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Notes on flux quantization and on the quantized Hall effect

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Abstract The paper rediscusses some points of two recently proposed theories'12, with the purpose of improving them. We argue that the quantum flux method of Ref.(l) is the natural manner of introducing the vector potential and the electric field to treat few-electron orbitals at low energy, whenever dynamics is dominated by the kinetic Hamiltonian. We discuss the wave function interpenetration in neighbouring orbitals, and the electrostatic repulsion among carriers of the same orbital, in the quantized Hall effect theory[2]. We also comment on the role of irnpurities, which is particularly important in the case of the quantized Hall effect in heterostructures.

In a recent paper, I have proposed a canonical method of flux quantization in quasi one-dimensional conductors[1]. The method was then used as one of the ingredients of a theory of the quantized Hall effect, formulated with M. simões2.

Here I add a few comments to supplement that method, and rediscuss some of the theory's points, in order to improve them.

1 - Let one take an ensemble of similar one-dimensional orbitals of length L, arranged in series along the x-direction, as it is shown in Figure **1,** supposing that there is one electron in every orbital. Fields are defined assuming periodic boundary conditions at the walls between next-neighbour orbitals.

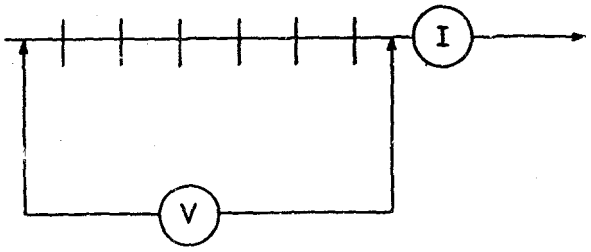

Fig. 1 - Ensemble of quasi one-dimensional orbitals arranged in series.

For a given orbital one introduces the following dynamical variables:

- The first mode of the longitudinal vector potential: A,.
- Its canonical pair is E_0, which I call the pure electromagnetic electrid field. E_0 is such that $[E_0, A_0] = i/Vol$. Vol is the volume ascribed to the considered orbital. S Being the orbital cros-section, there is âlso the associated (pure electromagnetic) electric flux $\phi_{E_0} = SE_0$, varying in the interval $\left(-\frac{e}{2}, +\frac{e}{2}\right)$ [1].
- The electron position x is in the interval $\left(-\frac{L}{2}, +\frac{L}{2}\right)$, and its generalized momentum is $p_g = -i\partial_x$.

The polarization electric field E_p, and the polarization flux ϕ_p are proportional to the electron position x,

$$E_p = -\frac{ex}{SL} \; ; \quad \text{and} \quad \phi_p = -\frac{ex}{\mathrm{L}} \tag{1}$$

The next step is to introduce two hybrid variables, namely, the electron physical momentum

$$p = p_g - e\, A_0 \; , \tag{2}$$

and the physical electric field E, together with the corresponding electric flux ϕ_E :

$$E = E_0 - \frac{\mathrm{ex}}{SL} \; ; \quad \text{and} \quad \phi_E = \mathrm{SE}, - \frac{\mathrm{ex}}{\mathrm{L}} \; , \tag{3}$$

observing that ϕ_E varies in the interval $(-e,+e)$.

E and p form a conjugate pair, in the sense that $[E,p] = i(2e/Vol)$.

In classical electromagnetism of material media, the trne electric flux is definined by the linear combination of the auxiliary field D, with the polarization electric field. So, the definition of the physical electric field E as a linear combination of a pure electromagnetic variable, E_0, with an electronic variable, E_p, is in remote analogy with that standard procedure of classical electromagnetism.

p and E are related to the two main observables of the ensemble of orbitals. The momentum expectation value is proportional to the current flowing through the ensemble, whereas the average electric field is proportional to the voltage between the system's ends.

The few lowest modes excited in low energy regime are not sensitive to the specification of the electron position. One cannot distinguish whether the electric flux is pure electromagnetic, or is due to the electronic polarization. This is why, in that regime, the relavant electric variable in orbital dynamics, should be the total electric field, denoted physical.

Whenever the electron kinetic Hamiltonian $p^2/2M$ can be taken as the leading part of the Hamiltonian in orbital dynamics, the conjugate variables, E and p, shall be the most relevant variables in that system. And this observation is to a large extent justified in terms of traditional procedures.

The mechanism of dissipation is discussed in reference 2.

2 - The problem concerning the appropriate manner of introducing the vector potential, to treat specific effects in specific materials, is not new.

Motivated by a question of Buckingham[3], about gauge invariance in BCS theory[4], Anderson[5] points out difficulties to establish basic differences among material~through the yroperties of the electromagnetic field inside them.

The method of Ref. 1, rediscussed above, is a proposal to solve that problem, in the case of low temperature systems, divided into few-electron orbitals, and if dynamics is dominated by the electron kinetic energy.

3 - The elementary orbital of the "electron-gas", where the quantized Hall effect occurs, is an example of a system with kinetic energy dominance; so that, the conjugate variables physical momentum/physical electric field are its main variables.

In low current regime, and if the system is in the middle of a plateau, the orbital state is almost a pure eigenstate of the physical momentum. And this is why the average elecric field drops to very low values, in that region.

This result explains the fall in longitudinal voltage, which is observed, whenever the Hall voltage is in a well-defined plateau.

4 - Wave function interpenetration among neighbouring orbitals

Electron wave-functions in the m^{th} orbital, are given by

$$e^{ikmr} f_n(y - y_m) \quad , \quad \text{where} \quad k = \frac{2\pi m}{eBL} \tag{4}$$

x is the variable along the direction of current flow, and L is the orbital's length along this direction. y is the transverse variable in the electron gas plane.

In formula (4), y_m is the central line of the m^{th} orbital, and $f_n(y - y_m)$ is the n^{th} eigensolution of the harrnonic oscilator of frequency $\omega_c = eB/M$.

Given an electron in the k^{th} level of that harmonic oscilator, the root mean square deviation of its position from the orbital central line is $\Delta y_0 = \sqrt{(k - 1/2)/eB}$.

Since the transverse width of a square orbital is $\sqrt{2\pi/eB}$, one notes that the orbital charge distribution penetrates the space of neighbouring orbitals. And the interpenetration shall be the more intense, the larger the number N of carriers per orbital.

The electrons dilute amidst the charge distribution of the other orbitals. However, since those N electrons occupy levels at the same orbital, they remain correlated.

The situation is in a sense similar to that of superconductivity theory[4,6], where it is supposed that many other electron also occupy the space filled by the two electrons of a Cooper pair. And, in spite of that, the two electrons are correlated.

5 - The Electrostatic repulsion in the orbital

One can verify that in the ground state the orbital charge distribution is nearly gaussian. The corresponding root mean square deviation turns out to be $\Delta y_0 = \sqrt{(N-1)/2eB}$.

This result allows one to estimate the electrostatic energy of the N-electron orbital. Without the logarithmic corrections, it is given by

$$\sim \frac{e^2}{16\pi}\frac{N^2}{\Delta y_0} \approx \frac{1}{4}\frac{e^2}{4\pi}\sqrt{eB}N^{3/2} \ . \tag{5}$$

Then, the orbital energy, as a function of the number of electrons, will be

$$\frac{1}{2}\frac{eB}{M}N^2 + \frac{1}{4}\frac{e^2}{4\pi}\sqrt{eB}N^{3/2} - e\,\tilde{V}_g N \ . \tag{6}$$

The first term in Eq.(6) comes from the packing of the N electrons, under the action of a magnetic field[2]. The last term refers to the orbital effective gate voltage[2].

As a requirement of the external conditions on the device, the average orbital electron nurnber N is given by $kV_g/2\alpha dB$, where a is the fine structure constant, d is the insulator thickness, and k its dielectric constant[2]. V_g is the gate voltage on the device.

Then minimizing expression with (6) with respect to the number of electrons, and taking N = N, one gets the effective gate voltage upon the orbital, $\tilde{V}_g$:

$$e\tilde{V}_g = \alpha^2 M\left(\alpha + \frac{3}{8}\sqrt{\rho}\right) \ , \tag{7a}$$

where

$$\rho = \frac{keV_g}{2\alpha^3 dM^2} \tag{7b}$$

The relative importance of the electrostatic energy, in the energy formula (6), is given by the parameter ρ. If $8\sqrt{\rho/3} >> 1$, then the electrostatic energy can be discarded. However, in the opposite case, $8\sqrt{\rho/3} << 1$, it is important.

In a device, where the insulator thickness is near 1.0 μm, and if the electron effective mass is 50 times smaller than the electron mass, then, the parameter p will be close to one, for a gate voltage of $\sim$ 20 Volt.

In the calculations of Ref. 2, the electrostatic repulsion was not included. If $\rho \approx 1$, one effect of the electrostatic energy is a slight shift in the position of the transition points between plateaus, relatively to the results of Ref. 2. The effect is small, because the transition point position, is determined mostly by the external conditions. Electrostatic repulsion also changes the energy levels of H_{charge} [2].

6 - The effect of impurities over the number of carriers

There are systems, like the heterostsuctures, where the number of carriers is determined by the impurity concentration. In this case, a necessary condition for observation of the effect is that the excitation energy of magnetic origin be much larger than the carrier's binding energy in the impurity center.

Let η be the carriers concentration in the doped layer of a heterostructure, and r the layer's thickness. The average area per carrier shall be given by $A \approx 1/r\eta$.

In order to observe the quantized Hall effect it is also necessary that the order of magnitude of the magnetic field be such that $B \approx 2\pi/eA \approx 2\pi r\eta/e$.

Besides, given the excitation energy of $eB/2M$, one concludes that the temperature where the effect starts being seen should be $T \approx eB/10M \approx \pi r\eta/5M$.

This means that variations of impurity concentration produce considerable changes in the characteristics of the effect, since they modify the concentration of carriers.

Let H_m be the Hamiltonian that describes the interaction of the electron with the magnetic field; and let H_{η} be a second term of the Hamiltonian accounting for the interaction between the carrier and the impurity center. A necessary condition for the occurrence of well defined plateaus of the quantized Hall effect in a heterostructure experiment, is the dominance of H_m over H_{ci}.

One can infer the relative importance between the two Hamiltonian terms, by comparing the characteristic magnetic energy eB/M with the impurity excitation energy ε_i in the absence of magnetic field.

As an example, I discuss the experiment of Paalanen, Tsui and Gossard[7]. They measured the Hall conductance as well as the longitudinal conductance, in a device[8] with a 0.07 μm thick Si-doped $Al_xGa_{1-x}As$ layer, at 50 mK, and with the

magnetic field varying up to 80 kG. For this field intensity, and taking the electron effective mass in that material to be $m_e/15$[8], one gets the value of 140°K for the ciclotron frequency.

Dingle, Stõrmer, Gossard and Wiegman[g] point out that in bulk GaAs, the binding energy of isolated Si donors is ~ 70°K. But accordingly with Lang, Jaros and Logan[10], in Si-doped $Al_xGa_{1-x}As$, the deep (Si) donor merges with conduction band at $x \leq 0.35$.

This indicates that the ionization energy of a Si impurity in AlGaAs should be much smaller than the ciclotron frequency, thus suggesting the H, dominance over H_{ci}, in the description of the experiment reported in ref. 7.

Then, concerning the influence of impurities on quantum transport in two-dimensional electron systems under magnetic field[11,12], I note that, in a first step, the concentration of impurities will affect the concentration of carriers, and the number of carriers per orbital. And this should be particularly true in the case of a. heterostructure.

In a second instance, a neighbouring impurity center will induce fliictuations of electron number in the orbita1[2], leading also to a certain modification in the electron wave function.

7 - On the length of the orbitals

In the system's ground state every orbitals is filled with electrons up to the level N – 1. Because of Pauli's principle, an electron in the level k of an orbital is not allowed to move to the same level in another orbital. So there is a gap in that system: $A \approx eB/M$.

The orbital has been introduced in the theory, in order that the electronic variable could be coupled to the flux variables. However, from the remark above, if the electron wave number passes from zero to $2\pi/L$, then the corresponding energy variation must equal the system's natural gap: $2\pi^2/ML^2 \approx A$.

These relations lead to an alternative estimate of the orbitals length L, which roughly agree with the value of L introduced in Ref. 2: $L' \approx 1.73L$.

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Resumo

O trabalho rediscute alguns pontos de duas teorias propostas recentemente[1,2], com o propósito de aperfeiçoá-los. Argumentamos que o método de quantização de fluxo da Ref. 1 é a maneira natural de introduzir o potencial vetor e o campo elétrico, para tratar orbitais de poucos elétrons a baixas energias, quando a dinâmica for dominada pela Hamiltoniana cinética. Discutimos a interpenetração das funções de onda de orbitais vizinhos, e a repulsão eletrostática entre os portadores de um mesmo orbital, na teoria do efeito Hall quantizado[2]. Comentamos também sobre o papel das impurezas, que é particularmente importante no caso do efeito Hall quantizado nas heteroestruturas.