xi(t, x_i, \dot{x}_i) \\ x_i' &= x_i + \epsilon\eta_i(t, x_i, \dot{x}_i) \end{aligned} \tag{5}$$

leaves invariant (up to a constant) the action

$$S = \int_1^2 L(t, x_i, \dot{x}_i)dt, \tag{6}$$

there exists an invariant for the system given by

$$I = \frac{\partial L}{\partial \dot{x}_i}(\eta_i - \dot{x}_i\xi) + \xi L - f(t, x_i, \dot{x}_i). \tag{7}$$

The conditions for the infinitesimal transformations (5) to be symmetry transformations of (6) are

$$L\frac{\partial \xi}{\partial \dot{x}_i} + \frac{\partial L}{\partial \dot{x}_j}\left(\frac{\partial \eta_j}{\partial \dot{x}_i} - \dot{x}_j\frac{\partial \xi}{\partial \dot{x}_i}\right) = \frac{\partial f}{\partial \dot{x}_i} \tag{8}$$

$$\begin{aligned} &\xi\frac{\partial L}{\partial t} + \eta_i\frac{\partial L}{\partial x_i} + L\left(\frac{\partial \xi}{\partial t} + \dot{x}_i\frac{\partial \xi}{\partial x_i}\right) + \\ &+ \frac{\partial L}{\partial \dot{x}_i}\left[\frac{\partial \eta_i}{\partial t} + \dot{x}_j\frac{\partial \eta_i}{\partial x_j} - \dot{x}_i\left(\frac{\partial \xi}{\partial t} + \dot{x}_j\frac{\partial \xi}{\partial x_j}\right)\right] = \\ &= \frac{\partial f}{\partial t} + \dot{x}_i\frac{\partial f}{\partial x_i} \end{aligned} \tag{9}$$

As the choice of ξ is free, we can made it equal to zero[17]. Furthermore, if we know an invariant, the associated symmetry can be obtained from

$$\eta_i = -g^{ij}\frac{\partial I}{\partial \dot{x}_j}, \tag{10}$$

where g_{ij} is defined by

$$\frac{\partial^2 L}{\partial \dot{x}_i \partial \dot{x}_j} g^{jk} = \delta_i^k. \tag{11}$$

For the two-dimensional systems analysed in this work, the lagrangian is

$$L = \dot{x}^2/2 + \dot{y}^2/2 - V(x,y). \tag{12}$$

In this case, the energy is the first invariant and the existence of a second independent invariant guarantees the integrability of the system, We will apply the Noether theorem for finding general conditions on the potential $V(x,y)$, which lead to the second invariant. The conditions (8) and (9), for this case, give us the following equations

$$\dot{x}\frac{\partial \eta_1}{\partial \dot{x}} + \dot{y}\frac{\partial \eta_1}{\partial \dot{x}} = \frac{\partial f}{\partial \dot{x}} \tag{13}$$

$$\dot{x}\frac{\partial \eta_1}{\partial \dot{y}} + \dot{y}\frac{\partial \eta_2}{\partial \dot{y}} = \frac{\partial f}{\partial \dot{y}} \tag{14}$$

$$\begin{aligned} &-\eta_1\frac{\partial V}{\partial x} - \eta_2\frac{\partial V}{\partial y} + \dot{x}\Big(\frac{\partial \eta_1}{\partial x}\dot{x} + \frac{\partial \eta_2}{\partial y}\dot{y}\Big) + \\ &+ \dot{y}\Big(\frac{\partial \eta_2}{\partial x}\dot{x} + \frac{\partial \eta_2}{\partial y}\dot{y}\Big) = \dot{x}\frac{\partial f}{\partial x} + \dot{y}\frac{\partial f}{\partial y}, \end{aligned} \tag{15}$$

where we take $f = f(x_i, \dot{x}_i)$ and $\eta_i(x_i, \dot{x}_i)$, due to our interest on explicitly time-independent invariants.

The compatibility condition between (13) and (14) leads to

$$\frac{\partial \eta_2}{\partial \dot{x}} = \frac{\partial \eta_1}{\partial \dot{y}} \tag{16}$$

We assume now that the invariant we look for has the following polynomial form

$$I = \sum_{n=0}^{N} F_n(\dot{x}, x, y)\dot{y}^n. \tag{17}$$

From (10) we get

$$\eta_1 = -\sum_{n=0}^{N} \frac{\partial F_n}{\partial \dot{x}} \dot{y}^n \tag{18}$$

$$\eta_2 = -\sum_{n=0}^{N} n F_n \dot{y}^{n-1}. \tag{19}$$

Condition (16) is satisfied by (18) and (19). From (13), (14), (18) and (19), we obtain

$$f = -\sum_{n=0}^{N} \left[\dot{x}\frac{\partial F_n}{\partial \dot{x}} + (n-1)F_n\right]\dot{y}^n. \tag{20}$$

By substituting (18), (19) and (20) in (15), we find the conditions to be verified by the F_n and by $V(x, y)$:

$$\frac{\partial V}{\partial x}\frac{\partial F_n}{\partial \dot{x}} + (n+1)F_{n+1}\frac{\partial V}{\partial y} - \dot{x}\frac{\partial F_n}{\partial x} - \frac{\partial F_{n-1}}{\partial y} = 0. \tag{21}$$

where $n = 0, 1, \ldots, N$.

Therefore, there are $(N+2)$ relations to be satisfied; the $(N+1)$ functions F_n can be determined from the first $(N+1)$ relations in (21). The last equation in (21) imposes restrictions on the F_n and $V(x, y)$. For example, the first condition, for $n = N$, leads to

$$F_N = h_N(x, \dot{x}); \tag{22}$$

the second condition, for $n = N-1$, furnishes

$$F_{N-1} = \frac{\partial h_N}{\partial \dot{x}} \int \frac{\partial V}{\partial x} dy - \dot{x}y\frac{\partial h_N}{\partial x} + h_{N-1}(x, \dot{x}) \tag{23}$$

and so on.

We can consider a more general lagrangian, with terms with a linear dependence on the velocities due to the presence, for instance, of a magnetic field:

$$L = \dot{x}^2/2 + \dot{y}^2/2 + A(x,y)\dot{x} + B(x,y)\dot{y} - V(x, y). \tag{24}$$

In this case, the conditions (21) will have the generalized form

$$\frac{\partial V}{\partial x}\frac{\partial F_n}{\partial \dot{x}} + (n+1)F_{n+1}\frac{\partial V}{\partial y} - \dot{x}\frac{\partial F_n}{\partial x} - \\ - \frac{\partial F_{n-1}}{\partial \dot{x}}\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) = 0, \tag{25}$$

with $n = 0, 1, ..., N$.

3. Some applications

The previous relations are too general, so we will take some particular cases. Let us start with $N = 1$. In this case, the invariant which we obtain will have a linear dependence on $\dot{y}$ and (17), *(20)* and *(21)* lead to

$$I = F_1\dot{y} + F_0 \tag{26}$$

with

$$\frac{\partial F_1}{\partial y} = 0 \tag{27}$$

$$\frac{\partial V}{\partial x}\frac{\partial F_1}{\partial \dot{x}} - \dot{x}\frac{\partial F_1}{\partial x} - \frac{\partial F_0}{\partial y} = 0 \tag{28}$$

$$\frac{\partial V}{\partial x}\frac{\partial F_0}{\partial \dot{x}} + F_1\frac{\partial V}{\partial y} - \dot{x}\frac{\partial F_0}{\partial y} = 0 \tag{29}$$

and

$$f = -\dot{x}\dot{y}\frac{\partial F_1}{\partial \dot{x}} - \dot{x}\frac{\partial F_0}{\partial x} + F_0. \tag{30}$$

Solving *(27)* and (28), we obtain

$$F_1 = h_1(x, \dot{x}) \tag{31}$$

$$F_0 = \frac{\partial h_1}{\partial \dot{x}}\int \frac{\partial V}{\partial x}\, dy - \frac{\partial h_1}{\partial x}\dot{x}y + h_0(x, \dot{x}). \tag{32}$$

The substitution of *(31)* and *(32)* in *(29)* will permit the determination of the potentials $V(x, y)$ with a invariant (17), when $N = 1$. They must satisfy the following condition:

$$\begin{aligned}
&\frac{\partial V}{\partial x}\left[\frac{\partial^2 h_1}{\partial \dot{x}^2}\int \frac{\partial V}{\partial x}dy - y\frac{\partial h_1}{\partial x} - \dot{x}y\frac{\partial^2 h_1}{\partial x \partial \dot{x}} + \frac{\partial h_0}{\partial \dot{x}}\right] + \\
&+ h_1\frac{\partial V}{\partial y} - \dot{x}\left[\frac{\partial^2 h_1}{\partial x \partial \dot{x}}\int \frac{\partial V}{\partial x}dy + \frac{\partial h_1}{\partial x}\frac{\partial}{\partial x}\left(\int \frac{\partial V}{\partial x}dy\right)\right] - \\
&- \dot{x}y\frac{\partial^2 h_1}{\partial x \partial \dot{x}} + \frac{\partial h_0}{\partial x} = 0.
\end{aligned} \tag{33}$$

We take our first example. The generalized Hénon-Heiles potential is given by

$$V = ax^2/2 + by^2/2 + dx^2y - (1/3)ey^3. \tag{34}$$

This potential, with $a = b = d = e = 1$, was introduced for modelling the motion of a star in an axial galaxy[18] and turned into an important model for the study of the integrability of two-dimensional systems.

From (34), (31) and (32):

$$F_1 = h_1(x, \dot{x}) \tag{35}$$

$$F_0 = axy\frac{\partial h_1}{\partial \dot{x}} + dx^2y\frac{\partial h_1}{\partial x} - \dot{x}y\frac{\partial h_1}{\partial x} + h_0(x, \dot{x}). \tag{36}$$

Condition (33) leads to the following expressions for F_1 and F_0 :

$$F_1 = C_2\dot{x}x^r \tag{37}$$

$$\begin{aligned} F_0 = {} & (azy + dxy^2)C_2x^r - \dot{x}^2yC_2rx^{r-1} - \frac{r(r-1)}{8d}C_2x^{r-3}\dot{x}^4 + \\ & + (1/4d)C_2\dot{x}^2[3ar - (b-a)]x^{r-1} + (1/2d)C_2a\left[\frac{3ar-(b-a)}{r+1}\right]x^{r+1} + \\ & + (1/(r+3))C_2dx^{r+3}, \end{aligned} \tag{38}$$

where $r = -\left(\frac{e+d}{5d}\right)$, and (33) will only be satisfied for the following values:

$$\begin{array}{lll} 1) & \underline{r=0} & e = -d \\ & & a = b \\ 2) & \underline{r=1} & e = -6d \\ 3) & \underline{r=3} & e = -16d \\ & & b = 16a \end{array} \tag{39}$$

The general form of the second invariant, from (26), is

$$\begin{aligned} I = {} & (azy + dxy^2)x^r - \dot{x}^2yrx^{r-1} + (1/4d)(3ar - b + a)x^{r-1} - \\ & - (1/8d)r(r-1)x^{r-3}\dot{x}^4 + \dot{x}\dot{y}x^r + \frac{a(3ar-b+a)x^{r+1}}{2d(r+1)} + \frac{dx^{r+3}}{r+3}. \end{aligned} \tag{40}$$

The three cases in (39) have been obtained separately, through the application of the Painlevé test and with the utilization of the direct method for the identification of polynomial invariants[19]. The process employed here has led us to obtain both of these integrable cases and we also obtain the explicit form for the generalized symmetries which are associated to the invariants (40). From (18), (19), (37) and (38), we obtain

$$\eta_1 = 2\dot{x}yrx^{r-1} - \dot{y}x^r - (\dot{x}/2d)(3ar - b + a)x^{r+1} + \frac{r(r-1)\dot{x}^3x^{r-3}}{2d} \tag{41}$$

$$\eta_2 = -\dot{x}x^r. \tag{42}$$

In our second example, we consider the perturbed Kepler potential (4). By using the eqs. (31), (32) and (33), two solutions arise:

1) For $N = 2$ and $a = b$

$$F_1 = -C_1 x$$
$$F_2 = C_1 \dot{x} y. \tag{43}$$

For this case, the second invariant of the system will be

$$I = C_1(y\dot{x} - x\dot{y}) \tag{44}$$

which is the expected conservation of the angular momentum. The associated symmetries are

$$\eta_1 = -C_1 y \quad ; \quad \eta_2 = C_1 x. \tag{45}$$

Of course, in this case, the more restricted geometrical form of the Norther theorem would be sufficient for the identifications of this invariant.

2) For $N = 2$ and $b = 4a$

$$F_1 = x\dot{x}$$
$$F_0 = -\dot{x}^2 y + (g/r)y + 2ayx^2 \tag{46}$$

and the second invariant is

$$I = \dot{x}(x\dot{y} - \dot{x}y) + (g/r)y + 2ayx^2 \tag{47}$$

with the following associated symmetries:

$$\eta_1 = 2\dot{x}y - x\dot{y}$$
$$\eta_2 = -\dot{x}x. \tag{48}$$

Our last example will be for a potential with the form

$$V = x^m y^n, \tag{49}$$

where m and n are different from zero. Eqs. *(31)* and *(32)* give us

$$F_1 = h_1(\dot{x}, x)$$
$$F_0 = (m/(n+1))x^{m-1}y^{n+1}\frac{\partial h_1}{\partial \dot{x}} - \dot{x}y\frac{\partial h_1}{\partial x} + h_0(x, \dot{x}) \tag{50}$$

and *(33)* leads to the following condition:

$$(m^2/(n+1))x^{2m-2}y^{2n+1}\frac{\partial^2 h_1}{\partial \dot{x}^2} - \Big[mx^{m-1}\frac{\partial h_1}{\partial x} +$$
$$+ (m/(n+1))(n+2)\dot{x}x^{m-1}\frac{\partial^2 h_1}{\partial x \partial \dot{x}} + (m/(n+1))(m-1)\dot{x}x^{m-2}\frac{\partial^2 h_1}{\partial \dot{x}^2}\Big]y^{n+1} +$$
$$mx^{m-1}\frac{\partial^2 h_0}{\partial \dot{x}}y^n + nx^m h_1 y^{n-1} + \dot{x}^2 y\frac{\partial^2 h_1}{\partial x^2} - \dot{x}\frac{\partial h_0}{\partial x} = 0, \tag{51}$$

with $n \neq -1$.

Only for the case $n = m = 1$ the condition *(51)* is satisfied and we derive

$$F_1 = C_1\dot{x}$$
$$F_0 = (C_1/2)(x^2 + y^2). \tag{52}$$

The second invariant is

$$I = \dot{x}\dot{y} + (1/2)(s^2 + y^2) \tag{53}$$

with the symmetries

$$\eta_1 = -\dot{y} \quad ; \quad \eta_2 = \dot{x}. \tag{54}$$

If we consider N = 2 or, in other words, if we admit the existence of **invariants** with a quadratic **dependence** on y, the system of eqs. (21) will have 4 equations whose solution is not trivial. However, the same procedure discussed here, for the case where N = 1, can be employed. For example, a nontrivial integrable case emerging from this analysis, for the potential (49), will occur **if** n = m = –1. **In** this case, the potential is

$$V = (1/xy) \tag{55}$$

and the second invariant will be

$$I = (1/2)y^2\dot{x}^2 + (1/2)x^2\dot{y}^2 - xy\dot{x}\dot{y} + (y/x) + (x/y), \tag{56}$$

with the following associated symmetries:

$$\begin{aligned} \eta_1 &= xy\dot{y} - y^2\dot{x} \\ \eta_2 &= xy\dot{x} - x^2\dot{y}. \end{aligned} \tag{57}$$

4. Concluding remarks

A method, based on the generalized form of the Noether theorem, has been presented in order to find integrable two-dimensional hamiltonian **systems.** We found the conditions for the existence of a polynomial second invariant on y and for the determination of the associated symmetries. The restricted application to hamiltonian systems and the fastidious calculations are the main limitations of this method. The advantage of the procedure is the possibility of obtaining, **jointly, all** the integrable cases, for a given N, and the determination of the associated symmetries, as we showed for the generalized Hénon-Heiles system. It suggests also a **deeper** analysis of the relations between the existence of symmetries, the integrability of the system and the verification of the Painlevé property.

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Resumo

Neste trabalho utilizamos a forma geral do Teorema de Noether como um procedimento sistemático para a identificação de sistemas bidimensionais integráveis. Algumas aplicações são feitas, entre as quais a análise dos casos integráveis do potencial de Hénon-Heiles generalizado.