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# $\Phi^4$ theory on a fractal lattice

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**Abstract** In this paper we perform the simulation of $\lambda\Phi^4$ theory on a fractal lattice. We present the results obtained for the behavior of this theory in a Sierpinski carpet when we vary the number of generations, for a fixed Hausdorf dimension, and when we keep the number of generations fixed, varying the Hausdorf dimension itself.

## Introduction

In the last few years, there has been much interest in the study of phase transitions on fractals. One of the reasons for such interest is the idea of using fractals, or certain kinds of fractals, to interpolate between integer dimensions in order to study the critical properties of statistical systems, since, until now, non integer dimensions only entered physics in a formal way,by means of continuous $\epsilon$ expansions near an integer dimension in the theory of critical phenomena[1].

The critica1 behavior of Ising-like models on fractal lattices has been studied by way of Real Space Renormalization Group[2,3,4], High Temperature expansions[5] and Numerical Simulations[6]. Both Z(3) gauge[7] and Ising gauge[8] theories have also been studied on fractal lattices by means of numerical simulations. Much effort has also been devoted, recently[9,10], to the question of how to characterize universality, if it exists, and also what is the more appropriate expression for lacunarity, which is a measure of the extent of the failure of a fractal to be translationally invariant [10].

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Renormalization Group techniques applied to Ising model spins on several self-similar lattices[2] have shown that the critical properties of these models depend not only on the fractal dimensionality but, also, on several topological factors[0], like the order of ramification, the connectivity Q, the lacunarity, etc ....

$\lambda\varphi^4$ theory, with integer dimension d ranging from 1 to 4, has been intensively studied in the last years. This theory, which provides an excellent laboratory for the study of more complex field theories, is known to have only one symmetric phase when $d = 1$ (Quantum Mechanics)[11], having two phases for $d > 1$ [12,13,14,15]. For $d = 2$ it has been shown that is has no finite temperature phase transition[12], that is, the critical temperature occurs at $T = 0$. As it is known that $\varphi^4$ is a non trivial theory for $d = 2$ and 3 and, probably, trivial for $d = 4$ [12,13,14], it would be very interesting to study the behavior of the renormalized coupling constant when $d$ varied from 3 to 4, if the critical properties of this theory could be smoothly interpolated between integer dimensions. Besides being an alternative method for studying the question of triviality, it has the advantage that, as in a fractal lattice the number of actíve sítes is smaller than in a normal lattice, the computations require less CPU time and memory storage. As there are many open points in the way of studying $\varphi^4$ on a fractal lattice, namely, how to define in an appropriate way the field derivatives and which boundary conditions are more adequate, we are going, first, to study this theory when d varies from 1 to 2, before going to the more difficult task of approaching the question of triviality of $\varphi^4_4$.

We will study in this work the lattice version of the Euclidean Lagrangian,

$$\mathcal{L} = \frac{1}{2}\left\{(\partial_0\phi)^2 + (\nabla_i\phi)^2 - m^2\phi^2\right\} + \frac{1}{4}\left(\lambda\phi^4 + \frac{m^4}{\lambda}\right) \tag{1}$$

Scaling up the scalar field, defining it by $\theta = \sqrt{\lambda}\frac{\phi}{m}$, then, the lattice partition function can be written as,

$$Z(j) = \mathcal{N}'\int[D\theta]\exp\left\{-\frac{m^2}{\lambda}\int d^2y\left[\frac{1}{2}(\partial_i\theta)^2 + \frac{1}{4}(\theta^2-1)^2 + j\theta\right]\right\} \tag{2}$$

where $y_i = mx_i$ and $j$ is the scaled current, given by $j = \frac{\sqrt{\lambda}}{m3}J$

## $\Phi^4$ *Thwry* on *a Fractal Lattice*

In the discrete version of this theory, fields *$(C)* are defined at the lattice sites, labelled by the nurnber vector *C*. The lattice sites are separated by a fixed distance $\underline{a}$. The lattice version of the derivative is,

$$\partial_i\theta = \frac{\theta(\vec{n}) - \theta(\vec{n}+\hat{e}_i)}{2a} + \frac{\theta(\vec{n}) - \theta(\vec{n}-\hat{e}_i)}{2a} \tag{3}$$

where $\hat{e}_i$ is the unit vector in the direction i.

If we identify $\beta = \frac{m^2}{\lambda}$, then, the lattice version of the partition function,

$$Z(j) = \mathcal{N}' \int [D\theta(\vec{n})] \exp\{-\beta\mathcal{H}\} \tag{4}$$

has the Euclidean Hamiltonian defined by,

$$\mathcal{H} = \frac{a^2}{2}\sum_{\vec{n}}\left\{-\frac{1}{a^2}\theta(\vec{n})\sum_{i=1}^{d}[\theta(\vec{n}+\hat{e}_i) + \theta(\vec{n}-\hat{e}_i)] + \frac{2d}{a^2}\theta^2(\vec{n})\right\} + $$
$$+\sum_{\vec{n}}\left\{\frac{1}{2}[\theta^2(\vec{n}) - 1]^2 + 2j(\vec{n})\theta(\vec{n})\right\} \tag{5}$$

in a space of integer dimension $\underline{d}$.

The expressions for the quantities that we will be interested in calculating,the average value of the fields (magnetization) and the two-point connected Green Function (which is the analogous of a susceptibility in a spin system), can be easily obtained by defining the connected partition function in the usual way,

$$Z = \exp(-W(j)) \tag{6}$$

Using the fact that we use a constant current j over the whole lattice, the derivative of the connected partition function as a function of $j$ yields,

$$\frac{\delta W(j)}{\delta j} = \beta N a^2 \sum_{\vec{n}} \frac{<\theta(\vec{n})>_j}{N} = \beta N a^2 \Theta, \tag{7}$$

where N is the number of sites on the lattice and $\sum_{\vec{n}} < \theta(\vec{n})/N >_j = \Theta$ is the average value of the fields over the entire volume.

The susceptibility $\chi(j,\beta)$ is given by,

$$\chi(j,\beta) \equiv -\frac{\delta\Theta}{\delta j} = \beta N a^2 \left( < \sum_{\vec{n}} \left[\frac{\theta(\vec{n})}{N}\right]^2 >_j - \Theta^2 \right) \tag{8}$$

and it is related to the second derivative of the effective potencial [15].

Sierpinski carpets are usually defined on an embedding two dimensional space, but they can be generalized to spaces of any dimensionality [10]. They are constructed, in an embedding space of dimension d, in the following way; we start with a hypercube in this space (the initiator), divide it in $b^d$ hypercubes and eliminate $l^d$ of these hypercubes from the central section. We repeat this procedure n times (each procedure in which a certain nurnber of hypercubes are decimated is called a generation, the initiator being the O-th generation of the Sierpinski carpet), and at each generation the carpet is rescaled, in order that the smaller hypercubes remain of unit volume. A fractal is obtained by making the number of generations $n \to \infty$ [2-5].

The Hausdorff dimension for a fractal, like the ones described above, is given by,

$$d_h = \frac{\ln (b^d - l^d)}{\ln b} \tag{9}$$

When studying $\lambda\varphi^4$ on a Sierpinski carpet, we have put a field $\theta_i$ at the center of each live unit cell on the carpet, and no fields where the cells have been eliminated. We could also have put the fields on the corners of the cells instead of the center. In the limit of infinite number of generations this will not make any difference[6]. As a consequence of this choice, we will have to eliminate from eq. (5) all the contributions to the Hamiltonian coming from the fields that have been decimated. In other words if, in a given direction, one or both nearest neighbors of a field $\theta_i$ are sitting on decimated cells, then their contribution to the field derivative, in that direction, must be equal to zero. This will modify eqs. (3) and (5).

When doing the Monte Carlo simulation we will move from live site to live site, so that in eqs. (7) and (8) the N appearing stands now for the total *number* of active *sites* in each generation.

We have tested both periodic (PBC) and Mean Field (MF) boundary conditions [16]. In this last type of boundary conditions the fields external to the border of the fractal are replaced by the average magnetization of the system and these values of the externa1 fields are updated each few Monte Carlo steps. To compare finite size effects caused by the introduction of each type of boundary condition, we have calculated the magnetization and susceptibility as functions of $\beta$ for the $\phi^4$ theory on a two dimensional square lattice with $64^2$ sites. The agreement between PBC and MF is very good, the only differencebeing that the susceptibility peak is less pronounced in the case of MF boundary conditions. During all the Monte Carlo simulations we have controlled the field configurations and we have not found any enhancement of the magnetization of the surface fields compared to that of the fields which are in the interior of the lattice. As a fractal is not a translationally invariant object and not a periodic one, we considered that the MF boundary conditions would be more appropriate to the investigation of these objects.

We have used the Metropolis algorithm[17] to implement the Monte Carlo numerical simulation on finite area lattices. We generated a new random field using the relation $\theta_i^N = (2s - 1) * A$, where is uniformly distributed in the interval $[0:1]$ and A is an empirically chosen constant. We have usually performed, for all points, from 4 to 7 runs with the same set of parameters, only changing the seed of the random number generator to check the consistency of our data. The error bars in each graph are obtained from runs with different seeds.

The value of the field at each site was updated once at a time and we ran over all sites in a sequential way until a Monte Carlo sweep was completed. We did 10 trials to change the field at a single site, before going to the next site. With this procedure, the correlation among different configurations was reduced, the rate of change of the fields at each sweep was increased and, in addition, the system converged to the equilibriurn at a faster rate.

We have used the variance reduction method[18] in order to reduce the statistical errors in the determination of the average value of the fields for each lattice configuration. The value of the new field at each site was taken to be the average over trials at each update. In this way, we obtain configurations with the same expected value for the fields, but with less deviation as compared with the procedure where the new field is taken to be the last trial value at each update.

In this numerical simulation we ran from 5000 to 80000 Monte Carlo steps with 10 updates at each lattice site (this is roughly equivalent to performing from 50000 to 800000 Monte Carlo steps). We discarded about 25% of the initial sweeps due to thermalization. The remaining configurations were separated in groups of 25. We stored the central configurations of each group to use in the measurements of physical quantities, and used the rest to do statistics and to measure correlations in the sample.

We have used a small constant external current ($j = 0.01$) over all the fields during the simulations. The introduction of this small ciirrent is very useful since it allows us to obtain reliable results without having to use so many Monte Carlo steps as one would have to use with $j = 0.0$. We have also used $m = 1$ and $a = 1$ during all our calculations.

In order to verify if our assumptions concerning the field derivatives and the boundary conditions were correct, we have studied the behavior of $\lambda\varphi^4$ theory in two different situations. We have first, for a fixed fractal dimension, varied the number of generations from 1 to 4. In figure 1 we show the behavior of the susceptibility against $\beta$ for a fractal with $b = 3$, $l = 1$ and $n = 1, 3$ and 4. We can observe in this figure that, as the number of generations grows, the susceptibility peak gets higher and the value of the critical coupling $\beta_c$ moves in the direction of higher values of $\beta$. For $n = 1$ it is nearly impossible at all to identify a peak in the susceptibility curve since it is too flat. As a complete study of the $n = 5$ case would involve a very large amount of CPU time, we have performed some runs in the case of $n = 5$, to check if the influence of finite size effects was still relevant. We have verified that the values of $\beta_c$ that come from the $n = 4$ and $n = 5$ lattices are, within the error bars, pratically identical and, as a consequence, we have not calculated the whole phase diagram for $n = 5$.

In a second step we have varied, for a fixed fractal generation, the fractal dimension and have studied the behavior of the susceptibility and the average value of the fields as a function of $\beta$ (which plays the role of the inverse of a "temperature", even though, strictly speaking, we are studying $\varphi^4$ at $T = 0$). In figures 2 and 3 we show the behavior of the magnetization and susceptibility as

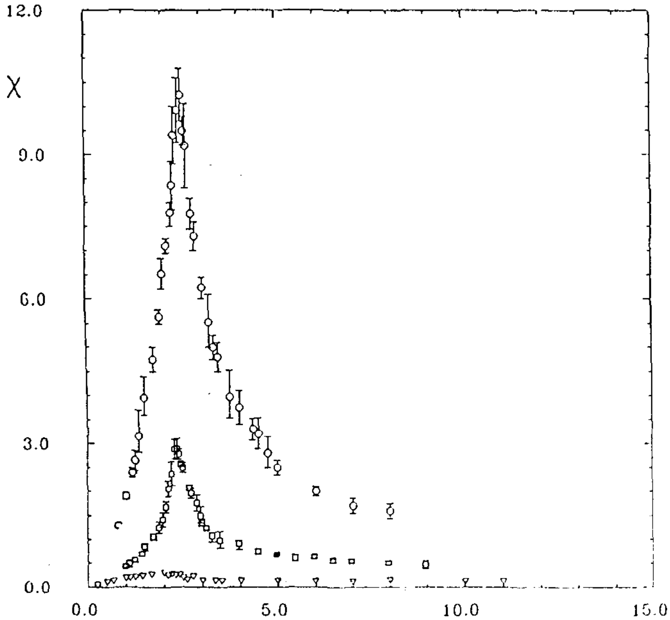


Fig. 1 - Behavior of the susceptibility as a function of the critical coupling, $\beta_c$, for a Sierpinski carpet with b = 3 and 1 = 1 and n = 1,3 and 4. ▽ stands for n = 1, □ for n = 3 and ○ for N = 4.

functions of $\beta$ when n = 2 for the following dimensions: d = 2, 1.976, 1.936 and 1.796. From both figures we can observe that as the d decreases from 2 to 1, $\beta_c \to \infty$. This behavior is in agreement with what one should expect if there were a kind of interpolation of the critical coupling between integer dimensions, since $\beta_c$ moves in the direction of larger values of $\beta$ from the $d = 2$ critical value to $\beta_c = \infty$, which is the critical coupling when d = 1, for in this case there is only a symmetric phase[11,12]. Even though we have used a small generation number, this fact will not modify qualitatively our results, since from fig. 1 we have verified that the effect of using lattices with larger generation number is to move the critical coupling in the direction of laiger values of $\beta$, that is, away from the value of $\beta_c$ obtained in the d = 2 lattice. When d = 2, $\beta_c$ will also move in the same direction, but it will be less shifted than the other ones because we have used a much larger lattice in this case. In this way, even though the true behavior of fractal is only achieved in the limit n $\rightarrow \infty$, we expect that we can obtain a good hint of the behavior of a $\varphi^4$ theory working with n = 2 fractals.

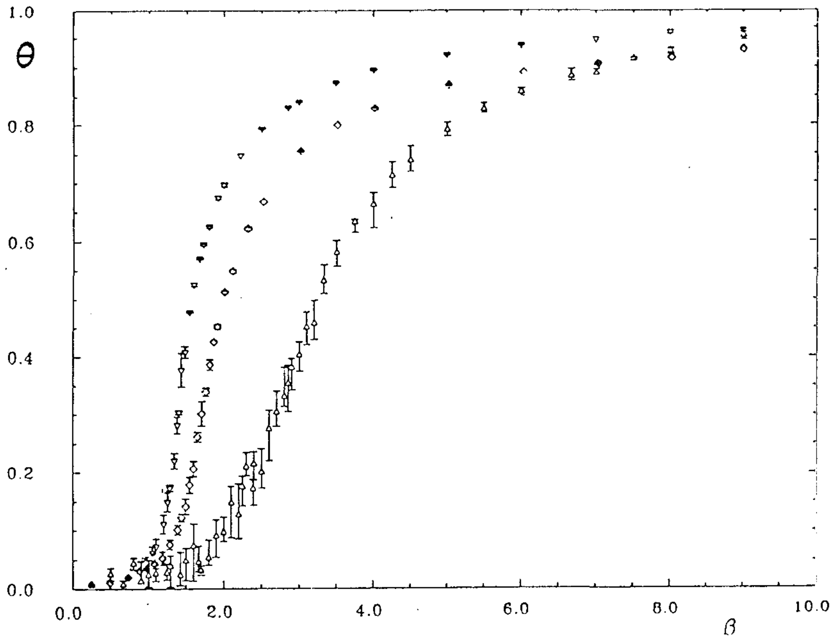


**Fig. 2 - Behavior of the magnetization as a function of the critical coupling, $\beta_c$, for different kinds of Sierpinski carpets when n = 2. a) $\nabla$ stands for b = 64 and $l$ = 0, O stands for b = 6 and l = 2 and $\triangle$ for b = 4 and $l$ = 2.**

In figure 4 we show the behavior of $\beta_c$ as a function of the fractal dimension $d_h$. As we can verify, even though we are using n = 2 we can perfectly observe that as we move away from d = 2, in the direction of d = 1, the values of $\beta_c$ move very fast in the right direction and that the results indicate that the critical coupling $\beta_c$ seems to interpolate smoothly between integer dimensions.

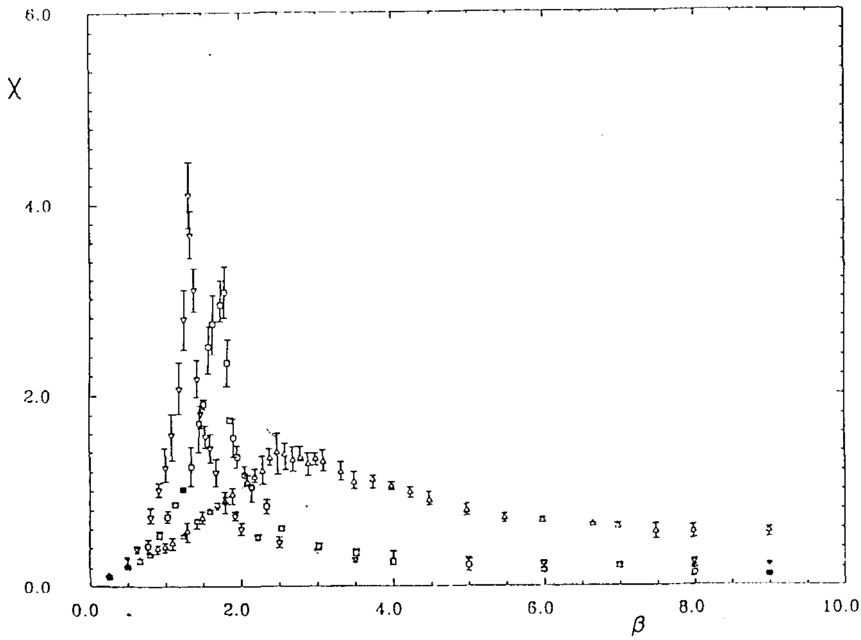


Fig. 3 - Behavior of the susceptibility as a function of the critical coupling, $\beta_c$, for different kinds of Sierpinski carpets, when n = 2. ▽ stands for b = 64 and $l = 0$, ◯ stands for b = 5 and $l = 1$, O stands for b = 6 and $l = 2$ and A stands for b = 4 and 1 = 2.

In the case of Ising models in Sierpinski carpets with dimensions ranging from 1 to 2, it was proposed in previous papers[6], that a smooth interpolation for the critical coupling $\beta$ and the critica1 exponent $\gamma$ (related to the susceptibility) between integer dimensions was better described if was better described if one used, instead of the usual Hausdorf dimension, a more physical dimension $d_{nn}$ defined via the average number of bonds connecting nearest neighbor live hypercubes, $d_{nn} = N^0$ of live bonds / $N^0$ of live hypercubes. In spite of the fact that there have been some suggestions for more adequate definition of the physical dimension[6,19], this question is still an open one. On the other hand, as we do not have enough data to discuss the appropriateness of different definitions, we will restrict ourselves to the Hausdorf dimension.

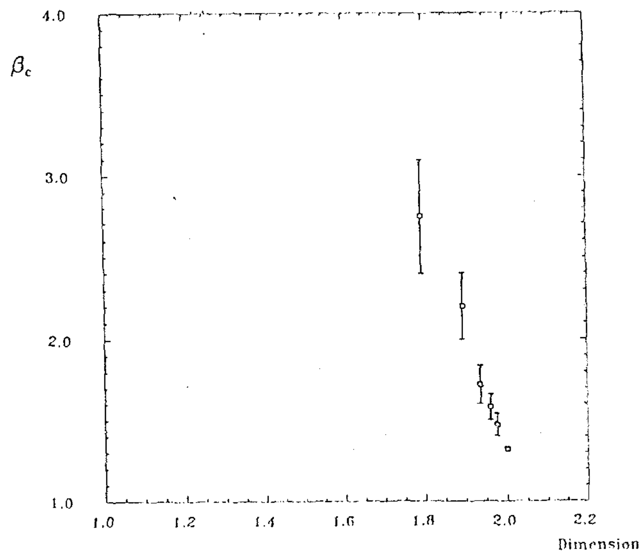


**Fig. 4 - Behavior of $\beta_c$ as a function of the Hausdorffdimension.**

Our next task is to do a more detailed study of this question of interpolation between higher integer dimensions, calculating the critical exponents and discussing the question of the triviality of $\varphi^4$ in four dimensions, as well as the question of the physical dirnension.

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## Resumo


Fazemos a simulação de Monte Carlo da teoria $\phi^4$ em uma rede fractal. Apresentamos os resultados obtidos para o comportamento desta teoria em um tapete

de Sierpinski quando variamos o número de gerações, para uma dada dimensão de Hausdorff, e quando variamos a dimensão de Hausdorff mantendo fixo o número de gerações.