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Gravitational motion of systems with mass quadrupole moments

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Abstract The equations of motion of a system with spin and mass quadrupole moments are derived from energy-momentum conservation using the method of Papapetrou.

1. Introduction

The equations of motion of a system with spin and quadrupole moments have been derived by Dixon[1] in a covariant approach using two-point tensors and for a more complete theory[2] using also the theory of vector bundles.

We shall derive the equations of motion following the moment method of Papapetrou[3]. Although non-covariant, this method is mathematically much more simple and straightforward. Instead of the arc ienght parameter s, the derivatives will be refered to an arbitrary parameter q which will then include the case for system[4] in which $ds^2 = 0$ and for which it seems that no covariant approach has yet being developed.

The method will give also the integrated form of the energy-momentum tensor density and of its first and second moments. Also one sees clearly the renormalization of the momentum and spin tensors by the second order moments.

Finally we mention that the definitions of the multipole moments, in this case of the spin and quadrupole moments, appear in a natural way and even using a quadrupole cuttoff one does not run into troubles of appropriate definitions of the multipoles, as discussed by Dixon[2] when analysing previous works on the subject by Madore[5] and Taub[6]. In a natural way the method obeys the requirement that the higher multipoles should couple to corresponding increasing derivatives of the field, which is necessary for the validity of using a multipole cuttoff for bodies small in comparison with the length scale of the external field.

2. The Equations of Motion

We start with the covariant version of the energy-momentum conservation law, which in terms of the density $T^{ij} = \sqrt{-g}\times$ energy-momentum tensor reads

$$\partial_j T^{ij} = -\Gamma^i_{mn} T^{mn}. \tag{1}$$

Following the method of Papapetrou[3] we consider an extended system with reference point $X^i(q)$ as a function of the path parameter q and velocity

$$u^i = \frac{dX^i}{dq} \tag{2}$$

We shall take moments of T^{ij} up to the second order about X^i and this will define our pole-dipole-quadrupole system. From eq. (1) we have the equations

$$\partial_k(x^j T^{ik}) = T^{ij} - x^j \Gamma^i_{mn} T^{mn}, \tag{3}$$

$$\partial_r(x^k x^i T^{jr}) = x^i T^{jk} + x^k T^{ji} - x^k x^i \Gamma^j_{mn} T^{mn}, \tag{4}$$

and

$$\partial_j(x^s x^r x^k T^{ij}) = x^r x^k T^{is} + x^s x^k T^{ir} + x^s x^r T^{ik} - x^s x^r x^k \Gamma^i_{mn} T^{mn}. \tag{5}$$

Following Papapetrou we integrate all these equations over the three-dimensional space volume of our system for constant t. The left-hand side of eq.(1) and of *(3)-(5)* become respectively,

$$\frac{d}{dt}\int T^{io} dV \quad , \quad \frac{d}{dt}\int x^j T^{io} dV,$$

$$\frac{d}{dt}\int x^k x^i T^{jo} dV \quad , \quad \frac{d}{dt}\int x^s x^r x^k T^{io} dV. \tag{6}$$

Next we write

$$x^i = X^i + \delta x^i \tag{7}$$

with $X^0 = t$, that is $\delta x^0 = 0$. Then we use this equation in eq.(6) and in the integrated expressions of the right-hand side of eqs.(3)-(5). After we have done that we expand $\Gamma^i_{mn}(x)$ around $X^i(q)$,

$$\Gamma^i_{mn}(x) = \Gamma^i_{mn}(X) + \delta x^j \partial_j \Gamma^i_{mn}(X) + \frac{1}{2}\delta x^j \delta x^k \partial_j \partial_k \Gamma^i_{mn}(X) + ... \tag{8}$$

where $\partial_j = \partial/\partial X^j$.

Then, omitting the X dependence of the Γ's, we obtain the following set of equations up to second order in δx^i ($u^0 = dt/ds$),

$$\frac{d}{dq}\frac{M^{io}}{u^0} + \Gamma^i_{mn}M^{mn} + \partial_k \Gamma^i_{mn}M^{kmn} + \frac{1}{2}\partial_j\partial_k\Gamma^i_{mn}M^{jkmn} = 0 \tag{9}$$

$$\frac{u^j}{u^0}M^{io} + \frac{d}{dq}\frac{M^{jio}}{u^0} = M^{ij} - \Gamma^i_{mn}M^{jmn} - \partial_k\Gamma^i_{mn}M^{jkmn}, \tag{10}$$

$$\frac{u^k}{u^0}M^{ijo} + \frac{u^i}{u^0}M^{kjo} + \frac{d}{dq}\frac{M^{kijo}}{u^0} = M^{ijk} + M^{kji} - \Gamma^j_{mn}M^{kimn} \tag{11}$$

and

$$u^s M^{rkio} + u^r M^{skio} + u^k M^{srio} = (M^{rkis} + M^{skir} + M^{srik})u^0, \tag{12}$$

where

$$M^{ij} = u^0 \int T^{ij} dV, \tag{13}$$

$$M^{ijk} = u^0 \int \delta x^i T^{jk} dV, \tag{14}$$

$$M^{ijkr} = u^0 \int \delta x^i \delta x^j T^{kr} dV. \tag{15}$$

Note that all moments are symmetric in the last two indices and the second order moment is also symmetric in the first two indices. Also, as $\delta x^0 = 0$,

$$M^{ojk} = 0 \quad , \quad M^{ojkr} = 0. \tag{16}$$

Eqs.(9)-(12) contain the equations of motion of the pole-dipole-quadrupole particle mixed with other relations that will give the integrated form of the energy-momentum tensor and of their moments. They are the next order extension of the pole-dipole equations of Papapetrou[3].

3. The Spin Equation

The proper definition of the momentum, the spin and quadrupole moment of the system arises in a natural way if you look first for the spin equation. For that purpose we interchange i and j in eq.(10) and subtract the resulting equation from eq.(10) itself. As $M^{ij} = M^{ji}$ we obtain

$$\frac{d}{dq}\sigma^{ij} = M^{io}\frac{u^j}{u^0} + \Gamma^i_{mn}M^{jmn} + \partial_k\Gamma^i_{mn}M^{jkmn} - (i,j) \tag{17}$$

where

$$\sigma^{ij} = \frac{M^{ijo} - M^{jio}}{u^0} = \int (\delta x^i T^{jo} - \delta x^j T^{io})dV \tag{18}$$

and *(i,j)* stands for the preceding terms with i and j interchanged. This quantity is the spin tensor in the dipole approximation.

Now we concentrate in eq.(11) and obtain an expression for M^{ijk} by the following procedure[3]. Add to eq.(11) the one obtained from it by exchanging j and k and subtract from it the equation obtained by exchanging i and j. Keeping in mind that M^{ijk} is symmetric in the first two indices we obtain

$$\begin{aligned} 2M^{ikj} &= (M^{jko} + M^{kjo})\frac{u^i}{u^0} + u^k\sigma^{ij} + u^j\sigma^{ik} \\ &+ (\Gamma^k_{mn}M^{ijmn} + \Gamma^j_{mn}M^{ikmn} - \Gamma^i_{mn}M^{jkmn}) + 2\frac{dL^{ijk}}{dq} \end{aligned} \tag{19}$$

where

$$L^{ijk} = \frac{1}{2u^0}(M^{ijko} + M^{ijko} - M^{jkio}) = L^{ikj}. \tag{20}$$

Substituting eq.(19) in eq.(17) and using $\Gamma\, dL/dq = d(\Gamma L)/dq - L\, d\Gamma/dq$ with $d/dq = u^i d_i$, we obtain

$$\begin{aligned} &\frac{dS^{ij}}{dq} + u^m(\Gamma^j_{mn}\sigma^{in} + \Gamma^i_{mn}\sigma^{nj}) + \frac{1}{2}\Gamma^i_{mn}\Gamma^j_{rs}(M^{mnrs} - M^{rsmn}) \\ &= \pi^i u^j + (\partial_k\Gamma^i_{mn} + \Gamma^i_{ka}\Gamma^a_{mn})M^{jkmn} - u^k\partial_k\Gamma^i_{mn}L^{jmn} - (i,j) \end{aligned} \tag{21}$$

where

$$S^{ij} = \sigma^{ij} - \Gamma^i_{mn}L^{jmn} + \Gamma^j_{mn}L^{imn} \tag{22}$$

and

$$\pi^i = \frac{1}{u^0}(M^{io} + \Gamma^i_{mn}M^{mno}) \tag{23}$$

S^{ij} is the spin tensor of our system in the quadrupole approximation and π^i is its momentum in the dipole approximation.

To calculate the third term in the left-hand side of eq.(21) we go to eq.(12). We add to this equation the one obtain from it by exchanging i and s, and subtract from it the two equations obtained from it by the exchange of r and i, and afterwards k and i. The final result is, using eq.(20),

$$M^{rkis} - M^{isrk} - u^k L^{ris} + u^r L^{kis} - u^i L^{srk} - u^s L^{irk} \tag{24}$$

Taking this result in (21) and using (22) we obtain

$$\frac{DS^{ij}}{Dq} = \pi^i u^j + (\partial_k \Gamma^i_{mn} + \Gamma^i_{ka}\Gamma^a_{mn})(M^{jkmn} - u^k L^{jmn}) - (i,j) \tag{25}$$

where

$$\frac{DS^{ij}}{Dq} = \frac{dS^{ij}}{dq} + u^m(\Gamma^i_{mn}S^{nj} + \Gamma^j_{mn}S^{in}) \tag{26}$$

is the covariant derivative of S^{ij}.

We shall now try to obtain from the second factor in the second term of the right-hand side of eq.(25) a quantity which is antisymmetric in k and n, which will them combine with the Riemann curvature tensor

$$R^i{}_{mnk} = \partial_k \Gamma^i_{mn} + \Gamma^i_{ka}\Gamma^a_{mn} - (n,k). \tag{27}$$

Using eq.(24) we have, remembering eq.(20),

$$\begin{aligned}\Gamma^a_{mn} M^{jkmn} = \Gamma^a_{mn}[M^{mnjk} &- \frac{u^n}{u^0}(M^{mjko} + M^{mkjo} - M^{jkmo}) \\ &+ \frac{u^k}{u^0}(M^{jmno} - \frac{1}{2}M^{mnjo}) + \frac{u^j}{u^0}(M^{kmno} - \frac{1}{2}M^{mnko})].\end{aligned} \tag{28}$$

Now we add to this the term $\Gamma^a_{mn}M^{jkmn}$ itself and also the one obtained from it by making use of eq.(12). In this way we obtain

$$\begin{aligned}\Gamma^a_{mn} M^{jkmn} = \Gamma^a_{mn}\Big[\frac{4}{3}J^{jmkn} &+ \frac{u^k}{u^0}\Big(M^{jmno} - \frac{1}{2}M^{mnjo}\Big) \\ &+ \frac{u^j}{u^0}\Big(M^{kmno} - \frac{1}{2}M^{mnko}\Big)\Big]\end{aligned} \tag{29}$$

where

$$\begin{aligned}J^{jmkn} = \frac{1}{4}\Big[M^{jkmn} &+ M^{mnjk} + \frac{u^m}{u^0}M^{jkno} - \frac{u^j}{u^0}M^{mkno} \\ &- \frac{u^k}{u^0}\Big(M^{njmo} - M^{nmjo}\Big) - (k,n)\Big].\end{aligned} \tag{30}$$

This quantity has the same symmetry properties of the Riemann curvature tensor. It is the mass quadrupole moment introduced by Dixon[1,2].

Using eq.(29) in eq.(25) we obtain

$$\frac{DS^{ij}}{Dq} = p^i u^j + \frac{4}{3}(\partial_k \Gamma^i_{mn} + \Gamma^i_{ka}\Gamma^a_{mn})J^{jmkn} - (i,j) \tag{31}$$

where

$$p^i = \pi^i + \frac{1}{u^0}(\partial_k \Gamma^i_{mn} + \Gamma^i_{ka}\Gamma^a_{mn})\Big(M^{mkno} - \frac{1}{2}M^{mnko}\Big). \tag{32}$$

This quantity is the momentum of the system in the quadrupole approximation.

As J^{jmkn} is antisymmetric in k,n we can write with the help of eq. (27),

$$\frac{DS^{ij}}{Dq} = p^i u^j - \frac{2}{3}R^i{}_{mnk} J^{jmnk} - (i,j). \tag{33}$$

This is the equation of motion for the spin tensor. It coincides with the equation obtained by Dixon[1,2]. From eqs.(22) and (32) one sees clearly the renormalization of the spin and momentum tensors of the system by the second order moments of T^{ij}.

4. The Momentum Equation

From eq. (10) we shall derive an expression for M^{ij} that will be used in eq. (9). Using eq.(19) in (10) and making use of eqs.(22), (23), (29) and (32) we obtain

$$\begin{aligned} M^{ij} &= p^i u^j + \frac{d}{dq}\Big(\frac{M^{jio}}{u^0} + \Gamma^i_{mn}L^{jmn}\Big) + \Gamma^i_{mn}u^m S^{jn} \\ &\quad \frac{4}{3}(\partial_k \Gamma^i_{mn} + \Gamma^i_{sk}\Gamma^s_{mn})J^{jmkn} - \frac{2}{3}\Gamma^i_{mn}\Gamma^j_{rs}J^{mrns}. \end{aligned} \tag{34}$$

By making use of eqs.(19) and (22) the third term on the left-hand side of eq.(9) can be written as

$$\begin{aligned} \partial_k \Gamma^i_{mn} M^{kmn} &= \partial_k \Gamma^i_{mn}\Big[u^m S^{kn} + \Gamma^k_{ab}\Big(u^m L^{nab} - \frac{1}{2}M^{mnab}\Big) \\ &\quad - \Gamma^n_{ab}\Big(u^m L^{kab} - M^{kmab}\Big)\Big] + \frac{1}{u^0}M^{mno}u^k \partial_k \Gamma^i_{mn} \\ &\quad + u^a \partial_a(\partial_k \Gamma^i_{mn} L^{kmn}) - L^{kmn} u^a \partial_a \partial_k \Gamma^i_{mn}. \end{aligned} \tag{35}$$

Substitution of eqs.(34) and (35) in eq.(9) give, using eq.(29),

$$\begin{aligned} &\frac{Dp^i}{Ds} + (\partial_k \Gamma^i_{mn} + \Gamma^i_{kr}\Gamma^r_{mn})u^m S^{kn} + \frac{2}{3}J^{jkmr}[(\partial_b \Gamma^i_{jr} + \\ &+ \Gamma^i_{ab}\Gamma^a_{rj})\Gamma^b_{mk} - 2\Gamma^i_{kb}(\partial_m \Gamma^b_{rj} + \Gamma^b_{am}\Gamma^a_{rj}) - 2\partial_m \Gamma^i_{kn}\Gamma^n_{jr} \\ &- \partial_m \partial_k \Gamma^i_{rj}) = 0. \end{aligned} \tag{36}$$

The term $\partial_m\partial_k\Gamma^i_{rj}$ indicates the presence of $\nabla_m R^i{}_{rjk}$ where V, is the covariant derivative with respect to m. Using the symmetry properties of J^{jkmr} we have

$$J^{jkmr}\nabla_m R^i{}_{rjk} = J^{jkmr}(\partial_m R^i{}_{rjk} + \Gamma^i_{mn}R^n{}_{rjk}$$
$$+ 2\Gamma^n_{mk}R^i{}_{rnj}) = 2J^{jkmr}[\partial_m\partial_k\Gamma^i_{rj} + 2\partial_m(\Gamma^i_{kn}\Gamma^n_{rj})$$
$$+ 2\Gamma^i_{kb}\Gamma^b_{ma}\Gamma^a_{rj} - (\partial_a\Gamma^i_{rj} + \Gamma^i_{ab}\Gamma^b_{rj})\Gamma^a_{mk}]. \tag{37}$$

Using this result in eq.(36) and remembering eq.(27) we obtain

$$\frac{Dp^i}{Ds} = \frac{1}{2}R^i{}_{mnk}u^m S^{nk} + \frac{1}{3}J^{jkmr}\nabla_m R^i{}_{rjk}. \tag{38}$$

By making using of the Rianchi identities we can write eq.(38) in a slightly different way. We have

$$\nabla_m R^i{}_{rjk} + \nabla_r R_m{}^i{}_{jk} + \nabla^i R_{rmjk} = 0. \tag{39}$$

Therefore eq.(38) can be written as

$$\frac{Dp^i}{Ds} = \frac{1}{2}R^i{}_{mnk}u^m S^{nk} + \frac{1}{6}J^{jkmr}\nabla^i R_{jkmr}. \tag{40}$$

This formula agrees with Dixon's equation for the momentum.

5. The Integrated Energy-Momentum Tensor and its Moments

Now we shall express the quantities written in eqs.(13)-(15) in terms of p^i, u^i, S^{ij} and J^{ikmj}.

We shall start with the second moment. We add to eq.(30) the one obtained from it by the exchange of m and k and use eq.(12) for $M^{jkmn} + M^{jmkn}$ and for this sum with j and n interchanged. Using afterwards eq.(24) for M^{jnkm} we obtain

$$J^{jmkn} + J^{jkmn} = -6(M^{kmjn} - u^m L^{kjn} - u^k L^{mnj}). \tag{41}$$

Putting k = 0 and using eq.(20) together with $\delta x^0 = 0$ then

$$J^{jmon} + J^{jomn} = -3\frac{u^m}{u^0}M^{jnoo} + 6u^0 L^{mnj}. \tag{42}$$

Putting m = 0 in this last equation then

$$J^{jono} = 3M^{jnoo}. \tag{43}$$

Taking this result in the previous equation one gets

$$L^{mnj} = \frac{1}{6u^0}\left[J^{jmon} + J^{jomn} + \frac{u^m}{u^0}J^{jono}\right]. \tag{44}$$

Substituting this result in eq.(41) we obtain the following expression for the second moment of the energy-moment tensor

$$M^{kmjn} = \frac{1}{6}\left[J^{jmnk} + \frac{u^m}{u^0}\left(J^{nkoj} + J^{nokj} + \frac{u^k}{u^0}J^{nojo}\right)\right] + (m,k). \tag{45}$$

Next we consider the first moment. Using eqs. (22) and (29) in eq. (19) we obtain

$$\begin{aligned} 2M^{ijk} &= (M^{jko} + M^{kjo})\frac{u^i}{u^0} + u^k S^{ij} + u^j S^{ik} \\ &+ \frac{4}{3}(\Gamma^k_{mn}J^{jmin} + \Gamma^j_{mn}J^{kmin} - \Gamma^i_{mn}J^{jmkn}) \\ &+ u^i(\Gamma^k_{mn}L^{jmn} + \Gamma^j_{mn}L^{kmn}). \end{aligned} \tag{46}$$

Putting k = 0 in this equation and using eqs. (18), (20) and (43) one gets

$$\begin{aligned} 2M^{ijo} &= u^i S^{jo} + u^0 S^{ij} + u^j S^{io} + \frac{4}{3}(\Gamma^0_{mn}J^{jmin} \\ &+ \Gamma^j_{mn}J^{omin} - \Gamma^i_{mn}J^{jmon}) - \frac{1}{3}\frac{u^i}{u^0}\Gamma^j_{mn}J^{mono}. \end{aligned} \tag{47}$$

Using this result and eq. (44) in eqs. (34) and (46) we obtain M^{ij} and M^{ijk} in terms of u^i, p^i, S^{ij} and J^{ijkm}.

References

1. W.G. Dixon, Gen. Rel. Grav. 4, 199 (1973).
2. W.G. Dixon, Phil. Trans. R. Soc. London 277, 59 (1974).
3. A. Papapetrou, Proc. R. Soc. London A209, 248 (1951).
4. M. Bailyn and S. Ragusa, Phys. Rev. 23, 1258 (1981).
5. J. Madore, Ann. Inst. Henri Poincaré 11, 221 (1969).
6. A.H. Taub, Proc. Galileo IV Centenary Conference, Florence, 77 (1964).

Resumo

As equações de movimento de um sistema com spin e momentos de quadrupolo de massa são obtidos da conservação do momentum-energia, usando o método de Papapetrou.