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An approach to infinite dimensional Lie groups

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Abstract We present a "pedestrian" approach to some aspects of the theory of infinite dimensional Lie groups which is currently applied in mathematical physics.

1. Introduction

Infinite dimensional Lie groups and Lie algebras play an increasing role in today's mathematical physics. In classical continuum physics it started with the seminal paper of Arnold on the hydrodynamics of incompressible fluids[1,2] and later on, an infinite dimensional Hamiltonian formalism was applied to plasma physics[3,4]. In the study of nonlinear completely integrable systems of the Lax type, a connection was discovered with Kirillov's orbit method in representation theory and provided more geometric insight[5,6]. In gauge field theory, symmetry groups must be extended by the group of gauge transforrnations and this led to a geometric interpretation of anomalies[7,8]. In two-dimensional conformally invariant quantum field theory, with its outstanding successes in statistical mechanics, the (extended) Virasoro algebra classifies possible models[9,10]. More specifically, in string theory the infinite dimensional Kähler structure of coset spaces of diffeomorphism groups provides a quantisation scheme for the (bosonic closed) string field theory[11,12,13].

From the mathematical point of view, a rigorous theory of infinite dimensional Lie groups requires powerful tools from functional analysis and group theory. Lie groups modelled locally on Banach or Hilbert spaces do not cover many interesting examples such as the diffeomorphism groups. These are locally modelled on Fréchet spaces and are studied as inverse limits of Banach or Hilbert spaces (introduced by Omori). We refer to Kirillov's and Milnor's lectures notes[14,15] and to the review article of Adams et al.[16] for details. The recent diffeology approach of Souriau and coworkers[17,18] seems to be more suited for quantisation, which is more interesting from a physicist's point of view.

Physics however cannot wait for a rigorous theory , and in this paper we present an alternative "pedestrian" way to recover known results and, hopefully, make them accessible to a wider class of physicists. For this purpose we do not worry about what is the right differentiability condition to impose or what is the right topology to use, but simply apply the formal differential calculus on generalised functions. Many results can be obtained in this way, using the theory of finite Lie groups in local coordinates replacing discrete by continuous indices and summation by integration.

From the examples cited above it appears that the most important infinite dimensional Lie groups arising in physics are:

a) the diffeormorphism group of a manifolf, Diff(M), with the composition of mappings as group law.

b) the current group, C(M,H), of *smooth* maps from a manifold M to a Lie group H with pointwise multiplication, which has the current algebra of quantum field theory as Lie algebra.

c) the group of fibre automorphism of a principal fibre bundle and its normal subgroup of gauge transformations (locally it is the semi-direct product of the two groups above).

We restrict ourselves to the $\mathrm{Diff}(M)$'s. They are introduced with their Lie algebra Diff (M) in §2. The adjoint and co-adjoint representations of Diff (M) are constructed in §3. We consider possible central extensions in §4 (see the review of Tuynman 19). In $5 we define the Poisson structure on $\mathrm{Diff}(M)^*$ through the symplectic structure on each co-adjoint orbit according the Kirillov-Kostant-Souriau prescription. Finally in $6 we apply the obtained formulae to $\mathrm{Diff}(S^1)$ extended by the Gelfand-Fuks-Bott cocycle[20,21].

A discussion of the classification of the generic orbits and their quantisation[13,22] is postponed to later work.

2. Diffeomorphism groups and their Lie algebrae

Ignoring all smoothness conditions, as explained in the introduction, let us call G = $\mathrm{Diff}_+(M)$ the group of all orientation preserving diffeomorphisms of the n-dimensional compact orientable manifold M. Elements of G are maps

$$\phi : M \to M : x \to y = \phi(x)$$

and a curve in G is given by a one-parameter family of diffeomorphism ϕ_t. Two curves ϕ_t and ψ_t are said to be equivalent at t = 0 if, for all

$$x : \phi_0(x) = \psi_0(x)$$

and

$$\frac{d}{dt}\phi_t(x)\Big|_{t=0} = \frac{d}{dt}\psi_t(x)\Big|_{t=0}$$

A tangent vector X_0 at ϕ_0 is an equivalence class of such curves. It is a map from M to the tangent bundle T M :

$$X_0 : M \to TM : x \to (y, u_y)$$

such that $\tau_M \circ X_0 = \phi_0$, where $\tau_M : TM \to M$ is the canonical projection map. Clearly the set fo all vectors at ϕ_0 form a vector space $T_0(G)$, the tangent space at ϕ_0 and formally we may construct the tangent bundle

$$TG = \bigcup_{\phi_0 \in G} (\phi_0, X_0)$$

where $X_0 \in T_0(G)$, with the canonical projection τ_G : TG $\to$ G. A vector field on G is a section of this tangent bundle i.e. a map X : G $\to$ TG : $\phi \to X(\phi)$ such that $\tau_G \circ$ X $= id_G$. $X(\phi)$ itself is the map $M \to TM$: $x \to (\phi(x), \vec{u}[\phi; x])$, where $u[\phi; x]$ is the value of a vector field on M at $\phi(x)$ depending functionally on ϕ. Vector fields can also be considered as functional derivatives acting on functionals $f[\phi]$:

$$(Xf)[\phi] = \int dx \sum_i u^i[\phi; x] \frac{\delta}{\delta\phi^i(x)} f[\phi] \ . \tag{1}$$

where dx is a suitable volume form on M and where $\phi^i(x)$ and $u^i[\phi; x]$ are coordinates and components in a coordinate neighbourhood. The functional derivative is defined as :

$$\frac{\delta}{\delta\phi^i(x)} f[\phi] = \lim_{\epsilon\to 0}(f[\phi + \epsilon\delta_i\delta_x] - f[\phi])$$

where

$$(\phi + \epsilon\delta_i\delta_x)^j(y) = \phi^j(y) + \epsilon\delta_i^j\delta(x,y) \ .$$

The Dirac density $\delta(x,y)$ is related to the choice of the volume dx by $\int dy\delta(x,y)g(y) = g(x)$ for any reasonable function g on M.

Introducing a mixed discrete-continuous index $\alpha = (i,x)$, with $i = 1,2,\ldots,n$ and $x \in M$,we write eq.(1) as :

$$X[\phi] = \sum_\alpha u^\alpha(\phi)\frac{\partial}{\partial\phi^\alpha} \tag{2}$$

The Lie bracket of two vector fields X and Y, given respectively by u^a and v^α , is the vector field $Z = [X,Y]$ given by

$$w^\alpha(\phi) = \sum_\beta \left(u^\beta(\phi)\frac{\partial}{\partial\phi^\beta}v^\alpha(\phi) - v^\beta(\phi)\frac{\partial}{\partial\phi^\beta}u^\alpha(\phi)\right) \ . \tag{3}$$

Left and right multiplication in G is defined by :

$$L_\phi : G \to G : \psi \to L_\phi(\psi) = \phi \circ \psi$$

$$R_\phi : G \to G : \psi \to R_\phi(\psi) = \psi \circ \phi$$

and their differentials at ψ are:

$$L_{\phi^*;\psi} : T_\psi(G) \to T_{\phi\circ\psi}(G) : X_\psi \to Y_{\phi\circ\psi}$$

$$R_{\phi^*;\psi} : T_\psi(G) \to T_{\phi\circ\psi}(G) : X_\psi \to Y_{\psi\circ\phi}$$

given explicitly by:

$$(L_{\phi^*;\psi})^{(i,x)}{}_{(j,y)} = \frac{\delta(\phi(\psi(x))^i}{\delta\psi^j(y)} = \frac{\partial\phi^i(z)}{\partial z^j}\Big|_{z=\psi(x)}\delta(x,y) \tag{4a}$$

$$(R_{\phi^*;\psi})^{(i,x)}{}_{(j,y)} = \frac{\delta(\psi(\phi(x))^i}{\delta\psi^j(y)} = \delta_j^i\ \delta(\phi(x),y) \tag{4b}$$

Left and right invariarit vector fields are defined by their value at the identity diffeomorphisrn id :

$$X_L(\phi) = L_{\phi^*;id} X_{id} \tag{5a}$$

$$X_R(\phi) = R_{\phi^*;id} X_{id} \tag{5b}$$

where $X_{id} \in T_{id}(G)$ is a vector field on M :

$$X_{id} : \mathrm{M} \to TM : x \to (x, \vec{u}(x)) \; .$$

The components of the left- and right-invariant vector fields are:

$$u_L^{(i,x)}(\phi) = \sum_j \frac{\partial \phi^i(x)}{\partial x^j} u^j(x) \tag{6a}$$

$$u_R^{(i,x)}(\phi) = u^i(\phi(x)) \tag{6b}$$

Lie brackets of (right-) left-invariant vector fields are (right-) left-invariant:

$$[X_L(\phi), Y_L(\phi)] = L_{\phi*;id} Z_{id} = Z_L(\phi) \tag{7a}$$

and, if X_L and Y_L are given by the vector fields $\vec{u}(x)$ and $\vec{v}(x)$ then Z_L is given by

$$\vec{w}(x) = -[\vec{u}(x), \vec{v}(x)] \; . \tag{8}$$

In the same way:

$$[X_R(\phi), Y_R(\phi)] = -R_{\phi*;id} Z_{id} \tag{7b}$$

with the same Z_{id} as above. The Lie algebra $\mathcal{G}$ of G, given by the algebra of the left invariant vector fields, is thus isomorphic to the Lie algebra of vector fields on M with the Lie product given by minus the usual Lie bracket of vector fields on M. A generalised local basis of the Lie algebra is given by :

$$E_{i,x}(\phi) = \int dz \sum_k \left\{ \frac{\partial \phi^k(z)}{\partial z^i} \delta(z,x) \right\} \frac{\delta}{\delta \phi^k(z)} \tag{9}$$

and any left-invariant vector field can be written as :

$$X_L(\phi) = \int dx \sum_i E_{(i,x)}(\phi) u^i(x) \tag{10}$$

The "structure constants" are defined by :

$$[E_\alpha, E_\beta] = \sum_\gamma E_\gamma f^\gamma_{\alpha\beta} \tag{11a}$$

with:

$$f^{(k,z)}_{(i,x),(j,y)} = \delta^k_i \,\delta(z,y)\frac{\partial\delta(z,x)}{\partial z^j} - \delta^k_j \delta(z,X)\frac{\partial\delta(z,y)}{\partial z^i} \,. \tag{11b}$$

The dual $\mathcal{G}^*$ of the Lie algebra is the space of linear functionals on $\mathcal{G}$, identified with the vector fields on M. Elements of $\mathcal{G}^*$, are covariant vector densities of weight one, in cooxdinates given by $\xi_i(x)$ and paired with elements of $\mathcal{G}$, through:

$$< \xi, \vec{u} >= \int dx \sum_i \xi_i(x) u^i(x) \tag{12}$$

This dual $\mathcal{G}^*$ can also be defined as the space of left-invariant one-forms on G. A one-form at ϕ is a linear functional on the vectors of $T_\phi(G)$:

$$\Xi(\phi) = \int dx \sum_i \xi_i[\phi; x]\delta\phi^i(x) \tag{13}$$

where the basis one-forms $\delta\phi^i(x)$ are duals of $\delta/\delta\phi^j(y)$, so that the pairing is:

$$< \Xi(\phi), X(\phi) >= \int dx \sum_i \xi_i[\phi; x] u^i[\phi; x] \tag{14}$$

The one-forms at ϕ generate the cotangent space $T^*_\phi(G)$ and the pull-back of L_ϕ is the map:

$$L^*_{\phi;\psi} : T^*_{\phi\circ\psi}(G) \to T^*_\psi(G) : \Xi(\phi\circ\psi) \to H(\psi)$$

which reads in components :

$$\begin{aligned} \xi_i[\phi\circ\psi; x] \to \eta_j[\psi; y] = \\ = \int dx \sum_{i} (L^*_{\phi;\psi})_{(j,y)}^{(i,x)} \xi_i[\phi\circ\psi; x] \end{aligned} \tag{15}$$

with

$$(L^*_{\phi;\psi})_{(j,y)}^{(i,x)} = (L_{\phi^*;\psi})^{(i,x)}{}_{(j,y)}$$

Left-invariant one-forms are given by:

$$\Xi^L(\phi) = L^*_{\phi^{-1};\phi}\, \Xi_{id} \tag{16a}$$

In components:

$$\xi_i^L[\phi;x] = \sum_j \frac{\partial(\phi^{-1}(z))^j}{\partial z^i}\Big|_{z=\phi(x)} \xi_j(x) \tag{16b}$$

where $\xi_j(x)$ are the components of the covariant vector density of weight one introduced above.

The dual **basis** of $\{E_{(i,x)}(\phi)\}$ is given by:

$$\Delta^{(i,x)}(\phi) = \int dz \sum_k \Big\{ \frac{\partial(\phi^{-1}(y))^i}{\partial y^k}\Big|_{y=\phi(z)} \delta(x,z) \Big\} \delta\phi^k(z) \tag{17}$$

and any left-invariant one-form **is** given by :

$$\Xi^L(\phi) = \int dx \sum_i \xi_i(x) \Delta^{(i,x)}(\phi) \tag{18}$$

3. The adjoint and co-adjoint representation

These representations of G in $\mathcal{G}$, respectively $\mathcal{G}$, are fundamental in Kirillov's **method** of orbits. Let $C_\phi = L_\phi \circ R_{\phi^{-1}}$ be the map $G \to G : \psi \to \phi \circ \psi \circ \phi^{-1}$, its derivative at the identity of G is :

$$C_{\phi^*;id} : T_{id}(G) \to T_{id}(G) : L_{\phi *;\phi^{-1}} \circ R_{\phi^{-1};id} \tag{19}$$

which is an automorphism of $\mathcal{G}$. The adjoint representation of G in $\mathcal{G}$ is:

$$Ad : G \to Aut(\mathcal{G}) : \phi \to Ad(\phi) = C_{\phi^*;id} \tag{20a}$$

In coordinates :

$$(Ad(\phi))^{(i,x)}{}_{(j,y)} = \frac{\partial\phi^i(y)}{\partial y^j} \delta(\phi^{-1}(x), y) \tag{20b}$$

The adjoint action of ϕ transforms the representative vector field $\vec{u}(x)$ of X_{id} into :

$$(Ad(\phi)\vec{u})^i(x) = \sum_j \frac{\partial\phi^i(y)}{\partial y^j} u^j(y)|_{y=\phi^{-1}(x)} \tag{20c}$$

The coadjoint representation is similarly defined **as:**

$$K : G \to Aut(\mathcal{G}^*) : \phi \to K(\phi) = C^*_{\phi^{-1};id} \tag{21a}$$

It satisfies :

$$< K(\phi)\Xi_L, X_L > = < \Xi_L, Ad(\phi^{-1})X_L > \qquad (21b)$$

The covariant density $\xi_i(x)$ is transformed as:

$$(K(\phi)\xi)_i(x) = \sum_j \xi_j(\phi^{-1}(x)) \frac{(\partial(\phi^{-1}(x))^j}{\partial x^i} \det\left\{\frac{\partial\phi^{-1}(x)}{\partial x}\right\} \qquad (21c)$$

The adjoint and coadjoint representations of the Lie algebra are given by the differentials of $Ad(\phi)$ and $K(\phi)$:

$$ad = Ad_{*;id} : \mathcal{G} \to End(\mathcal{G}) : X_L \to ad(X_L)$$

where $ad(X_L)$ is the map:

$$Y_L \to ad(X_L)Y_L = [X_L, Y_L] \ . \qquad (22a)$$

On the representative vector field $\vec{v}$, it acts as:

$$ad(\vec{u})\vec{v} = -[\vec{u}, \vec{v}] = -\mathcal{L}_{\vec{u}}\vec{v} \qquad (22b)$$

where $\mathcal{L}_{\vec{r}}\vec{v}$ is the Lie derivative along $\vec{u}$ of the vector field $\vec{v}$. The coadjoint representation is obtained in the same way and satisfies:

$$< k(\vec{u})\xi, \vec{v} > = - < \xi, ad(\vec{u})\vec{v} > \qquad (23a)$$

which yields:

$$k(\vec{u})\xi = -\mathcal{L}_{\vec{u}}\xi \qquad (23b)$$

with the Lie derivative along u of the vector density ξ :

$$(\mathcal{L}_{\vec{u}}\xi)_i(x) = \sum_j (u^j(x)\partial_j\xi_i(x) + \xi_j(x)\partial_i u^j(x) + \xi_i(x)\partial_j u^j(x) \ . \qquad (24)$$

4. Central extensions

A group $\hat{G}$ is a central extension of the group G by the abelian group A, + ,on which G acts trivially, if there is an exact sequence of group homomorphisrns:

$$0 \to A^{i} \xrightarrow{\iota} \hat{G} \xrightarrow{\kappa} G \to 1 \ , \quad \text{and} \quad \text{Ker}\ \kappa = \text{Im}(\iota)$$

such that $\iota(A)$ belongs to the centre of $\hat{G}$ and G $\simeq G/\ell(A)$, where κ : G $\to \hat{G}/\iota(A)$ is the canonical projection. An extension defines, and is defined by a cohomology class of $H^2(G,A)$ i.e. an equivalence class of two-cocycles differing by a coboundary. A two-cocycle is a map:

$$Z : G \times G \to A : (\phi,\psi) \to Z(\phi,\psi)$$

with zero coboundary:

$$\begin{aligned} \delta Z(\phi,\psi,\chi) : &= Z(\psi,\chi) - Z(\phi\circ\psi,\chi) + Z(\phi,\psi\circ\chi) - Z(\phi,\psi) \\ &= 0 \end{aligned} \qquad (25a)$$

and a two-coboundary is a two-cocycle of the form:

$$B(\phi,\psi) = \delta C(\phi,\psi) := C(\psi) - C(\phi\circ\psi) + C(\phi) \qquad (25b)$$

where the one-cochain C is a function on G with values in A. As a set, $\hat{G}$ is identified with G x A and has elements $\Phi = (\phi,p)$, $\Psi = (\psi,q)$; the group law is defined by:

$$\Phi * \Psi = (\phi\circ\psi, p+q-Z(\phi,\psi)) \ . \qquad (26)$$

The above cocycle condition guarantees that this is effectively a group law and it can be shown that two-cocycles differing by a coboundary yield equivalent extensions.

The cocycle condition implies that

$$Z_0 := Z(id,id) = Z(\phi,id) = Z(id,\phi) \quad \text{and} \quad Z(\phi^{-1},\phi) = Z(\phi,\phi^{-1}) \ .$$

The **neutral** element of $\hat{G}$ is $Id = (id, Z_0)$ and the inverse of the **group** element $\Phi = (\phi, p)$ is given by

$$(\phi, p)^{-1} = (\phi^{-1}, -p + Z_0 + Z(\phi, \phi^{-1})) \ . \tag{27}$$

Let $\{p^a\}$, $a = 1, 2, ..., d$, be coordinates of $p \in A$, then

$$\Phi^A = \{\phi^{(i,x)}, p^a\}$$

are coordinates of Φ and a vector X_Φ tangent to $\hat{G}$ at Φ is given by the derivation:

$$\begin{aligned} X_\Phi &= \int dx \sum_i u^i[\Phi; x] \frac{\delta}{\delta \phi^i(x)} + \sum_a u^a[\Phi] \frac{\partial}{\partial p^a} \\ &= \sum_A u^A[\Phi] \frac{\partial}{\partial \Phi^A} \ . \end{aligned} \tag{28}$$

All the considerations of the preceding paragraphs can be repeated with d more entries for the auxiliary matrices $L_{\Phi*;\Psi}$ and $R_{\Phi*;\Psi}$. The results for the relevant **matrix** elements are given below:

$$\begin{aligned} (L_{\Phi^*;\Psi})^{(i,x)}{}_{(j,y)} &= (L_{\phi^*;\psi})^{(i,x)}{}_{(j,y)} \\ (L_{\Phi^*;\psi})^{(i,x)}{}_{b} &= 0 \\ (L_{\Phi^*;\psi})^{a}{}_{b} &= \delta^a{}_b \\ (L_{\Phi^*;\psi})^{a}{}_{(j,y)} &= -\frac{\delta Z^a(\phi, \psi)}{\delta \psi^j(y)} \ . \end{aligned} \tag{29}$$

Left invariant vector fields are

$$X_L(\Phi) = L_{\phi^*;Id} X_{Id} \tag{30a}$$

given in component form by:

$$u_L^i[\Phi; x] = \sum_j \frac{\partial \phi^i(x)}{\partial x^j} u^j(x)$$

and

$$u_L^a[\Phi] = u^a - \int dy \sum_j \frac{\partial Z^a(\phi, \psi)}{\delta \psi^j(y)} \Big|_{\psi = id} u^j(y) \ . \tag{30b}$$

They are identified with $T_{Id}(\hat{G}) = \hat{\mathcal{G}}$, which is the corresponding extension of $\mathcal{G}$ by $\mathbf{R}^d$, the Lie algebra of A. Elements of $\hat{\mathcal{G}}$ are given by vector fields on M and by vectors of $T_{Z_0}(A)$, in components $u^A = (u^i(x), u^a)$. A generalised local basis of $\hat{\mathcal{G}}$ is given by the left invariant vector fields $E_A(\Phi)$:

$$E_{(i,x)}(\Phi) = \int dy \sum_j \left(\frac{\partial \phi^j(y)}{\partial y^i}\delta(y,x)\right)\frac{\delta}{\delta\phi^j(y)} - \sum_b \frac{\delta Z^b(\phi,\psi)}{\delta\psi^i(x)}\bigg|_{\psi=id}\frac{\partial}{\partial p^b} \tag{31a}$$

$$E_a(\Phi) = \frac{\partial}{\partial p^a}\ . \tag{31b}$$

They define the structure constants F^C_{AB} through:

$$[E_A, E_B] = \sum_C E_C F^C_{AB} \tag{32a}$$

$$F^{(k,z)}_{(i,x),(j,y)} = f^{(k,z)}_{(i,x),(j,y)} \tag{32b}$$

$$F^a_{(i,x),(j,y)} = -\left[\frac{\delta^2}{\delta\phi^i(x)\delta\psi^j(y)} - \frac{\delta^2}{\delta\phi^j(y)\delta\psi^i(x)}\right] Z^a(\phi,\psi)\bigg|_{\phi=\psi=id} \tag{32c}$$

The Lie bracket of two elements of $\hat{\mathcal{G}}$ defined by $(u^i(x), u^a)$ and $(v^i(x), v^a)$ is given by:

$$w^i(x) = -[\vec{u}, \vec{v}]^i(x) \tag{33a}$$

$$w^a = \sum_{i,j} \int dx \int dy F^a_{(i,x),(j,y)}\ u^i(x)\ v^j(y)\ . \tag{33b}$$

In fact, w^a is the value of a two-cocycle $Z(\vec{u}, \vec{v})$ on the Lie algebra $\mathcal{G}$ with values in $\mathbf{R}^d$, on which $\mathcal{G}$ acts trivially, and defines a Lie algebra extension $\hat{\mathcal{G}}$ of the Lie algebra $\mathcal{G}$, the commutation relations (11) being replaced by *(32)*.

The adjoint representation of $\hat{G}$ on $\hat{\mathcal{G}}$ is given in matrix forrn by:

$$Ad(\Phi)^{(i,x)}{}_{(j,y)} = \frac{\partial \phi^i(y)}{\partial y^j}\delta(\phi^{-1}(x), y)$$

$$Ad(\Phi)^{(i,x)}{}_b = 0 \qquad , \qquad Ad(\Phi)^a{}_b = \delta^a{}_b$$

$$Ad(\Phi)^a_{(j,y)} = -\sum_k \frac{\delta Z^a(\chi,\phi^{-1})}{\delta\chi^k(y)}\Big|_{\chi=\phi} \frac{\partial\phi^k(y)}{\partial y^j} - \frac{\delta Z^a(\phi,\psi)}{\delta\psi^i(y)}\Big|_{\psi=id} \quad . \tag{34}$$

A one-form on $\hat{G}$ at $\Phi = (4, p)$ is given by:

$$\Xi_\phi = \int dx \sum_i \xi_i[\Phi;x]\ \delta\phi^i(x) + \sum \xi_a[\Phi]\ \delta p^a \tag{35}$$

and left-invariant one-forms have components:

$$\xi_i^L[\Phi;x] = \sum_j \xi_j(x)\frac{\partial(\phi^{-1}(y))^j}{\partial y^i}\Big|_{y=\phi(x)} - \sum_b \xi_b \frac{\delta Z^b(\phi^{-1},\psi)}{\delta\psi^j(x)}\Big|_{\psi=\phi}$$
$$\xi_a^L[\Phi] = \xi_a \tag{36a}$$

An element of $\mathcal{G}^*$ is thus given by its components $\xi_A = \{\xi_i(x), \xi_a\}$, so that:

$$\Xi^L(\Phi) = \int dx \sum_i \xi_i(x)\ \Delta^{(i,x)}(\Phi) + \sum_a\ \xi_a\ \Delta^{(a)}(\Phi)\ . \tag{36b}$$

With the dual basis $\Delta^{(A)}(\Phi)$ of $E_{(A)}(\Phi)$ given by:

$$\Delta^{(i,x)}(\Phi) = \sum_j \frac{\partial(\phi^{-1}(y))}{\partial y^j}\Big|_{y=\phi(x)}\delta\phi^j(x)$$

$$\Delta^{(a)}(\Phi) = \delta p^a - \int dy \sum_j \frac{\delta Z^a(\phi^{-1},\psi)}{\delta\psi^j(y)}\Big|_{\psi=\phi}\delta\phi^j(y) \tag{37}$$

The coadjoint action of $\hat{G}$ on such an elernent transforms it into :

$$(K(\Phi)\xi)_i(x) = \det\left\{\frac{\phi^{-1}(x)}{\partial x}\right\}\sum_j \xi_j(\phi^{-1}(x))\frac{\partial(\phi^{-1}(x))^j}{\partial x^i}$$
$$-\sum_a \xi_a\left[\frac{\delta Z^a(\phi^{-1},\psi)}{\delta\psi^i(x)}\Big|_{\psi=id} + \sum_k \frac{\delta Z^a(\chi,\phi)}{\delta\chi^k(x)}\Big|_{\chi=\phi^{-1}}\frac{\partial(\phi^{-1}(x))^k}{\partial x^i}\right]$$

$$(K(\Phi)\xi)_a = \xi_a\ . \tag{38}$$

The adjoint and coadjoint representations of the Lie algebra $\mathcal{G}$, respectively in $\hat{\mathcal{G}}$ and $\hat{\mathcal{G}}^*$, are obtained taking derivatives. Let elements of $\hat{\mathcal{G}}$ and $\hat{\mathcal{G}}^*$ be given by $u^A = \{u^i(x), u^a\}$, $v^A = \{v^i(x), v^a\}$ and $\xi_A = \{\xi_i(x), \xi_a\}$, one obtains:

$$(ad(u)v)^i(x) = -(\mathcal{L}_{\vec{u}}\vec{v})^i(x)$$
$$(ad(u)v)^a = \int dx \int dy \sum_{i,j} F^a_{(i,x),(j,y)}\; u^i(x)\; v^j(y) \tag{39a}$$

and, for the coadjoint action:

$$(k(u)\xi)_i(x) = -(\mathcal{L}_{\vec{u}}\xi)(x) - \sum_b \xi_b \int dy \sum_j u^j(y) F^a_{(j,y),(i,x)}$$
$$(k(u)\xi)_a = 0\ . \tag{39b}$$

5. Symplectic structure on the coadjoint orbits

Let

$$\theta^L_\xi(\phi) = \int dx \sum_i \xi_i(x) \Delta^{(i,x)}(\phi)$$

be the left-invariant one-form on G corresponding to the covariant density $\xi \in \mathcal{G}^*$.

Fixing ξ, the coadjoint action of G defines the map:

$$K_\xi : G \to O_\xi : \phi \to \eta = K_\xi(\phi) := K(\phi)\xi \tag{40}$$

where O_ξ is the orbit of ξ in $\mathcal{G}^*$. Also let G_ξ be the isotropy group of ξ i.e.

$$G_\xi = \{\phi \in G; K(\phi)\xi = \xi\} \tag{41}$$

and

$$\pi_\xi : \mathrm{G} \to G/G_\xi : \phi \to [\phi] := \phi\, G_\xi$$

the canonical projection on the left coset space. The Lie algebra $\mathcal{G}_\xi$ of G_ξ is given by the Killing vector fields of the density ξ, since

$$k(\vec{v})\xi = -L_{\vec{v}}\xi = 0\ .$$

Going over to the guotient one obtains a diffeomorphism :

$$\tilde{K}_\xi = K_\xi \circ \pi_\xi^{-1} : G/G_\xi \to O_\xi : [\phi] \to \tilde{K}_\xi([\phi]) = K(\phi')\xi \tag{42}$$

where ϕ' is an element of the left coset $[\phi]$.

The **Kirillov-Kostant-Souriau** construction asserts the existence of a unique two-form ω_ξ on G/G_ξ such that:

$$\pi_\xi^* \omega_\xi = d\theta_\xi^L \ . \tag{43}$$

This two-form is non-degenerate and closed , so it defines a symplectic structure on G/G_ξ which is pushed forward by the diffeomorphism $\tilde{K}_\xi$ to a symplectic structure on the orbit O_ξ. An outline of this construction is given below.

First of all let us recall that tangent vectors at $[\phi]$ to the manifold G/G_ξ are equivalence classes $[X]_{[\phi]}$ on the tangent bundle TG, restricted to ϕG_ξ , with respect to the equivalence relation:

$$(\phi, X(\phi)) aeq (\phi', X'(\phi')) \tag{44}$$

if

$$\chi := \phi^{-1} \circ \phi' \in G_\xi$$

and

$$X'(\phi') = R_{\chi^*;\phi} X(\phi) + V_{\vec{v}}^L(\phi')$$

where

$$V_{\vec{v}}^L(\phi') = L_{\phi'^*;id} \vec{v}$$

is the left invariant vector field on G corresponding to a $\vec{v} \in \mathcal{G}_\xi$.

From the Maurer-Cartan structure equations it follows that:

$$d\theta_\xi^L(X,Y) = - < \xi, [\tilde{X}_\phi, \tilde{Y}_\phi] > \tag{45}$$

where $\tilde{X}_\phi = L_{\phi^{-1*};\phi} X(\phi)$ belongs to $\mathcal{G}$. Choosing another representative of $[X]_{[\phi]}$ one has:

$$\tilde{X}'_{[\phi']} = L_{\phi'^{-1*};\phi'} X'(\phi') = Ad(\chi^{-1}) \tilde{X}_\phi + \vec{v} \ ,$$

so that

$$\omega_\xi([X],[Y])_{[\phi]} = d\theta^L_\xi(X,Y)_\phi = d\theta^L_\xi(X',Y')_{\phi'} \tag{46}$$

is well defined on G/G_ξ.

It is clearly non-degenerate and

$$\pi^*_\xi\, d\omega_\xi = d\ \pi^*_\xi \omega_\xi = d^2\theta^L_\xi = 0$$

implies that it is closed, since π_ξ and its derivative are surjective.

The tangent space at $\eta = K(\phi)\xi$ to O_ξ is spanned by the infinitesimal generators of the coadjoint action on $\mathcal{G}^*$. As vector fields on the linear space $\mathcal{G}^*$ they can be identified with elements (vectors) of $\mathcal{G}^*$:

$$\gamma(\vec{u},\eta) := K_{\eta^*;id}\vec{u} = -L_{\vec{u}}\eta\ . \tag{47}$$

Now $Ker(K_{\eta^*;id}) = \mathcal{G}_\eta$ so that $\mathcal{G}/\mathcal{G}_\eta$ is isomorphic with $T_\eta(O_\eta)$ under the isomorphism:

$$\begin{aligned} \tilde{K}_{\xi^*;[\phi]} : T_{[\phi]}(G/G_\xi) \to T_\eta(O_\xi) :& \\ [X]_{[\phi]} \to \gamma(R_{\phi-1*;\phi}X(\phi),\eta) \end{aligned}$$

where $\eta = K(\phi)\xi$ and where $(\phi, X(\phi))$ is a representative of the equivalence class $[X]_{[\phi]}$.

The inverse isomorphism sends $\varsigma \in T_\eta(O_\xi)$ in the equivalence class $[X]_{[\phi]}$ of $(\phi, X(\phi))$ where 4 is a solution , modulo G_ξ ,of $\eta = K(\phi)\xi$ and where $X(\phi) = R_{\phi^*;id}\vec{u}$, with Ü solution of $\varsigma = -L_{\vec{u}}\eta$ defined modulo $\mathcal{G}_\eta$.

The push-forward of $\omega_\xi, \Omega_\xi = \tilde{K}_{\xi^*}\ \omega_\xi$, is defined on vectors of the form $\varsigma = \gamma(\vec{u},\eta)$ by:

$$\begin{aligned} \Omega_\xi(\varsigma_1,\varsigma_2)_\eta &= \omega_\xi([X_1],[X_2])_{[\phi]} = d\theta^L_\xi(X_1,X_2)_\phi \\ &= - < \xi, [\tilde{X}_{1\phi}, \tilde{X}_{2\phi}] > \ . \end{aligned} \tag{48}$$

where

$$[X_1]_{[\phi]} = (\tilde{K}^{-1}_\xi)_{*;\eta}\ \varsigma_1 \quad \text{and} \quad \tilde{X}_{1\phi} = Ad(\phi^{-1})\vec{u}_1$$

so that finally:

$$\Omega_\xi(\varsigma_1,\varsigma_2)_\eta = - < \eta, [\vec{u}_1, \vec{u}_2] > \tag{49}$$

This symplectic structure on each orbit defines a Poisson structure on $\mathcal{G}^*$, which is a Lie algebra structiire on the ring of functions on $\mathcal{G}^*$ such that the bracket is a derivation on the ring. It is defined by:

$$(f_1, f_2)_\xi = -\left\langle \xi, \left[\frac{\delta f_1(\xi)}{\delta \xi}, \frac{\delta f_2(\xi)}{\delta \xi}\right]\right\rangle , \tag{50}$$

where the functional derivative is viewed as a map:

$$\frac{\delta f}{\delta \xi} : \mathcal{G}^* \to \mathcal{G} : \xi \to \frac{\delta f(\xi)}{\delta \xi} ,$$

defined by:

$$\left\langle \eta, \frac{\delta f(\xi)}{\delta \xi} \right\rangle = \frac{d}{dt} f(\xi + t\eta))|_{t=0} \tag{51}$$

In components, the Poisson bracket reads:

$$(f_1, f_2)_\xi = -\int dx \sum_{i,j} \xi_i(x) \Big(\frac{\delta f_1(\xi)}{\delta f^j(x)}\, \partial_j \frac{\delta f_2(\xi)}{\delta \xi^i(x)} - \frac{\delta f_2(\xi)}{\delta \xi^j(x)}\, \partial_j \frac{\delta f_1(\xi)}{\delta \xi^i(x)} \Big) \tag{52}$$

Naturally, the symplectic structure on the coadjoint orbits of an element of the dual $\hat{\mathcal{G}}^*$ of the extended Lie algebra $\hat{\mathcal{G}}$ under the action of the extended group $\hat{G}$ can be constructed in a similar way with the help of the formulae of Chapter 4. In the next paragraph we apply this to the diffeomorphism group of the circle.

6. The circle S^1

Elements of the group Diff(S^1) are smooth functions from S^1 to S^1 with nowhere vanishing derivative (positive if orientation- preserving). The group has a non trivial central extension by R with Bott's two-cocycle:

$$Z(\phi, \psi) = c \int \ln(\phi \circ \psi)' d \ln(\psi)' \tag{53a}$$

where c is a real parameter and where $'$ denotes derivative. It can be rewritten as:

$$Z(\phi, \psi) = c \int_0^{2\pi} dx \ln(\phi'(x)) \frac{(\psi^{-1})''(x)}{(\psi^{-1})'(x)} \tag{53b}$$

The associated Gelfand-Fuks two cocycle on the Lie algebra of Diff(S^1) is calculated using (32c):

$$F^{\circ}(x,y) = c\Big(\frac{d^3}{dx^3}\,\delta(x,y) - \frac{d^3}{dy^3}\,\delta(y,x)\Big) \tag{54}$$

so that the two-cocycle (33b) reads:

$$\begin{aligned} Z(u,v) &= c\int dx(u(x)v'''(x) - v(x)u'''(x)) \\ &= -c\int dx(u'(x)v''(x) - v'(x)u''(x)) \ . \end{aligned} \tag{55}$$

The Lie algebra of the extended group and its dual have elements represented by $u = (u(z), u^o)$ and $\xi = (\xi(x), \xi_0)$, on which a group element $\Phi = (\phi, p)$ acts trough the adjoint and co-adjoint representation:

$$\begin{aligned} (Ad(\Phi)u)(x) &= \phi'(\phi^{-1}(x))\ u(\phi^{-1}(x)) \\ (Ad(\Phi)u)^{\circ} &= u^{\circ} - c\int dy\ S[\phi](y)\ u(y)\ , \end{aligned} \tag{56}$$

where

$$S[\phi] := \frac{2\phi'\phi''' - 3(\phi'')^2}{(\phi')^2} \tag{57}$$

is the Schwarzian of ϕ

$$\begin{aligned} (K(\Phi)\xi)(x) &= ((\phi^{-1})'(x))^2\ \xi(\phi^{-1}(x)) - cS[\phi^{-1}](x)\ \xi_0 \\ (K(\Phi)\xi))0 &= \xi_0\ . \end{aligned} \tag{58}$$

The property of the Schwarzian:

$$S[\phi^{-1}](x) = -(\phi'(y))^{-2}S[\phi](y)|_{y=\phi^{-1}(x)} \tag{59}$$

guarantees that $< K(\Phi)\xi, Ad(\Phi)u > = < \xi, u >$.

The isotropy group of a fixed element $\xi = (\xi(x), \xi_0)$ is given by all $\Phi = (\phi, p)$ solutions of

$$\xi(x) = ((\phi^{-1})'(x))^2\ \xi(\phi^{-1}(x)) - cS[\phi^{-1}](x)\ \xi_0\ . \tag{60}$$

Its Lie algebra is given by all $u = (u(x), u^o)$ solutions of $k(u)\xi = 0$, which according to (39b) becomes:

$$-(u\xi' + 2u'\xi) + 2cu'''\xi_0 = 0\ . \tag{61}$$

The orbit O_ξ is given by

$$\eta(x) = ((\phi^{-1})'(x))^2 \ \xi(\phi^{-1}(x)) - cS[\phi^{-1}](x) \ \xi_0$$
$$\eta_0 = \xi_0 \ , \quad \text{where} \quad \phi \in \text{Diff}(S^1) \ , \tag{62}$$

and vectors $\varsigma = (\varsigma(x), \varsigma_0)$, tangent to O_ξ at a point η of the orbit, are of the form

$$\varsigma(x) = -(u\varsigma' + 2u'\eta) + 2cu'''\eta_0](x)$$
$$\varsigma_0 = 0 \ . \tag{63}$$

The symplectic structure on the orbit O_ξ is defined by

$$\Omega_\xi(\varsigma_1, \varsigma_2)_\eta = \int dx \ \eta(x) \ (u_1(x)u_2'(x) - u_1'(x)u_2(x))$$
$$- \eta_0 c \int dx \ (u_1'(x)u_2''(x) - u_1''(x)u_2'(x)) \tag{64}$$

where u_1 and u_2 are defined by (63) up to an element of the Lie algebra of the isotropy group of η,i.e. a solution of

$$-(u\eta' + 2u'\eta) + 2cu'''\eta_0 = 0 \ .$$

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References

1. V. Arnold, Ann. Inst. Fourier (Grenoble) 16,319,1966.
2. D. Ebin and J. Marsden, Ann. Math 92,102,1970.
3. J. Marsden and A. Weinstein, Physica **7D,305,1983.**
4. J. Marsden and A. Weinstein , Physica **4D,394,1982.**
5. M. Adler, Invent. Math. 50, 219, 1979.

6. *A.* Reyman and Semenov-Tian-Shansky, Sov. Math. Doklady 21,630,1980.
7. L. Fadeev and S. Shatisvili, Theor. Math. Phys. 60,770,1984.
8. L. Alvarez-Gaumé and P. Ginsparg, Ann. Phys. **161,** 423,1985.
9. A. Belavin, A. Polyakov and A. Zamolodchikov, Nucl.Phys. **B241,** 333,1984.
10. P. Goddard and D. Olive, Int. J. Mod. Phys. **A1,** 303, 1986.
11. M. Bowick and S. Rajeev, Nucl. Phys. **B293,** 348, 1987.
12. A. Kirillov and D. Yurev, Funct. Anal. Appl **21,** 284,1987.
13. E. Witten,. Comm. Math. Phys. 114, 1,1988.
14. A. Kirillov, Springer Lect. Not. Math. 970, 101, 1982.
15. J. Milnor, in *Les Houches Session XL,1983*, ed. *B.* S. de Witt and R.Stora, Elsevier,1984.
16. M. Adams, T. Ratiu and R. Schmid , in *Infinite dimensional Lie groups with applications*, ed. *V.* Kac, Springer, 1985.
17. J. M. Souriau, CPT-86/P.1938, Marseille preprint, 1986.
18. P. Iglesias, CPT-88/P.2184, Marseille preprint, 1988.
19. G. Tuynman and W. Wiegerinkx, J. Geom. and Phys. 4,207,1987.
20. I. Gelfand and D. Fuks, Funct. Anal. Appl. 2, 342, 1968.
21. R. Bott, Enseign. Math. 23, 209,1977.
22. A. Alekseev and S. Shatisvili, Nucl. Phys. **B323,** 719, 1989.

Resumo

Apresentamos uma abordagem não rigorosa, mas simples de certos aspectos da teoria dos grupos de Lie infinitos que está sendo comumente usada em física matemática.