Revista Brasileira de Física, Vol. 20, nº 1, 1990

Random spin-1 Ising model on a Bethe lattice

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Received October 25, 1989

Abstract We formulate a random-exchange spin-1 Ising model on a Cayley tree of infinite coordination as a non-linear discrete mapping problem. The borders of stability of the paramagnetic and spin-glass phases are obtained from the fixed points of the mapping. Except for the location of a first-order boundary, obtained from a free-energy functional on the Bethe lattice, our results agree with replica-symmetric calculations for a spin-1 Sherrington-Kirkpatrick Ising glass. We also consider two replicas of the original system to analyze the stability of the fixed points under replica symmetry-breaking.

A replica-symmetric solution of an extension of the Sherrington and Kirkpatrick model for infinite-range Ising spin glasses, with spin 1 and the inclusion of a crystalline anisotropy, has been shown to display continuous and first-order phase transitions, with a tricritical point[1,2]. To make contact with these calculations, we formulate a gaussian random exchange spin-1 Ising model, on a Cayley tree of infinite coordination, as a discrete non-linear mapping problem. The fixed points of the mapping give the equations of state on the Bethe lattice, which turn out to be equivalent to the replica-symmetric solutions of Sherrington and collaborators[1,2]. The regions of stability of the paramagnetic and spin-glass phases are given by the criteria of stability of the corresponding fixed points. The thermodynamic first-order boundary, however, comes from the consideration of a suitable free-energy functional on the Bethe lattice.

According to an idea of Thouless[3,4], we consider two replicas of the original system to investigate the possibility of replica symmetry-breaking. As in the case of the spin $-\frac{1}{2}$ Ising model[4], in zero field, we show the existence of an unstable, symmetric, and a stable, symmetry-breaking, spin-glass fixed point. We also perform numerical calculations to show that the symmetric spin-glass fixed point is unstable along the line of first-order phase transitions, in agreement with results of Lage and de Almeida[5] for the generalized SK model. Finally, according to a paper by Fedorov[6], we make some remarks about the possibility of using the hierarchical character of the tree to introduce a more general symmetry-breaking order parameter.

Consider a spin-1 Ising model, given by the hamiltonian

$$\mathcal{H} = -\sum_{(ij)} J_{ij} S_i S_j + D \sum_i S_i^2 - H \sum_i S_i \; , \tag{1}$$

where $S_i = +1, 0, -1$, for all sites i, and the first sum is over nearest-neighbor pairs of spins on the sites of a Cayley tree of ramification r. The exchange parameters J_{ij} are independent, identically distributed, random variables. The partition function can be calculated as a sum over configurations of spins belonging to successive generations of the tree. It is then quite natural to write recursion relations between an effective field, L_j, and an effective anisotropy, Δ_j (j= $1, 2, ...,$ r labels the sites of a certain generation), and the corresponding effective quantities, L_0 and Δ_0, in the next generation. Introducing the more convenient variables

$$m_j = \frac{2 \sinh \; L_j}{2 \cosh \; L_j + \exp(-\Delta_j)}, \tag{2a}$$

and

$$q_j = \frac{2 \cosh \; L_j}{2 \cosh L_j + \exp(-\Delta_j)} \; , \tag{2b}$$

which are associated with effective values of the magnetization and the quadrupolar parameter per spin, we can write the recursion relations

$$m_0 = f(x, y) = \frac{2 \sinh(\beta H + x)}{2 \cosh(\beta H + x) + \exp(\beta D - y)} \; , \tag{3a}$$

and

$$q_0 = g(x, y) = \frac{2 \cosh(\beta H + x)}{2 \cosh(\beta H + x) + \exp(\beta D - y)} \; , \tag{3b}$$

where

$$x = \sum_{j=1}^{r} \quad \text{and} \quad y = \sum_{j=1}^{r} y_j \ ,$$

with

$$x_j = \frac{1}{2} \ln \frac{(\cosh t_j - 1)q_j + m_j \sinh t_j + 1}{(\cosh t_j - 1)q_j - m_j \sinh t_j + 1} \ , \tag{4a}$$

$$y_j = \frac{1}{2} \ln \left\{ [(\cosh t_j - 1)q_j + 1]^2 - m_j^2 \sinh^2 t_j \right\} \ , \tag{4b}$$

mhere $t_j \equiv \beta J_{0j}$, $\beta \equiv (k_B T)^{-1}$, and T is the absolute temperature.

To calculate the expected values of mo, q_0, and their moments, let us consider the random variables x and y, with a joint probability density, $p(x, y)$, such that

$$p(x,y) = \int \frac{dk_1}{2\pi} \int \frac{dk_2}{2\pi} \exp(-ik_1 x - ik_2 y) F(k_1, k_2) \ . \tag{5}$$

The Fourier transform, $F(k_1, k_2)$, is given by

$$F(k_1,k_2) = \int \int \exp(ik_1 x + ik_2 y) p(x,y) dx \ dy =$$

$$< \exp(ik_1 x + ik_2 y) >= \prod_{j=1}^{r} < \exp(ik_1 x_j + ik_2 y_j) > \tag{6}$$

where the last equality comes from the cycle-free structure of the Cayley tree. As the random variables are identically distributed, we have

$$F(k_1, k_2) =< \exp(ik_1 x_j + ik_2 y_j) >^r \tag{7}$$

In the limit of infinite coordination, given by r $\rightarrow \infty$, with r $< J_{0j} >= J_0$, r $< J_{0j}^2 >= J^2$, and r $< J_{0j}^n >= 0$, for $n \geq 3$, we perform a cumulant expansion to obtain

$$F(k_1, k_2) = \exp\left[i\beta J_0 m k_1 - \frac{1}{2}\beta^2 J^2 Q k_1^2 + \frac{1}{2} i\beta J^2 (q - Q) k_2 \right] \ , \tag{8}$$

where m $\equiv < m_j >$, Q $\equiv < m_j^2 >$, and q $\equiv < q_j >$. From eqs. (5) and (8), we have the probability density

$$p(x,y) = (2\pi\beta^2J^2Q^2)^{-1/2}\exp\Big[-\frac{(x-\beta J_0 m)^2}{2\beta t^2 J^2 Q}\Big]\delta\Big[y+\frac{\beta^2J^2}{2}(q-Q)\Big] \ . \qquad (9)$$

As the probability density depends on the first and second moments of m_j and only on the first moment of q_j, we have the three-dimensional mapping

$$m' \equiv < m_0 >= \int_{-\infty}^{+\infty}\frac{dz}{(2\pi)^{1/2}}\exp\Big(-\frac{z^2}{2}\Big)\frac{\sinh(\beta H+\beta J_0 m+\beta J Q^{1/2}z)}{M} \ , \qquad (10a)$$

$$Q' \equiv < m_0^2 >= \int_{-\infty}^{+\infty}\frac{dz}{(2\pi)^{1/2}}\exp\Big(-\frac{z^2}{2}\Big)\Big[\frac{\sinh(\beta H+\beta J_0 m+\beta J Q^{1/2}z)}{M}\Big]^2 \ , \qquad (10b)$$

and

$$q' \equiv < q_0 >= \int_{-\infty}^{+\infty}\frac{dz}{(2\pi)^{1/2}}\exp\Big(-\frac{z^2}{2}\Big)\frac{\cosh(\beta H+\beta J_0 m+\beta J Q^{1/2}z)}{M} \ , \qquad (10c)$$

where

$$M = \cosh(\beta H+\beta J_0 m+\beta J Q^{1/2}z)+\frac{1}{2}\exp\Big[\beta D-\frac{1}{2}\beta^2J^2(q-Q)\Big] \ . \qquad (11)$$

It should be mentioned that relations of this form, in the context of a more general spin-1 Ising model, including random biquadratic exchange interactions, have already been obtained by Thompson and collaborators[7], without, however, a detailed analysis of the fixed points. In this paper, to make contact with calculations for the generalized SK model[1,2], we restrict our considerations to the pure spin glass ($J_0 = 0$) in zero field (H = O). Under these circumstances, the magnetization vanishes, and the mapping becomes two-dimensional, in terms of the second moment, Q, and the quadrupolar term, q. The fixed points, Q' = Q = Q*, and $q' = q = q^*$, correspond to the replica-symmetric solutions of Ghatak and Sherrington[1] for the generalized version of the long-range SK Ising spin glass.

In zero field, for $J_0 = 0$, there is a trivial paramagnetic fixed point, $Q^* = 0$, and $q^* \neq 0$, coming from the equation

$$q^* = \left[1 + \frac{1}{2}\exp\left(\frac{d}{t} - \frac{q^*}{2t^2}\right)\right]^{-1} , \tag{12}$$

where we define $d \equiv D/J$, and $t \equiv (\beta J)^{-1}$. With the change of variables $u^* = 2q^* - 1$, eq. (12) can be rewritten in the familiar form

$$u^* = \tanh \frac{1}{8t^2}(u^* - 4td + 1 + 4t^2 \ln 2) , \tag{13}$$

from which we see that, for $t < \sqrt{2}/4$, there may be three distinct values of q^*. The region of stability of this paramagnetic fixed point is given by the conditions:

$$\lambda_1 \equiv \left(\frac{\partial Q'}{\partial Q}\right)_* = \left(\frac{q^*}{t}\right)^2 < 1 , \tag{14}$$

and

$$\lambda_2 \equiv \left(\frac{\partial q'}{\partial q}\right)_* = \frac{(1-q^*)q^*}{2t^2} < 1 , \tag{15}$$

Let us consider the $d-t$ phase diagram. For $t > \sqrt{2}/4$, there is always a single value of q^*. The paramagnetic fixed point is well defined, and the border of stability is given by $\lambda_1 = 1$ (as $q^* > \frac{1}{3}$, $\lambda_1 > \lambda_2$). For $\frac{1}{3} < t < \sqrt{2/4}$, with decreasing values of d, there appear three distinct values of q^*. The stable paramagnetic fixed point is associated with the smallest value of q^*, and the border of stability is again given by $\lambda_1 = 1$, with $q^* > \frac{1}{3}$. At the special temperature $t = \frac{1}{3}$, the stability border is given by $\lambda_1 = \lambda_2 = 1$, with $q^* = \frac{1}{3}$. For $t < \frac{1}{3}$, with decreasing values of d, eq.(12) still displays three distinct roots, but the smallest value of q^* is always less than $\frac{1}{3}$. In this range of temperatures, $\lambda_2 > \lambda_1$, and the stability border is given by $\lambda_2 = 1$, with the smallest root of eq.(12). From these considerations, the paramagnetic borders are given by the analytic expressions

$$d_1(t) = \frac{1}{2} + t \ \ln \frac{2(1-t)}{t} , \tag{16}$$

for $\frac{1}{3} \le t \le 1$, and

$$d_2(t) = \frac{1}{2}[1 - (1 - 8t^2)^{1/2}] - 2t \ \ln \frac{1 - (1 - 8t^2)^{1/2}}{4t} , \tag{17}$$

for $0 \leq t \leq \frac{1}{3}$, as depicted in fig. 1, in agreement with the results of Mottishaw and Sherrington[2].

At low temperatures, there is a spin-glass fixed point, $Q^* \neq 0$ and $q^* \neq 0$. For $t > \frac{1}{3}$, the stability borders of the spin-glass and the paramagnetic regions are given by eq. (16), the phase transition being of second order. For $0 < t < \frac{1}{3}$, we have performed a numerical calculation to find the dashed line depicted in fig. **1.** In this range of temperatures, there is an overlap between the regions of stability of the spin-glass and the paramagnetic fixed points. The phase transition is then discontinuous, with a tricritical point at $t_t = \frac{1}{3}$, and $d_t = \frac{1}{2} + \frac{2}{3}\ln 2$. The thermodynamic line of first-order transitions can be obtained from an expression for the free energy associated with the spin-glass model on the Bethe lattice. From an integration of the equations of state, given by the fixed points of the mapping, we have the free-energy functional

$$\mathcal{F}(Q^*,q^*) = +\frac{q^{*2}-Q^{*2}}{4t} - t\int_{-\infty}^{+\infty}\frac{dz}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}z^2\right)\ln\left\{1+ \right.$$
$$\left. + 2\exp\left(-\frac{d}{t}+\frac{q^*-Q^*}{2t^2}\right)\cosh\left(\frac{z\,Q^{*1/2}}{t}\right)\right\}, \qquad (18)$$

whih is equivalent to an expression obtained from the pair approximation developed by Katsura and collaborators[8]. Using eq. 18), we calculate the first-order line shown in fig. **2.** These results, which can also be obtained from the replica-symmetric free energy, are quantitatively differentfrom the first-order line depicted in the paper by Ghatak and Sherrington[1].

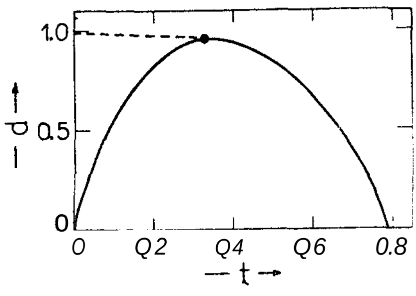

Fig. 1 - The solid line, given by eqs. (16) and (17), represents the border of stability of the pararnagnetic fixed point. For $t > 1/3$, it coincides with the border of stability of the spin-glass fixed point. For $t < 1/3$, however, the spin-glass fixed point is stable up to the dashed line.

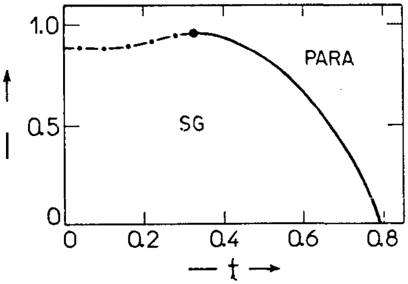

Fig. 2 - Paramagnetic and spin-glass regions in the d − t phase diagram of the pure spin-1 Ising glass in zero field. The solid line represents second-order transitions. The dot-dashed line of first-order transitions was obtained from the free-energy functional of eq. (18).

According to a previous work for the spin $-\frac{1}{2}$ model[4], let us consider two replicas of the original system. They may be represented by the old variables m_j and q_j, and by a new set of variables, $\tilde{m}_j$ and $\tilde{q}_j$. Inserting these new variables into eqs. (4a) and (4b), we obtain $\tilde{x}$, and $\tilde{y}$, from which we calculate $\tilde{m}_j = f(5, \tilde{y})$, and $\tilde{q}_j = g(\tilde{x}, \tilde{y})$. We then have a joint probability density, $p(x, y, \tilde{x}, \tilde{y})$, whose Fourier transform, in the limit of infinite coordination, is given by

$$F(k_1, k_2, \tilde{k}_1, \tilde{k}_2) = \Big\langle \exp(ik_1x + ik_2 + i\tilde{k}_1\tilde{x} + i\tilde{k}_2\tilde{y}) \Big\rangle =$$
$$= \exp\Big[i\beta J_0(k_1 + \tilde{k}_1)m + \frac{1}{2}i\beta^2 J^2(q - Q)(k_2 + \tilde{k}_2) -$$
$$- \frac{1}{2}\beta^2 J^2(Qk_1^2 + 2Sk_1\tilde{k}_1 + Q\tilde{k}_1^2)\Big], \qquad (19)$$

where $m \equiv < m_j > = < \tilde{m}_j >$, $q \equiv < q_j > = < \tilde{q}_j >$, $Q \equiv < m_3^2 > = < \tilde{m}_3^2 >$, and there is a replica symmetry-breaking variable, $S \equiv < m_j \, \tilde{m}_j >$. Restricting to the case $J_0 = 0$, we use eq. (19) to obtain $p(x, y, \tilde{x}, y)$, from which we can write the three-dimensional mapping

$$Q' = F(Q, q, Q) \qquad (20a)$$

$$q' = G(Q, q) \qquad (20b)$$

and

$$S' = F(Q, q, S) \qquad (20c)$$

where

$$G(Q, q) = \int_{-\infty}^{+\infty} \frac{dz}{(2\pi)^{1/2}} \exp\Big(-\frac{z^2}{2}\Big) \frac{\cosh\Big(\frac{Q^{1/2}z}{t}\Big)}{\cosh\Big(\frac{Q^{1/2}z}{t}\Big) + A} \quad , \qquad (21)$$

and

$$F(Q,q,S)=$$
$$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{dz_1dz_2}{2\pi}\exp\left(-\frac{z_1^2+z_2^2}{2}\right)\left[\frac{\sinh\left(\frac{\xi_+}{t}\right)}{\cosh\left(\frac{\xi_+}{t}\right)+A}\right]\left[\frac{\sinh\left(\frac{\xi_-}{t}\right)}{\cosh\left(\frac{\xi_-}{t}\right)+A}\right] \tag{22}$$

with

$$A=\frac{1}{2}\exp\left(\frac{d}{t}-\frac{q-Q}{2t^2}\right), \tag{23}$$

and

$$\xi_\pm=\left(\frac{Q+S}{2}\right)^{1/2}z_1\pm\left(\frac{Q-S}{2}\right)^{1/2}z_2\,. \tag{24}$$

From eqs. (20), $Q=S$ implies $Q'=S'$. So, to reach a fixed point $Q^*\neq S^*$, we have to consider the possibility of symmetry-breaking boundary conditions from the outset. The paramagnetic fixed point is given by $Q^*=S^*=0$. The spin-glass fixed point, however, is either replica-symmetric ($Q^*=S^*\neq 0$) or replica symmetry-breaking ($Q^*\neq 0$, $S^*=0$). The stability of the replica-symmetric fixed point is governed by the eigenvalue

$$\lambda_S\equiv\left(\frac{\partial S'}{\partial S}_*\right)=\frac{1}{t^2}\int_{-\infty}^{+\infty}\frac{dz}{2\pi^{1/2}}\exp\left(-\frac{z_1^2}{2}\right)\frac{\left[1+A\cosh\left(\frac{Q^{1/2}z}{t}\right)\right]^2}{\left[A+\cosh\left(\frac{Q^{1/2}z}{t}\right)\right]^4}, \tag{25}$$

with A given by eq. (23), for $Q=Q^*$ and $q=q^*$ given by the fixed points of eqs. (20). For $t\geq\frac{1}{3}$, $\lambda_S=1$ at the paramagnetic border, and $\lambda_S>1$ in the spin-glass region. Along the line of first-order transitions, we have verified that $\lambda_S>1$ for the spin-glass fixed point. We then come to the conclusion that the spin-glass phase cannot be represented by a single symmetric fixed point[5].

In a recent publication, Fedorov[6] has taken advantage of the hierarchical character of the Cayley tree to work with more general boundary conditions. Considering the surface of a tree of ramification r, for each cluster of r spins interacting with a certain spin of the next generation, it is possible to choose C initial values

$S_0 = Q_0 \neq 0$, and $r - \ell$ values $S_0 < Q_0 \neq 0$. It is then convenient to define a quantity $x = \ell/r$, which measures the distance between replicas ($x = 1$ corresponds to a complete coincidence of the replicas, and $x = 0$ to their minimal overlap). According to Fedorov's construction[6], the recursion relation given by eq. (20c) should be rewrittem as

$$S'(x) = F[Q, q, Qx + (1-x)S(x)] \; , \tag{26}$$

where, in the infinite coordination limit, $S(x)$ is continuous on $x \in [0,1]$. For $t > \frac{1}{3}$, in the spin-glass phase, the stable fixed point, $S^*(x)$, is a monotonic non-decreasing function of x, with $S^*(0) = 0$, and $S^*(1) = Q^* \neq 0$.

We thank helpful discussions with Mário J. de Oliveira and Carlos S.O. Yokoi.

References

1. S.K. Ghatak and D. Sherrington, J.Phys. C **10,** 3149 (1977).
2. P.J. Mottishaw and D. Sherrington, J. Phys. C 18, 5102 (1985).
3. D.J. Thouless, Phys. Rev. Lett. 56, 1082 (1986).
4. M.J. de Oliveira and S.R. Salinas, Phys. Rev. B 35, 2005 (1987).
5. E.J.S. Lage and J.R.L. de Almeida, J. Phys. C 15, L1187 (1982).
6. Ya.D. Fedorov, JETP Lett. 46, 340 (1987).
7. C.J. Thompson, S. Inawashiro, and N.E. Frankel, Progr. Theor. Phys. Suppl. 87, 155 (1986).
8. S. Katsura, Prog. Theor. Phys. Suppl. 87, 139 (1986).

Resumo

Formulamos o problema de um modelo de Ising de spin 1, com parâmetro de troca aleatória, definido numa árvore de Cayley no limite de coordenação infinita, como um mapeamento discreto não linear. Obtivemos as fronteiras de estabilidade das fases paramagnética e vidro de spin a partir dos pontos fixos do mapeamento. Exceto a localização da fronteira de primeira ordem, obtida a partir de um funcional energia livre na rede de Bethe, os nossos resultados concordam com estudos baseados na solução com simetria de réplicas de um análogo do modelo de Sherrington-Kirkpatrick para o vidro de spin 1 de Ising. Consideramos ainda duas réplicas do sistema original a fim de analisar a estabilidade dos pontos fixos face à quebra de simetria das réplicas.