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# Supersymmetric quantum mechanics and new potentials

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**Abstract** Using supersymmetric quantum mechanics we generalize some exactly solvable potentials: the particle in the box, Poschl-Teller and Rosen-Morse. We evaluate the new potentials and indicate their eigenfunctions and spectra.

## 1. INTRODUCTION

We know that the number of Schrödinger equations that have analytic solutions is quite small. In recent years some works have tried to increase this number, starting from potentials whose solutions are known (e.q. Abraham and Moses[1] and Pursey[2]). Supersymmetric quantum mechanics (SQM) has also been used for that purpose. The superalgebra is used to construct a hierarchy of Hamiltonians[3] and to build new Hamiltonians from a Ricatti equation[4,5,6].

The method to construct new potentials from known potentials using SQM, which we use in this paper, was proposed by Nieto[4] and Alves and Drigo Filho[5]. It is based on the factorisation method which was applied by Mielnik[7] to the harmonic oscillator and by Fernandez[8] to the Coulomb potential. This method is also applicable to spatially limited potentials. We will see it through the example of the particle in the box.

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Here, we use the superalgebra to construct new potentials from exactly solvable potentials. This construction was used to generalize the Coulomb and the harmonic oscillator potentials[5], as well as the Morse potential[6]. Firstly, we present the method for a general potential in sec. 2. Then, we apply it to the simple potential of a particle in a box (sec. 3). We treat the Poschl-Teller and Rosen-Morse potentials in sec. 4 and we comment on the results in sec. 5.

## 2. Generalization method

In SQM[9,10] we have two supersymmetric charges $Q$ and $Q^+$; they satisfy the anticommutation relations

$$\{Q, Q^+\} = H_{ss'} \;;\; (Q,Q) = 0 \;,\; (Q^+, Q^+) = 0 \tag{1}$$

A simple realization of this algebra is

$$Q = \begin{pmatrix} 0 & 0 \\ d^- & 0 \end{pmatrix} \quad \text{and} \quad Q^+ = \begin{pmatrix} 0 & d^+ \\ 0 & 0 \end{pmatrix} \tag{2}$$

and we have

$$H_{ss} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} = \begin{pmatrix} d^+d^- & 0 \\ 0 & d^-d^+ \end{pmatrix} \tag{3}$$

$H_-$ is called the supersymmetric partner of $H_+$; they have the same spectrum except for the zero-energy ground state which belongs to $H_+$ only. We note that

$$Q \begin{bmatrix} \psi_+ \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ d^-\psi_+ \end{bmatrix} \quad \text{and} \quad Q^+ \begin{bmatrix} 0 \\ \psi_- \end{bmatrix} = \begin{bmatrix} d^+\psi_- \\ 0 \end{bmatrix} \tag{4}$$

i.e., Q and $Q^+$ induce transformations between the "bosonic" sector $(\psi_+)$ and the "fermionic" sector $(\psi_-)$. Then, the $H_-$ eigenfunctions can be written in terms of $H_+$ eigenfunctions $(\psi_- \propto d^-\psi_+)$. The reciprocal is also true, i.e. $\psi_+ \propto d^+\psi_-$ are the eigenfunctions of $H_+$ with the exception of the ground-state. With operators $d^\pm$ writtem in the usual form

$$d^\pm = \mp \frac{d}{dx} + \frac{dW}{dx}(x) \tag{5}$$

the supersymmetric Hamiltonian is written as

$$H_{ss} = -\frac{d^2}{dx^2} + \left[\frac{dW}{dx}(x)\right]^2 + \sigma_3 \frac{d^2}{d^2x} W(x) \tag{6}$$

where $\sigma_3$ is the Pauli matrix, $W(x)$ is called superpotential and is associated with the $H+$ ground state eigenfunctions $\psi_{+,0}$

$$W(x) = -\ln \psi_{+,0}(x) \tag{7}$$

We can construct the new potentials from a generalization of the $d^{\pm}$ operators

$$D^{*} = \mp \frac{d}{dx} + F(x) \tag{8}$$

The function $F(x)$ is determined when we impose that

$$H_- = D^- D^+ \tag{9}$$

and we obtain the Ricati equation

$$F^2(x) + \frac{d}{dx} F(x) = \left[\frac{d}{dx} W(x)\right]^2 + \frac{d^2}{dx^2} W(x) \tag{10}$$

The commutator of the new operators is

$$[D^-, D^+] = 2\frac{d}{dx} F(x) \tag{11}$$

that defines a new Hamiltonian

$$\mathcal{H}_+ = D^+ D^- = D^- D^+ - [D^-, D^+] = D^- D^+ - 2\frac{d}{dx} F(x) \tag{12}$$

$\mathcal{H}_+$ gives a new potential which is different from the $H+$ and $H_-$ potentials. However, from the supersymmetric algebra we know that the $\mathcal{H}_+$ spectrum is the same as that of H– and the $\mathcal{H}_+$ eigenfunctions $(\Psi_+)$ are

$$\Psi_+ = D^+ \psi_- = D^+ d^- \psi_+ \tag{13}$$

This map is not complete, because it excludes the $\mathcal{H}_+$ ground state. It is obtained by the equation

$$D\Psi_{+,0} = 0 \tag{14}$$

## 3. Particle in the box

The Hamiltonian of the one-dimensional **particle** in the **box**[1] **is**

$$H_{+} = -\frac{d^2}{dx^2} - 1 \quad ; \quad -\frac{1}{2}\pi \leq x \leq \frac{1}{2}\pi \tag{15}$$

H+ **has** eigenfunctions and eigenvalues given by

$$\psi_{+,n}(x) = \begin{cases} \sqrt{\frac{2}{\pi}} & \sin(nx) \qquad n \quad \text{even} \\ \sqrt{\frac{2}{\pi}} & \cos(nx) \qquad n \quad \text{odd} \end{cases} \tag{16}$$

$$E_n = n^2 - 1 \qquad n = 1, 2, 3 \tag{17}$$

The constant term in (15) **only** displace the spectrum. It sets the eigenvalue of the ground state to zero, $E_1 = 0$.

We note that **eq.(15)** can be factorized, $H_+ = d^+ d^-$, by

$$d^{\pm} = \mp\frac{d}{dx} + \frac{d}{dx}W(x) = \mp\frac{d}{dx} - \frac{d}{dx}\ln\psi_{+,1}(x) = \mp\frac{d}{dx} + \tan x \tag{18}$$

Thus, the **supersymmetric** partner of H+ is

$$H_{-} = d^- d^+ = -\frac{d^2}{dx^2} + \frac{1 + \sin^2 x}{\cos^2 x} \tag{19}$$

Defining new operators $D^{\pm}$, as in eqs. (8), we obtain $F(x)$ given by

$$F(x) = \tan x + \frac{4\cos^2 x}{\sin 2x + 2x + 4\Gamma} \equiv \tan x + \phi(x) \tag{20}$$

Thus, the new Hamiltonian **is**

$$\mathcal{H}_{+} = D^+ D^- = D^- D^+ - [D^-, D^+] = -\frac{d^2}{dx^2} - 1 - \frac{32\cos x[(x + 2\Gamma)\sin x + \cos x]}{(+\sin 2x + 2x + 4\Gamma)^2} \tag{21}$$

which corresponds to the generalized potential

$$\mathcal{V}_+(x) = -1 - \frac{32\cos x[(x+2\Gamma)\sin x + \cos x]}{(\sin 2x + 2x + 4\Gamma^2)} \tag{22}$$

The constant $\Gamma$ is arbitrary and is chosen to avoid singularitues; here we choose: $\Gamma < -\frac{\pi}{4}$ or $\Gamma > \frac{\pi}{4}$. The spectrum of this Hamiltonian is the same of the particle in the box *(17)*. Its eigenfunctions are

$$\Psi_{+,n}(x) = D^+\psi_{-,n}(x) = D^+d^-\psi_{+,n}(x) \tag{23}$$

with $\psi_{+,n}$ given by *(16)* and the ground state $(\tilde{\Psi}_+)$ is evaluated by using *(14)*

$$\tilde{\Psi}_+ \alpha \cos^{-1} x \mp \left[\int_{-\frac{\pi}{2}}^{x} \phi(\bar{x})d\bar{x}\right] \tag{24}$$

## 4. Other potentials

Using the Pöschl-Teller potential[11] we write the Hamiltonian

$$H_+^{(\mathrm{PT})} = -\frac{d^2}{dz^2} + \frac{k(k-1)}{\sin^2 z} + \frac{\lambda(\lambda-1)}{\cos^2 z} - (k-\lambda)^2 \quad , \quad z = \alpha x \tag{25}$$

where $\alpha, k$ and $\lambda$ are constants. The eigenfunctions and eigenvalues are[12]

$$\psi_{+,n}^{\mathrm{PT}} = N(\alpha,k,\lambda,n)(\sin z)^k(\cos z)^\lambda P_n^{(k-\frac{1}{2},\lambda-\frac{1}{2})}(1-2\sin^2 z) \tag{26}$$

$$E_n = (k+\lambda+2n)^2 - (k+\lambda)^2 \tag{27}$$

where

$$P_n^{(k-\frac{1}{2},\lambda-\frac{1}{2})}(1-2\sin^2 z)$$

are the Jacobi polynomials and $N(\alpha, k, \lambda, n)$ is the normalization constant. The factor $[-(k+\lambda)^2]$ in *(25)* sets the ground state eigenvalue to zero.

The Hamiltonian *(25)* is factorized by

$$d_{\mathrm{PT}}^{\mp} = \pm\frac{d}{dz} + \frac{d}{dz}W_{\mathrm{PT}}(z) = \mp\frac{d}{dz} - k\cot z + \lambda\tan z \tag{28}$$

and the supersymmetric partner is

$$
\begin{aligned}
H_{-}^{(\mathrm{PT})} &= d_{\mathrm{PT}}^{-} d_{\mathrm{PT}}^{+} = d_{\mathrm{PT}}^{+} d_{\mathrm{PT}}^{-} - [d_{\mathrm{PT}}^{-}, d_{\mathrm{PT}}^{+}] = \\
&= -\frac{d^2}{dz^2} + \frac{k(k+1)}{\sin^2 z} - \frac{\lambda(\lambda+1)}{\cos^2 z} - (\lambda+k)^2
\end{aligned}
\tag{29}
$$

The generalization of the operators (28), as indicated in (8), leaves us with

$$
\begin{aligned}
F_{\mathrm{PT}}(z) &= -k \,\mathrm{cotg} z + \mathrm{Xtg}\, z + \frac{[(\sin z)^{2k} + (\cos z)^{2\lambda}]^{-1}}{\Gamma_{\mathrm{PT}} + \int_0^z [(\sin \bar{z})^{2k} + (\cos \bar{z})^{2\lambda}]^{-1} d\bar{z}} \\
&\equiv -k \cot z + \lambda \tan + \phi_{\mathrm{PT}}(z)
\end{aligned}
\tag{30}
$$

We choose $\Gamma_{\mathrm{PT}} > 0$ to avoid singularities . Thus, the new Hamiltonian is

$$
\begin{aligned}
\mathcal{H}_{\mathrm{PT}} &= D_{\mathrm{PT}}^{+} D_{\mathrm{PT}}^{-} = D_{\mathrm{PT}}^{-} D_{\mathrm{PT}}^{+} - [D_{\mathrm{PT}}^{-}, D_{\mathrm{PT}}^{+}] = \\
&= -\frac{d^2}{dz^2} + \lambda(\lambda-1)\frac{1}{\cos^2 z} + \frac{k(k-1)}{\mathrm{sen}^2 z} - (\lambda+k)^2 - 2\frac{d}{dz}\phi_{\mathrm{PT}}(z)
\end{aligned}
\tag{31}
$$

and the potential is

$$
\mathcal{V}_{\mathrm{PT}}(z) = \frac{\mathrm{X(A} + 1)}{\cos^2 z} + \frac{k(k+1)}{\sin^2 z} + (\lambda+k)^2 - 2\frac{d}{dz}\phi_{\mathrm{PT}}(z) \tag{32}
$$

This Hamiltonian has the spectrum given by (27) and its eigenfunctions are

$$
\Psi_n^{\mathrm{PT}}(z) = D_{\mathrm{PT}}^{+} d_{\mathrm{PT}}^{-} \Psi_n^{\mathrm{PT}} \tag{33}
$$

and the ground state $(\tilde{\Psi}_{\mathrm{PT}}(z))$ is

$$
\tilde{\Psi}_{\mathrm{PT}}(z) \alpha (\sin z)^{-k} (\cos z)^{-\lambda} \exp\left\{ \int_0^z \phi_{\mathrm{PT}}(\bar{z}) d\bar{z} \right\} \tag{34}
$$

The other potential that we treat is the Rosen-Morse one[13]. It was recently studied by Nieto[12] and Aragão de Carvalho[14]. Its Hamiltonian be written in the form

$$
H_{+}^{\mathrm{RM}} = k^2 \varphi_0^2 \left\{ -\frac{d^2}{d^2 z} + \beta \tanh z - \gamma_{+} \mathrm{sech}^2 z \right\} + \mu^2 + \varphi_0 \tag{35}
$$

where $z = k\varphi_0 x$; $\beta = 2\mu/k^2\varphi_0$; $\gamma_+ = (1 + k)/k^2$; $k, \varphi_0$ and $\mu$ are constants. The eigenvalues and the eigenfunctions of this Hamiltonian are

$$\psi_{+,n}^{(RM)} = N(k,\varphi_0,\mu;n)e^{az}\cosh^{-b}(z)F(-n,(4\gamma+1)^{1/2} \\ -n; a+b+1; 1/2[1+tghz]) \tag{36}$$

$$E_n = -K^2\varphi_0^2(a^2+b^2)+\mu^2+\varphi_0^2 \ ; \ a = \frac{-\beta}{(4\gamma_+ + 1)^{1/2} - 2n - 1}$$

and

$$b = \frac{1}{2}[(4\gamma_+ + 1)^{1/2} - 2n - 1] \tag{37}$$

We can factorize the Hamiltonian *(35)* by the operators

$$d_{RN}^{\pm} = \mp k\varphi_0 \frac{d}{dz} \pm \varphi_0 \tanh z + \mu \tag{38}$$

that satisfy the commutation relation

$$[d_{RM}^-, d_{RM}^+] = 2k\varphi_0^2 \mathrm{sech}^2 z \tag{39}$$

The supersymmetric partner of *(35)* is

$$H_-^{RM} = d_{RM}^+ d_{RM}^- == k^2\varphi_0^2\left\{-\frac{d^2}{dz^2} + \beta\tanh z - \gamma_- \mathrm{sech}^2 z\right\} + \mu^2 + \varphi_0^2 \tag{40}$$

where

$$\gamma_- = \frac{1-k}{k^2}$$

From the generalized operators (8) we obtain

$$F_{RM}(z) = \varphi_0 \tanh z + \mu + \frac{e^{2\mu}(\cosh z)^{-2\varphi_0}}{\Gamma_{RM} + \int_0^z e^{-2\mu \bar{z}}(\cosh \bar{z})^{-2\varphi_0} d\bar{z}} \equiv \\ \equiv \varphi_0 \tanh z + \mu + \phi_{RM}(z) \tag{41}$$

To avoid singularities we choose $\Gamma_{RM} > 0$. Then, the new Hamiltonian is

$$\mathcal{H}_{\text{RM}} = D^{+}_{\text{RM}} D^{-}_{\text{RM}} = D^{-}_{\text{RM}} D^{+}_{\text{RM}} [D^{-}_{\text{RM}}, D^{+}_{\text{RM}}] =$$
$$= k^2 \varphi_0^2 \left\{ -\frac{d^2}{dz^2} + \beta \tanh z - \gamma_{+} \operatorname{sech} z - \frac{2}{k\varphi_0} - \frac{d}{dz} \phi_{\text{RM}}(z) \right\} + \mu^2 + \varphi_0^2 \tag{42}$$

whose potential is

$$\mathcal{V}_{\text{RM}} = 2\varphi_0 \mu \tanh z - \varphi_0 (k+1) \operatorname{sech}^2 z - 2k\varphi_0 \frac{d}{dz} \phi_{\text{RM}}(z) \tag{43}$$

We note that the new Hamiltonian (42) has the spectrum given by (37) and tis eigenfunctions are given in terms of the functions $\psi^{(\text{RM})}_{+,n}$ (36):

$$\Psi^{(\text{RM})}_n(z) = D^{+}_{\text{RM}} d^{-}_{\text{RM}} \psi^{(\text{RM})}_{+,n} \tag{44}$$

with ground state

$$\tilde{\Psi}_{\text{RM}}(z) \alpha (\cosh z)^{1/k} e^{\mu z / k\varphi_0} \exp \left\{ \int_0^z \frac{\phi(\bar{z})}{k\varphi_0} d\bar{z} \right\} \tag{45}$$

## 5. Conclusion

From the potentials studied (particle in the box, Pöschl-Teller and Rosen-Morse) we obtained new potentials (eq. (22), (32) and (43)), which are different from the original ones, but whose spectra and eigenfunctions are known. The relation between the old system and the new one is established through the SQM.

As the spectrum of one potential is the same as that of its generalized version, some papers have appeared trying to distinguish these systems through the scattering produced by them. Cooper et al[15] and Kare and Sukhatme[16] have worked in this direction but they use a generalization method different from the one we used here.

Nieto[4] explored the link between the generalization of the potential from supersymmetry and from the inverse scattering method. Using this result, the potential obtained in (22) should be the same (up to integration constants) as the potential found by Abraham and Moses[1] for the particle in the box. Unfortunately, these potentials are different. However, we can see that in ref. 1 eq. (44) is not a solution of eq. (10) and also that eq. (45) is not derived from eq. (44); these mistakes justify the difference between the results.

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### Resumo

Usando a mecânica quântica supersimétrica vamos generalizar os potenciais: da partícula em uma caixa, Pöchl-Teller e Rosen-Morse. Calculamos os novos potenciais e indicamos suas respectivas autofunções e espectro.