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Ambiguities on electric and magnetic fields for an extended gauge model

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Abstract Generalized electric and magnetic fields in a system containing N-potential fields in the same $U(1)$ - group are obtained. Bianchi identities, eqiiations of motions, conserved charges and Lorentz forces are developed in association to each of these fields. Such facts confirm that the same parameter $\alpha(x)$ is able to organize the presence of distinct fields. The physics generated from the minimal action principle is independent of the initial definition for the electric (magnetic) field. Nevertheless, such a choice reveals differences in the Bianchi identity context.

1. Introduction

The idea of symmetry under continuous transformations has entered the formulation of physical theories through the notion of a particular type of topological groups, namely, the Lie groups. However, the algebra, rather than the group itself, is the structure physicists usually deal with, in implementing the symmetry in theory. Despite the one-to-one correspondence between the group structure and the associated group manifold, the non-uniqueness of the group associated to a given Lie algebra may justify the fact that the group underlying a physical theory does not fix the number of physical particles displayed in its spectrum. It is the algebra, through the conserved currents, charges and field transformations, that organizes the spectrum and interactions of particles. But, being simply a local piece of information of the group as a whole, the latter does not offer enough constraints to completely fix the number of particles described by the theory under discussion.

Gauge theories have been guided by the condition that the number of potential fields associated to a single simple group must be equal to the dimension of the adjoint representation of the group[1]. However, a possible enlargement of the context, where a same Lie algebra is attached to a non-defined number of quanta, is under development[2]. Proofs based on different physical aspects, as the counting of the number of degrees of freedom, the local Noether theorem indications, the presence of a geometrical origin based on Kaluza-Klein arguments, considerations from supersymmetry with no relaxed constraints, all work in order to confirm the assumption of the presence of N-potential fields in a same simple group. Thus, transformations such as

$$\begin{aligned}
&A_\mu \rightarrow A_\mu + \partial_\mu \alpha(x) \\
&B_\mu \rightarrow B_\mu + \partial_\mu \alpha(x) \\
&\vdots \\
&N_\mu \rightarrow N_\mu + \partial_\mu \alpha(x)
\end{aligned} \tag{1}$$

generate N distinct fields. This fact can always be confirmed by calculating the poles corresponding to the two-point Green functions. This explicit confirmation is necessary in order to avert any suspicion that the fields involved in (1) be branches of subsidiary fields as copy fields, auxiliary fields or Lagrange-multiplier fields. The results from[2] are building up a viewpoint where the N-fields should be understood as fundamental fields. This is in the sense that they are defined from first principles: developed directly from symmetry.

Thus (1)shows that a same parameter $\alpha(x)$ contains possibilities for generating different quantum numbers. Although each field is born from a same group symmetry, they develop distinct dynamics. Therefore, the system contains N-independent dynamics. Then, a meaning for the parameter $\alpha(x)$ is to induce different quanta in theory.

This property, of distinct fields being originated from a same symmetry, inverts the traffic. From Picadilly's station, where Faraday did his famous experiments, until the $U(1)$-synthesis, a physical line of reasoning was built up, carrying important facts. In this image, the question to be asked is: getting on at $U(1)$ station which physics may we derive from (1)? Perhaps another line can be developed. However the objective in this work is not to build up physical models. It is just to systematize some classical properties under this extended $U(1)$ symmetry.

This work is organized as follows. In section two, electric and magnetic fields are defined through four characteristics. They are: gauge invariance, dimensional analysis, conserved charge and Lorentz force. However such definitions should be writtem in terms of physical fields. Section three works to systematize this extended abelian model. Section four studies an intended generalization for electric and magnetic fields.

2. Generalization for electric and magnetic fields

The introduction of more potential fields in a same $U(1)$-group induces a generalization of the QED-Lagrangian. Redefining (1) as

$$D_\mu(x) \to D_\mu(x) + N\partial_\mu \alpha(x) \tag{2}$$

$$X^i_\mu(x) \to X^i_\mu(x), \tag{3}$$

where the fields $D_\mu(x)$ and $X^i_\mu(x)$ ($i = 1, ..., N-1$) were obtained by adding and subtracting in pairs the original fields, it yields,

$$\mathcal{L} = Z_{\mu\nu}Z^{\mu\nu} + \frac{1}{2}m_{ij}X^i_\mu X^{\mu j} + \text{ matter} \tag{4}$$

with

$$Z_{\mu\nu} = dD_{\mu\nu} + \alpha_i X^i_{\mu\nu} + \beta_i \Sigma^i_{\mu\nu} + \rho_i g_{\mu\nu}\Sigma^{i\alpha}_\alpha + \gamma_{ij}X^i_\mu X^j_\nu \tag{5}$$

and

$$D_{\mu\nu} = \partial_\mu D_\nu - \partial_\nu D_\mu \tag{6}$$

$$X^i_{\mu\nu} = \partial_\mu X^i_\nu - \partial_\nu X^i_\mu \tag{7}$$

$$\Sigma^i_{\mu\nu} = \partial_\mu X^i_\nu - \partial_\nu X^i_\mu \tag{8}$$

$Z_{\mu\nu}$ represents the most generalized gauge scalar that this extended abelian gauge model develops. The coefficients $d, \alpha_i, \beta_i, \rho_i$ and γ_{ij} that theory generates are called free coefficientes. They correspond to each gauge scalar that the model contains. Under such an extension the gauge fields transforming as (2) develops other terms. Besides the normal QED case, (2) in (4) generates a mixing kinetic term, with X^i_μ fields, and also an interacting term, involving the D_μ-field. Another relevance is the appearance of a mass matrix, m_{ij} associated to the vector matter fields.

Under a classical viewpoint two physical entities need to be explored. They are the spin sectors and the theoretical entities that offer experimental contact. From the Lorentz group one gets the information that a potential field is carrying both spin-1 and spin-0 structures. Rewriting (4) as

$$\mathcal{L} = Z_{[\mu\nu]}Z^{[\mu\nu]} + Z_{(\mu\nu)}Z^{(\mu\nu)} + \frac{1}{2}m_{ij}X^i_\mu X^{\mu j} + \text{ matter}, \tag{9}$$

one gets that the dynamics for the vectorial and scalar pieces are controlled by the antisymmetric and symmetric parts.

Now, the next stage to be formulated is about the entities for describing a measurement. Classically, the best known structures to estimate the gauge theorg dynamics are the electric and magnetic fields. Therefore, a primordial investigations for (9) is about the possibility of carrying an enlargement for the concept of electric-magnetic field. Four properties are basic for extending the Maxwell approach for electric field. They are: gauge invariance, dimension $[M]^2$, association to a source determined through a continuity equation, and association to a Lorentz force which will measure the corresponding generalized charge.

The first two aspects to coordinate the establishment of a generalized electric and rnagnetic fields are immediate in (5). However, for understanding the sources involved, it is necessary to interpret the equations that the theory develops. They are N-equations of motion, Bianchi identities and the local Noether theorem. From the homogeneous equation we get that such generalized electric fields are, at least, a linear combination between the strength fields $D_{\mu\nu}$ and $X^i_{\mu\nu}$. Otherwise, each of them would work as a source for the other. Nevertheless, the sources are well determined through the minimal action principle. Thus, deriving the equations of motion and taking their second derivatives, one gets for D_μ-field,

$$\partial_\mu j^\mu_{D-\text{matter}} = 0 \ . \tag{10}$$

Similarly for a X^i_μ–fields,

$$\partial^\mu j_{\mu i} = 0 \tag{11}$$

where

$$j_{\mu i} = j^{X_i}_{\mu i} + j^{\text{mass-term}} + j^{X_i-\text{matter}}_{\mu i} \tag{12}$$

with

$$j^{X_i}_{\mu i} = -\beta_i \partial_\alpha Z^{(\alpha}{}_{\mu)} - \rho_i g_\mu{}^\alpha \partial_\alpha Z_\eta{}^\eta + \gamma_{[ij]} Z_{[\mu\alpha]} X^{\alpha j} + \gamma_{(ij)} Z_{(\mu\lambda)} X^{\alpha j} \tag{13}$$

$$j^{\text{mass-term}}_{\mu i} = (m_{ij} + m_{ji}) X_{\mu j} \tag{14}$$

$j^\mu_{D-\text{matter}}$ and $j^\mu_{X_i-\text{matter}}$ means the external sources coupled to each field, respectively.

Thus, from (10) and (11), one concludes that a generalized electric field must appear as a linear combination between the strength fields $D_{\mu\nu}$, $X^i_{\mu\nu}$ and $\gamma_{[ij]} X^i_\mu X^j_\nu$. The Noether conserved current is compatible with such proposal. As an example, generalized electric and magnetic field associated to D_μ-field dynamics are

$$\vec{E}_D = Z^{oi}_D \ \ ; \ \ \vec{B}_D = \frac{1}{2} \epsilon_{imn} Z^{mn}_D \tag{15}$$

where

$$Z^{\mu\nu}_D = \alpha_1 D_{\mu\nu} + \alpha_{2i} X^i_{\mu\nu} + \alpha_3 \gamma_{[ij]} X^i_\mu X^j_\nu \tag{16}$$

α_1, α_{2i} and α_3 are coefficients determined through D_μ-equations of motion. Similarly, but with a different linear combination, the generalized fields for $X_{\mu i}$ are obtained.

However, the electric fields determined in (15) are still not structures to be measured. This is because, although they are associated to conserved charges, they should also be associated with physical masses. Nevertheless, the terms m_{ij} are not necessarily the physical masses, but just mass parameters. This means that the real generalized electric fields that will be measured in the laboratory must be obtained from physical fields that correspond to the pole of a two-point Green function.

3. Physical fields

Besides the four prescription analysed in section 2, physical excitations must also correspond to physical masses. Thus, it becomes necessary to rotate the system of coupled equations, in (4), in order to identify the physical masses with the constraint $p_\mu p^\mu = m^2$. From[3], one gets that the physical fields corresponding to the physical masses are obtained from a tranformation described by a U-matrix,

$$\begin{bmatrix} G_1 \\ G_2 \\ \vdots \\ G_N \end{bmatrix} = U \begin{bmatrix} D \\ X_1 \\ \vdots \\ X_{n-1} \end{bmatrix} \tag{17}$$

with the condition

$$(U^t)^{-1} = U \tag{18}$$

(18) guarantees that the matrix U does not have rows and columns equal or proportional. This means that the number of constructor fields must be equal to the number of physical fields. In components, **(17)** is written as

$$G_{\mu I} = u_{IO} D_\mu + u_{Ii} X^i_\mu \tag{19}$$

where the parameters u_{IO}, u_{Ii} are elements of the U-matrix. They are originated from the free coefficients of theory and calculated as the eigenvectors necessary to rotate a piece of Lagrangian whose eigenvalues represent the physical masses.

(19) results in the following gauge transformation

$$G_{\mu I}(x) \to G_{\mu I}(x) + u_{IO}\partial_\mu \alpha(x) \ . \tag{20}$$

This expression indicates that there is just one group, and the fiels fransform with different weights acting in the same group parameter $\alpha(x)$. Thus, a gauge group contains the possibility to build up physical fields $G_{\mu I}$ which transform under the same space-time dependent gauge transformation, but with different weights u_{OI}.

The inverse transformation from (17) are expressed as

$$D_\mu = \bar{u}_{OI} G^I_\mu \tag{21}$$

$$X_{\mu i} = \bar{u}_{iI} G^I_\mu \tag{22}$$

where

$$\bar{u}_{mI} u_{In} = \delta_{mn} \tag{23}$$

A consequence of (20) and (23) is the generation of a gauge invariant mass term

$$\mathcal{L}_m = m^2_{ij} \bar{u}_{iI} \bar{u}_{jJ} G^I_\mu G^{\mu J} \tag{24}$$

However, though the above expression associates a mass term to each fields, it is only apparent that all these fields would be massive. Remember that terms m_{ij} are the only mass parameter to determine the physical masses. From Goldstone's theorem or through explicit calculations, we note that at least one physical field is massless. This fact can also be hinted by writing (24) in terms of constructor fields and is confirmed by the information from Ward identities. Nonetheless, although the presence of (24) carries a massless particle, it is useful to give another framework for the infrared problem. This means that, though gauge theories have a massless particle as a constraint, the natural advent of mass parameters m_{ij} propitiate a mechanism to regularize, in principle, some infrared difficulties.

In its most general treatment, a gauge theory is defined via the action. Thus, from the condition that S be stationary, results

$$\delta S = \int d^4x \left\{ \delta\Phi_i \left[\frac{\partial L}{\partial \Phi_i} - \partial_\mu \frac{\partial L}{\partial \partial_\mu \Phi_i} \right] + \partial_\mu \left[\delta\Phi_i \frac{\partial L}{\partial \partial_\mu \Phi_i} \right] \right\} = 0$$

where

$$\mathcal{L} \equiv \mathcal{L}(\Phi, \partial\Phi)$$

$$\Phi = \{\Phi_i\} \ . \tag{25}$$

However, a property obtained from this enrichment of symmetry, with the introduction of more fields, is the appearance of a different basis to control physics. Thus, the field- Φ is representing the constructor fields basis, $\Phi \equiv (D_\mu, X_{\mu j})$, or may be associated with the physical fields, $\Phi \equiv (G_{uI})$, or, even, to represent any other field basis.

Studying the mapping between these two variables systems, $[D, X_i]$ and $[G_I]$, we have

$$\mathcal{L}'(G_I) = \mathcal{L}(D(G) \;\; ; \;\; X(G))$$

and

$$\mathcal{L}(D, X_i) = \mathcal{L}'(G_I(D; X_i)) \tag{26}$$

where L' means a different functional form of the same Lagrangian. Observe that L]' and L] exhibit different functional dependence on the fields. Thus, it becomes necessary to compare these two systems of variables. $L(D, X_i)$ represents a coupled system with N-equations of motion. $\mathcal{L}'(G_I)$ also represents a coupled system with N-equations of motions. Thus, the first question is about whether the dynamics (with its temporal evolution) is manifested independently of the reference system. From (9) we get

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu D_\nu} - \frac{\partial \mathcal{L}}{\partial D_\nu} = u_{OI} \left(\partial_\mu \frac{\partial \mathcal{L}'}{\partial \partial_\mu G^I_\nu} - \frac{\partial \mathcal{L}'}{\partial G_\nu} \right) = 0 \tag{27}$$

and

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu X_{\nu_i}} - \frac{\partial \mathcal{L}}{\partial X_{\nu_i}} = u_{iI} \left(\partial_\mu \frac{\partial \mathcal{L}'}{\partial \partial_\mu G^I_\nu} - \frac{\partial \mathcal{L}'}{\partial G_\nu} \right) = 0 \tag{28}$$

(27) and (28) show that a reparametrization of the fields does not alter the equations of motion: the on-shell information is preserved. Thus, if the physical fields are on shell, then the constructor fields are also on shell.

A second information and consistency between these two frames comes from Noether's theorem

$$\delta \mathcal{L}(D, X_i) = \partial_\mu J^\mu(D, X_i) = \delta \mathcal{L}'(G_I) = \partial_\mu J^{\mu'}(G_I) = 0 \tag{29}$$

with its respective Noether current being related as

$$J'_\mu(G_I) = J_\mu(D(G_I) \;;\; X(G_I)) + \partial^\nu L_{\mu\nu}(D, X) \tag{30}$$

where $L_{\mu\nu}(D, x)$ is antisymmetríc.

A third aspect to be explored is about the N-conserved currents obtained from the N-equations of motion. The D_μ-field dynamics gives,

$$\partial_\mu J^\mu_D = 0 \tag{31}$$

From X^i_μ-field dynamics,

$$\partial_\mu J^\mu_{X_i} = 0\ . \tag{32}$$

Similarly, the physical-sector manifests,

$$\partial_\mu J^{\mu'}_{G_I} = 0\ . \tag{33}$$

Then, from (23) and (28), a mapping relating these conservation laws give,

$$\partial_\mu J^\mu_D = u_{OI} \partial_\mu J^{\mu I'}_{G_I} \tag{34}$$

$$\partial_\mu J^\mu_{X_i} = u_{iI} \partial_\mu J^{\mu I'}_{G_I} \tag{35}$$

The canonical momenta are also related, depending on the basis associated,

$$\pi^\mu_D = u_{IO} \pi^\mu_{G_I} \tag{36}$$

$$\pi^\mu_{X_i} = u_{Ii} \pi^\mu_{G_I} \tag{37}$$

Thus, a next discussion that is still possible to be included before calculating any specific equation of rnotion is about how to identify the physical poles. There are two methods to obtain the physical masses. They are: to determine the propagator poles or to express the equations of motion in a Klein-Gordon type form. From the Lorentz group we know that a potential field $G_{\mu I}$ belongs to the representation $\left(\frac{1}{2}, \frac{1}{2}\right)$. Thus, it carries a quanturn with spin-1 and an other with spin-0. Therefore, an important theoretical task is to study the possibilities of determining a theory spectrum containing these two quanta. This means to be able to identify the quantum numbers corresponding to the transverse and longitudinal pieces, without violating physical rules. Our intention here is to study the possibility of splitting the mass sectors as.

$$(p_\mu p^\mu)_T = M^2_T \tag{38}$$

$$(p_\mu p^\mu)_L = M_L^2 \tag{39}$$

For this we should note that, given a Lorentz transformation,

$$x'_\mu = \Lambda_\mu^n u x_\nu \tag{40}$$

the transverse and longitudinal components do not mix between themselves.

$$(G'_T)_\mu = (\Lambda^{-1})_\mu^\rho (G_T)_\rho \tag{41}$$

$$(G'_L)_\mu = (\Lambda^{-1})_\mu^\rho (G_L)_\rho \tag{42}$$

Thus, (41) and (42) guarantee that under Lorentz transformation the decomposition in transversal and longitudinal parts is covariant. Therefore such a decomposition as made on (38) and (39) is consistent with Poincaré. This means that, at the formal level, the theory admits the interpretations

$$(\Box + m_T^2)G_T = 0 \tag{43}$$

$$(\Box + m_L^2)G_L = 0 \tag{44}$$

for (38) and (39) respectively. Thus, the splitting (38)-(43), (39)-(44) show that it healthy to read off, in the spectrum, the existence of different masses associated to each spin. However, the fields associated to them are non-local.

A still fundamental step to be understood, whenever more than one potential field rotates under the same group, is to understand to which extension symmetry penetrates on these fields. As we know, from the usual gauge theory, sometimes a symmetry can exist at classical level but not at quantum level. Physically, it is important to detect the action of symmetry over each field separately. This means that, although the symmetry acts over all N-fields as a whole, it should also be able to act on each of those fields separately. This is a consistency test to be made for assuming the presence of more fields in a same group. Coming to facts, it becomes necessary to explore the constraints and relation that symmetry specifies for each field.

Thus, the next entities for categorizing the physical fields $G_{\mu I}$ without calling a Lagrangian are the Bianchi identities and the local Noether theorem. They represent the closest relations generated from the so-called impulse of symmetry. Then, our effort will be to test how these entities appear to confirm the possibility of introducting more fields in a same group. Considering the following covariant derivative,

$$\nabla_{\mu I} = \partial_\mu \rightarrow gG_{\mu I} \tag{45}$$

and the Jacobi identity, one gets

$$\partial_\mu G^{JK}_{\nu\sigma} + \partial_\sigma G^{IG}_{\mu\nu} + \partial_\nu G^{KJ}_{\sigma\mu} = 0 \tag{46}$$

where

$$G^{IJ}_{\mu\nu} = \partial_\mu G^I_\nu - \partial_\nu G^J_\mu \tag{47}$$

(46) contains $4N^3$ homogeneous identities. Observe that most of them are not gauge invariant, unless $u_{IO} = u_{JO} = u_{KO}$. Another aspect of (46) is that it involves not only antisymmetric terms but also symmetric. However, observing in more detail, most of these identities appear as a consequence of only óne kernel given by

$$\partial_\gamma G^I_{[\mu\nu]} = \partial_\mu \mathcal{G}^I_{(\mu\gamma)} - \partial_\nu \mathcal{G}^I_{(\gamma\mu)} \tag{48}$$

where $\mathcal{G}^I_{[\mu\nu]}$ and $\mathcal{G}^I_{(\mu\nu)}$ are antisymmetric and symmetric tensors. Thus, we have to interpret the synthesis that (48) represents. An identity may be physical or occasional. Analising the right hand side, one notes that it is not a source for the left hand side, but just one way of rewriting. Thus, the two sides are not independent. Consequently, the stipulated $4N^3$ homogeneous identities are not all independent. There survive just N-independent gauge invariant Bianchi identities. They are

$$\partial_\mu G^I_{[\nu\gamma]} + \partial_\gamma G^I_{[\mu\nu]} + \partial_\nu G^I_{[\gamma\mu]} = 0 \tag{49}$$

(49) manifests a relevant physical information. It shows that such an extended model associates to each field a corresponding Bianchi identity. Thus, there are independent pairs of electric and magnetic fields to differentiate each involved field.

Complementing the discussion, another possible Bianchi identity is one corresponding to the $U(1)$-system as a whole. From

$$\nabla_\mu = \partial_\mu + g_I G_{\mu I} \tag{50}$$

it yields,

$$\partial_\mu G_{[\nu\gamma]} + \partial_\gamma G_{[\mu\nu]} + \partial_\nu G_{[\gamma\mu]} = 0 \tag{51}$$

where

$$G_{[\mu\nu]} = g_I(\partial_\mu G_{\nu I} - \partial_\nu G_{\mu I}) \tag{52}$$

However although (51) can have physical consequences, it is just a linear combination of (49).

A second identity, that is not derived from dynamics but a consequence of symmetry, is the local Noether theorem. Consider

$$\Phi \equiv (G_1, G_2, ..., G_N; \psi_1 ... \psi_S; \bar{\psi}_1 ... \bar{\psi}_S) \tag{53}$$

where the potentials fields transform as (20) and different matter fields transform under the same $U(1)$-group as

$$\psi_F \to e^{i\alpha(x)}\psi_F \tag{54}$$

where F means the flavour of the fermion field. Due to the arbitrariness of the function $\alpha(x)$, the local Noether theorem provides three identities containing on-shell informations. They are

(i)

$$\partial_\mu J^\mu = 0$$

where

$$J^\mu = i\Big[\frac{\partial \mathcal{L}}{\partial\partial_\mu\psi_F}\psi_F - \frac{\partial \mathcal{L}}{\partial\partial_\mu\bar{\psi}_F}\bar{\psi}_F\Big] \tag{55}$$

(ii)

$$\Big[u_{IO}\partial_\mu\frac{\partial \mathcal{L}}{\partial\partial_\mu G_{\nu I}} + J^\nu\Big]\partial_\nu\alpha(x) = 0 \tag{56}$$

(iii)

$$u_{IO}\frac{\partial \mathcal{L}}{\partial\partial_\mu G_{\nu I}}\partial_\mu\partial_\nu\alpha(x) = 0 \tag{57}$$

Observe that (56) and (57) contain local conditions. In terms of the fields (D, X_i), (57) indicates that terms involving D_μ-fields must be antisymmetric. However, a difference appears when we are working with the physical system, where (57) doe not necessarily imply antisymmetry for the derivatives involving the physical fields. Thus, a symmetric field strenght involving physical fields should exist. Another consequence is that the expression

$$u_{IO}\frac{\partial \mathcal{L}}{\partial\partial_\mu G_{\nu I}} = 0 \tag{58}$$

does not necessarily mean that the field- $G_{\mu I}$ is not dynamical. Thus, through the presence of more fields we obtain that the requirement of a weaker solution is not compulsory. In[4] it was verified that, by using such property of introducting more potential fields in a same symmetry, it results that the Coulomb-Gauss law does not work as a necessary constraint in gauge theories.

We proceed now to construct the Lagrangian for the physical potential fields. Primary requirements are Lorentz and gauge invariance. From (4) and (17), one gets

$$\mathcal{L}_G = Z_G^{\mu\nu} Z_{G\mu\nu} + d_{IJ} G_{\mu I} G_J^\mu \tag{59}$$

where

$$Z_G^{[\mu\nu]} = \alpha_I G_I^{\mu\nu} + \rho_{[IJ]} G_I^\mu G_J^\nu \tag{60}$$

$$\begin{aligned} Z_G^{(\mu\nu)} = {} & \beta_I G_I^{\mu\nu} + \rho_I g^{\mu\nu} \mathcal{G}_{I\alpha}^\alpha + \rho_{(IJ)} G_I^\mu G_J^\nu + \\ & + \sigma_{IJ} g^{\mu\nu} G_{\alpha I} G_J^\alpha \end{aligned} \tag{61}$$

with

$$G^{[\mu\nu]} \equiv \partial^\mu G_I^\nu - \partial^\nu G_I^\mu \tag{62}$$

$$\mathcal{G}_I^{(\mu\nu)} \quad \partial^\mu G_I^\nu + \partial^\nu G_I^\mu \tag{63}$$

(60) and **(61)** are gauge invariant field strength. Rewriting **(59)** in terms of kinetic and interacting terms, we have

$$\mathcal{L}_G = \mathcal{L}_K + \mathcal{L}_I$$

where

$$\begin{aligned} \mathcal{L}_K = {} & a_{IJ}\partial_\mu G_{\nu I} \cdot \partial^\mu G_J^\nu + b_{IJ}\partial_\mu G_{\nu I} \cdot \partial^\nu G_J^\mu + \\ & + c_{IJ}\partial_\mu G_I^\mu \cdot \partial_\nu G_J^\nu + d_{IJ} G_{\mu I} G_J^\mu \ . \end{aligned}$$

$$\mathcal{L}_I = a_{IJK}(\partial_\mu G_{\nu I})G_9^\mu G_K^\nu + b_{IJK}(\partial_\mu G_{\nu I})G_{\nu J}G_K^\nu + \\ + a_{IJKL}G_{\mu I}G_{\nu J}G_K^\mu G_L^\nu \ . \tag{64}$$

The coefficients are related through

$$a_{IJ} = 2(\alpha_I\alpha_J + \beta_I\beta_J)$$

$$b_{IJ} = 2(\beta_I\beta_J - \alpha_I\alpha_J)$$

$$c_{IJ} = 8\rho_I(\beta_J + 2\rho_J)$$

$$d_{IJK} = 4(\alpha_I\gamma_{[JK]} + \beta_I\gamma_{(JK)})$$

$$b_{IJK} = 4\rho_I(\gamma_{(JK)} + 4\sigma_{(JK)})$$

$$a_{IJKL} = \gamma_{[IJ]}\gamma_{[KL]} + \gamma_{(IJ)}\gamma_{(KL)} + \\ + 2\gamma_{(IK)}\sigma_{(JL)} + 4\sigma_{(IK)}\sigma_{(JL)} \ . \tag{65}$$

Gauge invariance is obtained by comparing (59), (64) with (4). Then, it yields the following relationships between the coefficients

$$a_{IJ} = 2d^2\bar{u}_{IO}\bar{u}_{JO} + 2d(\alpha_j\bar{u}_{0I}\bar{u}_{jJ} + \alpha_i\bar{u}_{0I}\bar{u}_{iJ} + 2(\alpha_i\alpha_j + \beta_i\beta_j)\bar{u}_{Ii}\bar{u}_{Jj}$$

$$b_{IJ} = -2d^2\bar{u}_{IO}\bar{u}_{JO} - 2d(\alpha_j\bar{u}_{0I}\bar{u}_{jJ} + \alpha_i\bar{u}_{0J}\bar{u}_{iI} + 2(\beta_i\beta_j - \alpha_i\alpha_j)\bar{u}_{Ii}\bar{u}_{Jj}$$

$$c_{IJ} = 8(\beta_i + 2\rho_i)\rho_j\bar{u}_{Ii}\bar{u}_{Jj}$$

$$d_{IJ} = \frac{1}{2}m_{ij}^2\bar{u}_{Ii}\bar{u}_{Jj}$$

$$a_{IJK} = 2d\gamma_{ij}\bar{u}_{IO}\bar{u}_{Ji}\bar{u}_{Kj} + 4(\alpha_i\gamma_{[jk]} + \beta_i\gamma_{(jk)}\bar{u}_{Ii}\bar{u}_{Jj}\bar{u}_{Kk}$$

$$b_{IJK} = 4\rho_i(\gamma_{jk} + 4\sigma_{(jk)}\bar{u}_{Ii}\bar{u}_{Ij}\bar{u}_{kk}$$

$$a_{IJKL} = [\gamma_{[ij]}\gamma_{[k\ell]} + \gamma_{(ij)}\gamma_{(k\ell)} + 2\gamma_{(ik)}\tau_{(j\ell)} + \\ + 4\sigma_{(ik)}\sigma_{(j\ell)}]\bar{u}_{Ii}\bar{u}_{Jj}\bar{u}_{Kk}\bar{u}_{L\ell} \ . \tag{66}$$

This model contains N-equations of motion with their respective conservation laws. Calculating the k-th equation of motion, we have

$$\mathcal{D}^{(1)}_{\mu I} Z^{[\mu\nu]}_{G} + \mathcal{D}^{(2)}_{\mu I} Z^{(\mu\nu)}_{G} + g^{\mu\nu}\mathcal{D}^{(3)}_{\mu I} Z^{\alpha}_{G\alpha} + \frac{1}{2} d_{IJ} G^{\nu}_{J} = 0$$

where

$$\begin{aligned}
\mathcal{D}^{(1)}_{\mu I} &= \alpha_I \partial_\mu - \rho_{[IJ]} G_{\mu J} \\
\mathcal{D}^{(2)}_{\mu I} &= \beta_I \partial_\mu - \rho_{(IJ)} G_{\mu J} \\
\mathcal{D}^{(3)}_{\mu I} &= \gamma_I \partial_\mu + \sigma_{IJ} G_{\mu J} \ .
\end{aligned} \tag{67}$$

From (65) and (66) one gets that (67) is gauge invariant.

The dynamics also confirms the conjecture that a group allows the presence of N-potential fields. This is so because one can obtain, from the dynamics, a continuity equation associated to each proposed field. It gives N-conserved currents

$$\partial_\nu J^{\nu}_{G_I} = 0$$

where

$$\begin{aligned}
J^{\nu}_{G_I} = {} & \beta_I \partial_\mu Z^{(\mu\nu)}_{G} + \rho_I \partial^{\nu} Z^{\alpha}_{G\alpha} + \\
& + \rho_{[IJ]} Z^{[\mu\nu]}_{G} G_{\mu J} - \gamma_{(IJ)} Z^{(\mu\nu)}_{G} G_{\mu J} + \\
& - \sigma_{IJ} Z^{\alpha}_{G\alpha} G^{\nu}_{J} - d_{IJ} G^{\nu}_{J}
\end{aligned} \tag{68}$$

(68) also shows that a $U(1)$-group has the property of the gauge field working as its own source.

Since such an extended gauge model is both Lorentz and gauge invariant, a next step will be to analyse the conditions for having a positive-definite Hamiltonian. Any model based on (59) will require a stable vacuum.

A theory with a non-positive Hamiltonian contains drawbacks at the quantum level, because it is necessary to establish a stable background in order to quantize it. Thus, for analysing whether the corresponding Hamiltonian is bounded from below it becomes necessary only to study the free condition. From such simplification, where no interaction and source terms are necessary to analyse the vacuum, we will calculate the energy-momentum tensor by just considering the kinetic part plus mass terms. Thus, in our present motivation, it is necessary to split the free-energy momentum tensor. For the free-antisymmetric part, one gets

$$T_A^{F,\mu\nu} = -\frac{1}{2} Z^F_{[\alpha\beta]} Z_F^{[\alpha\beta]} \eta^{\mu\nu} + 2 Z_F^{[\mu\alpha]} Z_{F[\alpha}{}^{\nu]} \tag{69}$$

where $Z^F_{[\mu\nu]}$ is the free part in (60). It shows a positive Hamiltonian given by

$$\mathcal{H}_A = \vec{E}^2_A(G) + \vec{B}^2_A(G) \tag{70}$$

where

$$\vec{E}_A(G) = \alpha_I G_I^{oi} \quad \text{and} \quad \vec{B}_A(G) = \frac{\alpha_I}{2} \epsilon_{ijk} G_I^{jk} \tag{71}$$

Similarly the free-symmetric part plus the mass contribution gives,

$$\begin{aligned} T_S^{F,\mu\nu} = & -\frac{1}{2} Z^F_{(\alpha\beta)} Z_F^{(\alpha\beta)} \eta^{\mu\nu} + 2 Z_F^{(\mu\alpha)} Z_{F(\alpha}{}^{\nu)} + \\ & -\frac{1}{2} d_{IJ} G_{I\alpha} G_J^{\alpha} \eta^{\mu\nu} + 2 d_{IJ} G_I^{\mu} G_J^{\nu} \end{aligned} \tag{72}$$

It yields,

$$\begin{aligned} \mathcal{H} = & \vec{E}^2_S(G) + \vec{M}^2_S(G) + \frac{1}{2}(Z_F^{(ii)})^2 - \frac{3}{3}(Z_F^{(oo)})^2 + \\ & + \frac{3}{2} d_{IJ} G_I^o G_J^o + \frac{1}{2} d_{IJ} G_I^i G_J^i \end{aligned} \tag{73}$$

with

$$\vec{E}_S(G) = Z_F^{(oi)} \quad ; \quad \vec{M}_S(G) = \frac{1}{2} d_{ijk} Z_F^{(jk)} \tag{74}$$

d_{ijk} is completely symmetric in its indices and i $\neq j \neq$ k. Observe that the symmetric piece is not entirely negative. The off diagonal terms of the symmetric strength field contribute positively. Spurious terrns such as $Z_F^{(00)}$ must be controlled. For this, we should introduce appropriate subsidiary conditions. **A** first candidate is

$$\beta^I \partial_0 G_I^0 + \gamma^I \partial_\mu G_I^\mu = 0 \tag{75}$$

(75) violates the manifest Lorentz covariance, similarly to the Coulomb gauge. Observe that this constraint belongs to the type of general gauge fixing condition[5]. *A* hypothesis with $\beta_I = 0$ would not be convenient because it would cancel all the

symmetric part of the Lagrangian. Finally, one can point out that gauge invariance can be restored in (75) for

$$\beta_I = \gamma_I = \bar{u}_{OI} \tag{76}$$

Considering this analysis for the free part, the potential fields can be expanded in a plane wave basis,

$$G_{\mu I} = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_{k_I}}} [a_{\mu I}(k) f_{kI}(x) + a^+_{\mu I}(k) f_{kI}(x)]$$

where

$$a_{\mu I}(k) = \sum_{\lambda=0}^{3} \alpha(k,\lambda)\epsilon_\mu(k,\lambda)$$

$$f_{kI}(x) = \frac{1}{\sqrt{(2\pi)^3 2\omega_{k_I}}} e^{-ik_I \cdot x} \quad , \quad \omega^2_{k_I} = |\vec{k}_I|^2 + m^2_I \tag{77}$$

where the indice I on the right hand side is not summed over.

Thus, an interpretation for the subsidiary condition can be obtained. Substituing (77) in (75) one gets the following collective expression,

$$\beta^I[k_0 a_0]_I - \gamma^I[k^\mu a_\mu]_I = 0 \tag{78}$$

Thus, a meaning for (78) can be obtained by choosing a frame where the G_1-field, the photon, is considered as moving along the third axis, $k^\mu_{\text{photon}} = (k, o, o, k)$. Then, by selecting a suitable set of basis for the photon one can re-discuss the meaning of cancelling the negative portion of the Hamiltonian. It shows that, while in classical QED the longitudinal and time-like photon average contributions to the Hamiltonian cancel each other, for (78) the number of these two kinds of photons are not necessarily the same but just proportional. The ensemble of particles that the $U(1)$-group englobes controls this proportion. It will correspond to a distribution depending on parameters, masses and momentiim of the other particles that are sharing, with of photon, the $U(1)$-symmetry. In a further work, consequence of (78) will be studied in more detail. It is still necessary to exploit the Fock space about the possibility that it contains states with different kinds of quanta.

4 - Generalized electric and magnetic fields

The inclusion of N-gauge bosons in a same group $U(1)$ propitiates an extension in the expression for electric and magnetic fields. Following the conditions discussed in section 2, a possible definition is

$$\vec{E}_{G_I} = Z_I^{[oi]} \tag{79}$$

$$\vec{B}_{G_I} = \frac{1}{2}\epsilon_{ijk} Z_I^{[jk]} \tag{80}$$

where

$$Z_{\mu\nu} = \sum_I Z_{\mu\nu I} \tag{81}$$

For instance,

$$Z_{\mu\nu 1} = \alpha_1 G^1_{\mu\nu} + \rho_{[1J]} G^1_\mu G^J_\nu \tag{82}$$

(79) and *(80)* satisfy gauge invariance and the usual mass canonical dimension. Thus, they are also possible experimental entities. This physical fact turns them into good candidates for the roles electric and magnetic fields. However, there is a new information, with respect to the Maxwell case. It is the extension due to the presence of an antisymmetrical interacting term, $\gamma_{[IJ]} G_{\mu I} G_{\nu J}$. It is analogous to the non-abelian case.

(79) and *(80)* should be understood as one of the channels that the theory contains for measurements. However, such a physical channel is not univocal. One can define electric fields in the theory differently from *(79)* and (80). Therefore it is important to work consistently. This amounts to defining the variables and the dynamics that the corresponding channel represents. Thus we have to characterize the correspondent Bianchi identities, equations of motion, conserved charges and Lorentz forces for the channel of the generalized fields (79) and (80). There are N-Bianchi identities corresponding to each antisymmetric field strenght given by *(81)*. They are repespectively associated to each physical field $G_{\mu I}$

$$\partial_\lambda Z_{[\mu\nu]I} + \text{ cyclic } = \rho_{[IJ]} \partial_\lambda (G_{\nu J}) + \text{ cyclic} \tag{83}$$

where the indice **I** on the right hand side is not to be understood as summed over.

The N-equations of motion belong to the same QED functional structure. **Thus,** the corresponding dynamics for each field is

$$\alpha_I \nabla \cdot \vec{E}(G) = \rho_{G_I} \tag{84}$$

$$\alpha_I \left(\frac{\partial}{\partial t} \vec{E}(G) - \nabla \times \vec{B}(G) \right) = \vec{J}_{G_I} \tag{85}$$

with the following conservation law

$$\frac{\partial}{\partial t} \rho_{G_I} + \nabla \cdot \vec{J}_{G_I} = 0 \tag{86}$$

$\vec{E}(G)$ **and** $\vec{B}(G)$ are fields defined in terms of the N-collective system. They are

$$\vec{E}(G) = Z^{[oi]} \tag{87}$$

$$\vec{B}(G) = \frac{1}{2} \epsilon_{ijk} Z^{[jk]} \tag{88}$$

The charge and the currents corresponding to each physical field (no externa1 sources were considered) are

$$\begin{aligned} -\rho_{G_I} = {} & \beta_I \nabla \cdot \vec{\mathcal{E}}(G) + \rho_{[IJ]} \vec{E}(G) \cdot \vec{G}_J + \rho_{(IJ)} \vec{\mathcal{E}}(G) \cdot \vec{G}_J + \\ & + \sigma_{IJ} Z_i^i G_J^o + \frac{1}{2} d_{IJ} G_J^o \end{aligned} \tag{89}$$

$$\begin{aligned} -\vec{J}_{G_I} = {} & \beta_I \frac{\partial}{\partial t} \vec{\mathcal{E}}(G) - \beta_I \nabla \times M(G) + \\ & + \rho_{[JI]} (\vec{E}(G) G_{oJ} + \vec{B}(G) \times \vec{G}_J) + \\ & \rho_{(IJ)} [\vec{\mathcal{E}}(G) G_{oJ} + (\vec{M}(G) \times \vec{G}_J)_S] + \\ & + \sigma_{IJ} [Z_o{}^o + Z_i{}^i] \vec{G}_J + \frac{1}{2} d_{IJ} \vec{G}_J \end{aligned} \tag{90}$$

where

$$(\vec{A} \times \vec{B})_S = \frac{1}{2} d_{ijk} A^j B^k$$

with

$$\vec{\mathcal{E}}(G) = Z^{(oi)} \tag{91}$$

$$\vec{M}(G) = \frac{1}{2} d_{ijk} Z^{(jk)} \tag{92}$$

Observe that (91) and **(92)** are not gauge invariant and that each equation in (84) and (85) are differentiated by coefficients as α_I, β_I and so on.

The symmetric energy-momentum tensor derived from (59) is

$$\begin{aligned} T_G^{\mu\nu} = & -\frac{1}{2} Z_G^{[\alpha\beta]} Z_{G[\alpha\beta]} g^{\mu\nu} + 2 Z_G^{[\mu\alpha]} Z_{G[\alpha}{}^{\nu]} + \\ & -\frac{1}{2} Z_G^{(\alpha\beta)} Z_G(\alpha\beta) g^{\mu\nu} + 2 Z_G^{(\mu\alpha)} Z_{G(\alpha}{}^{\nu)} + \\ & -\frac{1}{2} d_{IJ} G_{\alpha I} G_J^{\alpha} g^{\mu\nu} + 2 d_{IJ} G_I^{\mu} G_J^{\nu} \end{aligned} \tag{93}$$

(93) includes contributions from kinetic, interacting and massive terms. The corresponding Lorentz density four-vector will be

$$f_G^{\nu} = \partial_\mu T_G^{\mu\nu} \tag{94}$$

The energy-momentum tensor and the Lorentz force corresponding to each physical field are

$$T_G^{\mu\nu} = \sum_I T_I^{\mu\nu} \tag{95}$$

$$f_{G_I}^{\nu} = \partial_\mu T_I^{\mu\nu} \tag{96}$$

Finally, we conclude by observing that dynamical entities as charges, currents, energy-momentum, Lorentz forces carry a physics that does not depend on the channel (place characterized by one definition of the electric and magnetic fields). This means that each channel is only writing differently the same physics. However, a different conclusion is obtained when the Bianchi identities are considered. **(83)** shows that depending on how the measurable electric and magnetic fields are defined, symmetry generates equations either with a source or sourceless.

5. Conclusions

The idea of introducing more potential fields generates a new kind of collective behaviour for the particles. Their dynamics are different, but with a connection as they share a same parameter $\alpha(x)$. The objective of this work was to study a classical viewpoint, where generalized electric and magnetic fields appear.

The preparation of theoretical entities that the theory develops for being measured is not immediate. Requirements are stipulated to define the physical masses and the physical sources. From Einstein'e energy equation, the physical masses are defined as the poles of a two-point Green function. From this calculation we also obtain the associated physical fields. Nevertheless, it is still necessary to prove how much theory is elastic enough incompass the presence of N-physical fields. For this, a first verification is to test if there are N-Bianchi identities associated. Another test is via the local Noether theorem. They explicitly inform about the possibility of the theory to lodge N-fields. Then, after such consistency steps, a practical attitude is to take the equation of motion corresponding to each physical field. Now, there appear paths for the entities to be measured. This is because for each equation of motion one gets more than one way of writing them in terms of experimental variables. Theory provides a richness for defining more than one electric (magnetic) field. Thus, under a classical viewpoint although the dynamics is unique, it can be represented in different experimental ways.

Physics is an experimental science. The proposed Lagrangian in (59) is Lorentz covariant and gauge invariant. Its corresponding Hamiltonian is covariant[4] and positively defined under certain conditions (75). Therefore, such a Lagrangian contains some basic conditions for being able to do physics. However, classicaly, it contains more than one way for measuring its dynamics. This means that there are two definitions for electric (magnetic) fields. They were characterized in the text as two channels,

$$\vec{E}(G_I) = \partial^o G_I^i - \partial^i G_I^o \tag{97}$$

$$\vec{E}_{G_I} = \vec{E}(G_I) + \rho_{[IJ]} G_I^o G_J^i \tag{98}$$

The physics derived from the minimal action principle is independent on the channel. Equations of motion, conserved charges, energy momentum tensor, Lorentz forces can be written in terms of different experimental variables corresponding to each channel but derived from common expressions based on $G_{\mu I}$ fields. However, for Bianchi identites such channel vision results in different physical significances.

While one experimental via as (97) is sourceless, (98) is similar to the monopole approach.

As a Iast comment, we would note that two types of dynamics are generated with the presence of more then one potential field in a same group: one dynamics corresponding to each particle considered and the other dynamics involving this collective system of particles as a whole. The importance is that the group symmetry develops both structures. In the first case experimental fields $\vec{E}_{G_I}$ and $\vec{B}_{G_I}$ are generated, while the collective system can be narrated through $\vec{E}(G)$and $\vec{B}(G)$.

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Resumo

Uma generalização para campos elétricos e magnéticos é obtida para um sistema contendo N-potenciais de campo dentro de um mesmo grupo-U(1). Identidades de Bianchi, equações de movimento, cargas conservadas e forças de Lorentz aparecem associadas a cada um destes potenciais de campo. Tais fatos confirmam sobre que um mesmo parâmetro de calibre, $\alpha(x)$, é capaz de organizar a presença de campos distintos. A Física gerada a partir do princípio de mínima ação deve ser independente das definições iniciais de campos elétricos (magnéticos). Entretanto, o uso de tal possibilidade de escolha ocasiona diferenças que se refletem no contexto das identidades de Bianchi.