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Quantum field theory in non-stationary coordinate systems and Green functions

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Abstract In this paper we study a neutral massive scalar field in a two-dimensional Milne space time. The quantization is made on hyperboles which are Lorentz invariant surfaces. The expansion for the field operator is carried on using a complete set of orthonormal modes which have definite positive and negative dilatation frequence. We have calculated the advanced and retarded Green functions and proved that the Feynman propagator diverges in the usual sense.

1. Introduction

Interesting possibilities were revealed when attempts were made to quantize the gravitational field. Although up to now these attempts have all failed, there have been other important results in quantum field theory in curved space and in curvilinear coordinate systems. One of the most important of these results was achieved by Hawking[1], who proved that a black hole radiates a thermal spectrum.

Before Hawking's results Fulling[2] showed that a uniformly accelerating observer constructs an operator algebra representation different from the representation constructed by an inertial observers.

Unruh has further demonstrated, using a detector model, that a uniformly accelerating observer in a Minkowski space-time observes a thermal spectrum, while an inertial observer measures the field in its vacuum state. We will not go over the detector problem since this subject has been widely discussed in the literature[3]. We will deal only with the formal part of the quantization of a neutral scalar field. This will be done in a two-dimensional flat space-time using a particular curvilinear coordinate system.

In 1975 Kalnins[4] proved that in a two-dimensional flat space-time there are only ten coordinate systems in which the Klein-Gordon equation has separable variables. In one of these systems, the Lorentz invariant surfaces ($x^2 =$ const) arise naturally. Fubini, Hansen and Jackíw[5] quantized a massless neutral scalar field using this type of surface. di Sessa[6], Sommerfield[7] and Rothe et al.[8] did the same with a massive neutral scalar field, but only di Sessa deals with the problem of the associated Green function. In this paper we use the same coordinate system and quantization as Sommerfield (massive neutral scalar field). The Pauli-Jordan and the advanced and retarded Green functions will be calculated and the divergence of the Feynman propagator will be demonstrated.

In section 2, after a brief exposition of the two-dimensional Milne and Rindler space-times we display the Klein-Gordon equation in the Milne system. Two sets of mode solutions are presented. In section 3 two criteria of choosing positive and negative frequency modes are discussed and the Sommerfield criterium is adopted. In section 4 we calculate the Pauli-Jordan function, the advanced and retarded Green functions and we demonstrate that the Feynman propagator diverges. The convergence and evaluation of certain integrals in the comples plane is discussed in Appendix A. The addition theorem for the cylinder functions will be generalized in Appendix B.

In this paper we use the convention $\hbar = c = k_B = 1$.

2. Massive scalar field in Milne's universe

Let us consider a two-dimensional Minkowski space-time with line element

$$ds^2 = (dy^0)^2 - (dy^1)^2 \ . \tag{2.1}$$

We shall use the following coordinate transformation

$$y^0 = \eta \sinh \xi \qquad 0 < \eta < \infty$$
$$y^1 = \eta \cosh \xi \qquad -\infty < \xi < \infty \tag{2.2}$$

In this case the line element eq. (2.1) becomes

$$ds^2 = \eta^2 d\xi^2 - d\eta^2 \tag{2.3}$$

This transformation covers only the region $y^1 > |y^0|$. The (ξ, η) coordinate system is called Rindler's coordinates. It can be shown that this system is one naturally suited to an observer with constant proper acceleration[g].

As this system does not cover the whole Minkovski space-time, we shall select the following additional coordinate transformation (see Fig. 1)

$$\begin{cases} y^0 = -\eta \sinh \xi \\ y^1 = -\eta \cosh \xi \end{cases} \qquad \text{Region II (Rindler)} \tag{2.4a}$$

$$\begin{cases} y^0 = \eta \cosh \xi \\ y^1 = \eta \sinh \xi \end{cases} \qquad \text{Region F (Milne)} \tag{2.4b}$$

$$\begin{cases} y^0 = -\eta \cosh \xi \\ y^1 = -\eta \sinh \end{cases} \qquad \text{Region P (Milne)} \tag{2.4c}$$

The four coordinate transformation eqs. (2.2), (2.4a), (2.4b) and (2.4~)together cover all Minkovski space time.

The coordinate systems that cover the region inside the light cone are a two-dimensional Milne Universe.

Using the transformation eq. (2.4b) the line element eq. (2.1) becomes

$$ds^2 = d\eta^2 - \eta^2 d\xi^2 \tag{2.5}$$

Observers who perceive the universe expanding from $y^0 = 0$ have world lines ξ=const. The surfaces η =const are hyperboles where we postulate the commutation relation between the fields.

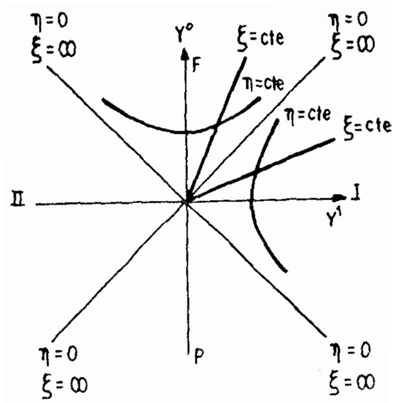

Fig. **1**

It is useful to define new variables 7, r in the region (F)

$$\xi = a\gamma \qquad a > 0 \tag{2.6a}$$

$$\eta = \frac{1}{a}e^{a\tau} \qquad -\infty < \gamma, \tau < \infty \tag{2.6b}$$

In order to quantize a neutral massive scalar field it is necessary to solve the Klein-Gordon equation in the Milne Universe.

It becomes

$$\left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\gamma^2} + e^{2a\tau}m^2\right)\Phi(\tau,\gamma) = 0 \tag{2.7}$$

The time-dependent part of the Klein-Gordon equation is a Bessel equation

$$\left(\frac{d^2}{d\eta^2} + \frac{1}{\eta}\frac{d}{d\eta} + m^2 + \frac{\lambda^2}{\eta^2}\right)\chi_\lambda(m\eta) = 0 \tag{2.8}$$

The set of solutions $\phi_\lambda \propto e^{i\lambda\xi}\chi_\lambda$ ***(mq)*** and ϕ^*_λ ***a*** $e^{-i\lambda\xi}\chi^*_\lambda$ ***(mq)*** is complete and the Φ field can be expanded in the form

$$\Phi(\eta,\xi)=\int_{-\infty}^{\infty}d\lambda\Big(a(\lambda)\phi_\lambda+a^\dagger(\lambda)\phi_\lambda^*\Big) \tag{2.9}$$

We shall take the scalar product

$$(\phi_1,\phi_2)=-i\int_\Sigma\sqrt{-g_\Sigma}\,d\Sigma^\mu\phi_1(x)\overleftrightarrow{\partial_\mu}\phi_1^*(x) \tag{2.10}$$

where $d\Sigma^\mu=\eta^\mu d\Sigma$, with η^μ a future-directed unit vector orthogonal to the space-like hypersurface C, and $d\Sigma^\mu$ is the volume element in C.

The Klein-Gordon eq.(2.7) posesses two distinct complete sets of orthonormal mode solutions (orthonormal under the scalar product (2.10)).

$\{u_\lambda,u_\lambda^*\}$ and $\{v_\lambda,v_\lambda^*\}$ namely

$$u_\lambda(\eta,\xi)=-\frac{i}{2\sqrt{2}}e^{\pi\lambda/2}e^{i\lambda\xi}H_{i\lambda}^{(2)}(m\eta) \tag{2.11a}$$

$$u_\lambda^*(\eta,\xi)=\frac{i}{2\sqrt{2}}e^{\pi\lambda/2}e^{-i\lambda\xi}H_{-i\lambda}^{(1)}(m\eta) \tag{2.11b}$$

and

$$v_\lambda(\eta,\xi)=-\frac{i}{2}(\sinh\pi|\lambda|)^{-1/2}e^{i\lambda\xi}J_{-i|\lambda|}(m\eta) \tag{2.12a}$$

$$v_\lambda^*(\eta,\xi)=\frac{i}{2}(\sinh\pi|\lambda|)^{-1/2}e^{-i\lambda\xi}J_{i|\lambda|}(m\eta) \tag{2.12b}$$

$H_{i\lambda}^{(1)}$ and $H_{i\lambda}^{(2)}$ are the Bessel functions of the third kind or Hankel functions of imaginary order. $J_{i\lambda}$ is the Bessel function of first kind with imaginary order[10].

Positive and negative frequency modes must be distinguished in the quantization in order to identify $a(\lambda)$ and $a^\dagger(\lambda)$ as annihilation and creation operators of quanta of the field. If the space-time has a stationary geometry there exists a time-like Killing vector K. This vector generates a one parameter Lie group of isometries, and the orthonormal modes satisfy

$$L_K u=-i\omega u \tag{2.13}$$

where L_K is the Lie derivative with respect to K. In this case there is a natural way of defining positive and negative frequency modes.

The vacuum associated with these modes is called trivial or Killing vacuum''. However, the line element eq. (2.5) is time (η) dependent and the curves ξ=const are not integral curves of a time-like Killing vector K. There is no simple way of definind positive and negative frequency modes. Different solutions for this problem were presented by Sommerfield and di Sessa. For each way of defining positive and negative modes we have different quantizations.

3. The di Sessa and Sommerfield quantization

(a) di Sessa Criterion

This author claims that the concept of positive frequency requires for its definition a complexification of the real Lorentzian manifold. In this situation the positive frequency modes are those which vanish when $t \to -i\infty$. It is easy to see that

$$\lim_{\eta \to -i\infty} H^{(2)}_{i\lambda}(m\eta) = 0 \tag{3.1}$$

Then eqs. (2.11a) and (2.11b) are positive and negative frequency modes respectively.

$J_{i\lambda}$ and $J_{-i\lambda}$ do not vanish when $\eta \to -i\infty$, so (2.12a) and (2.12b) do not have definite positive or negative frequency in the di Sessa criterion.

The vacuum associated with eqs. (2.11a) and (2.11b) will be represented by $|0>$.

(b) Sommerfield Criterion

The operator

$$D = \frac{1}{2}\int_{-\infty}^{\infty} d\gamma\Big(\Big(\frac{\partial}{\partial\tau}\,\Phi\Big)^2 + \Big(\frac{\partial}{\partial\gamma}\,\Phi\Big)^2 + e^{2a\tau}m^2\Phi^2\Big) \tag{3.2}$$

generates translation in r, and is called dilatation generator. It satisfies the Heisenberg equation

$$\Big[\Phi(\tau,\gamma), D\Big] = i\,\frac{\partial}{\partial\tau}\,\Phi\ . \tag{3.3}$$

Sommerfield used this fact and the additional fact that in the light cone ($\eta \to 0$ or $\tau \to -\infty$) we have

$$\lim_{\substack{\eta\to 0\\ \text{or } \tau\to -\infty}} J_{i\lambda}(m\eta) \propto \frac{e^{ia\lambda\tau}}{2^{i\lambda}\Gamma(1+i\lambda)} \tag{3.4}$$

to choose eqs. (2.12a) and (2.12b) as positive and negative dilatation frequency modes respectively.

Using eqs.(2.9), (2.12a), (2.12b), (3.2) and (3.4) we obtain

$$\lim_{\tau\to-\infty} D(\tau) \propto \frac{1}{2}\int_{-\infty}^{\infty} d\lambda|\lambda|(a(\lambda)a^{\dagger}(\lambda) + a^{\dagger}(\lambda)a(\lambda)) \tag{3.5}$$

So the Fock space can be constructed and the associated vacuum will be represented by $|\bar{0}>$. The problem is to find the Green functions associated with the modes (2.12a), (2.12b).

The Feynman propagator of the modes (2.11a) and (2.11b) has already been calculated[6].

It will be shown that the Feynman propagator associated with the modes (2.12a) and (2.12b) diverges. The other propagators, the retarded and advanced Green functions G_R and G_A are defined respectively by

$$G_R(x,x') = -\Theta(x^0 - x^{0'})G(x,x') \tag{3.6}$$

$$G_A(x,x') = \Theta(x^{0'} - x^0)G(x,x') \tag{3.7}$$

where $G(x,x')$ is known as the Pauli-Jordan function, which is defined as the expected value of the commutator of the field in the vacuum state.

$$iG(x,x') =< \bar{O}|[\Phi(x),\Phi(x')]|\bar{O}> \tag{3.8}$$

The Feynman propagator G_F is defined as the time ordered product of fields

$$iG_F(x,x') =< \bar{O}|T\Phi(x)\Phi(x')|\bar{O}>= \\ \Theta(z^0 - x^{0'})G^+(z,x') + \Theta(x^{0'} - z^0)G^-(x,x') \tag{3.9}$$

where

$$\Theta(x^0) = \begin{cases} 1 & x^0 > 0 \\ 0 & x^0 < 0 \end{cases} \tag{3.10}$$

and $G^+(x,x')$ and $G^-(x,x')$ are the Wightman functions.

4. The Green function of the fields

The Pauli-Jordan function of the fields will be calculated in this section using the set **(2.12a)**, (2.12b) and the result is the same as that obtained using the complete set **(2.11a)**, **(2.11b)** (di Sessa modes).

This is not a trivial result. We know that if we have two sets of orthonormal modes $\{u_\lambda, u_\lambda^*\}$ and $\{v_\lambda, v_\lambda^*\}$, the Pauli-Jordan function calculated using the two sets will be the same if the two sets are complete. Studying the sets (2.11) and (2.12) we see that the zero modes of (2.12) can not be defined. So, it is straightforward to conclude that the modes u_0 and u_0^* can not be expressed using the set eq. (2.12). The two sets are not equivalent and the set eq. (2.12) is not complete in any space of functions which contains all the elements of eq. (2.11).

The calculation of the Bogoliubov coefficients between the sets (2.11) and (2.12) results is

$$\alpha_{\mu\nu} = (v_\mu, u_\nu) = \left(\frac{e^{\pi\nu}}{2\sinh\pi\nu}\right)^{1/2} \delta(\mu - \nu)$$

$$\beta_{\mu\nu} = (v_\mu, u_\nu^*) = \left(\frac{e^{-\pi\nu}}{2\sinh\pi\nu}\right)^{1/2} \delta(\mu - \nu)$$

It is not allowed $\nu = 0$. When $\mu = 0$, $\alpha_{\mu\nu} = \alpha_{0\nu} \equiv 0$. The same holds for $\beta_{\mu\nu}$. So we do not have a Bogoliubov transformation (in strictum sensum) between the sets (2.11) and (2.12).

We will also demonstrate that the Feyman Propagator diverges. The Pauli-Jordan function can be split into its positive and negative frequency parts as

$$iG(x,x') = G^+(x,x') - G^-(x,x') \tag{4.1}$$

where

$$G^+(x,x') = \int_{-\infty}^{\infty} d\lambda\, v_\lambda^+(x)(v_\lambda^+(x'))^* \tag{4.2a}$$

and

$$G^-(x,x') = \int_{-\infty}^{\infty} d\lambda v_\lambda^-(x)(v_\lambda^-(x'))^* \qquad (4.2a)$$

Substituting eqs. (2.12a), (2.12b) in (4.2a) and (4.2b) we have

$$G^+(\eta,\xi;\eta',\xi') = \frac{1}{4}\int_{-\infty}^{\infty} \frac{d\lambda}{\sinh \pi|\lambda|} e^{i\lambda(\xi-\xi')} J_{-i|\lambda|}(m\eta) J_{i|\lambda|}(m\eta') \qquad (4.3a)$$

$$G^-(\eta,\xi;\eta',\xi') = \frac{1}{4}\int_{-\infty} \frac{d\lambda}{\sinh \pi|\lambda|} e^{-i\lambda(\xi-\xi')} J_{i|\lambda|}(m\eta) J_{-i|\lambda|}(m\eta') \qquad (4.36)$$

The Feyman propagator (3.9) diverges because the integrals (4.3a) and (4.3b) calculated individually are divergent since when $\lambda \to 0^+, 0^-$ the integrand behaves like $\frac{1}{\pi|\lambda|} J_0(m\eta) J_0(m\eta') + \rho$, $|\rho|$ bounded near the origin ($\lambda = 0$). Let us study this divergence.

When $\eta > \eta'$ $G_F(x,x') = G^+(x,x')$, then:

$$G_F(x,x') = \frac{1}{2}\int_0^{\infty} d\lambda \quad \cos\lambda(\xi - E') J_{-i\lambda}(m\eta) J_{i\lambda}(m\eta') \quad \eta > \eta' \qquad (4.4)$$

This is an improper integral and its value is obtained when we make the lower (and upper) limit tend to zero (and ∞).

Then

$$G_F(\eta,\xi;\eta',\xi') = \lim_{\substack{\epsilon\to 0^+ \\ R\to w}} \frac{1}{2}\int^{R} \frac{d\lambda}{\sinh \pi\lambda} c \quad \lambda(\xi - \xi') J_{-i\lambda}(\text{mil}) J_{i\lambda}(m\eta') \qquad (4.5)$$

$$= \lim_{\epsilon\to 0^+} \frac{1}{2}\int^{1} \frac{d\lambda}{\sinh \pi\lambda} \cos\lambda(E - \xi') J_{-ix}(m\eta) J_{i\lambda}(\text{mil}') +$$

$$\lim_{R\to\infty} \frac{1}{2}\int_1^{R} \frac{d\lambda}{\sinh \pi\lambda} \cos\lambda(\xi - E') J_{-i\lambda}(\text{mil}) J_{i\lambda}(\text{mil}') \qquad (4.6)$$

The second integral in eq. (4.6) does converge if $\eta'/\eta \neq e^{\xi-\xi'}$ and $\eta/\eta' \# e^{\xi-\xi'}$.

This is equivalent to choosing points separated by a space-time interval not equal to zero. The first integral in eq. (4.6) diverges whenever $J_0(m\eta)$ and $J_0(m\eta')$ differ from zero. In fact, the value of the first integral in eq. (4.6) is $\log\frac{1}{\epsilon} J_0(m\eta) J_0(m\eta') + \Omega(\epsilon)$ where $\Omega(\epsilon)$ is bounded. The behavior of the Feynman propagator is ilustrated below.

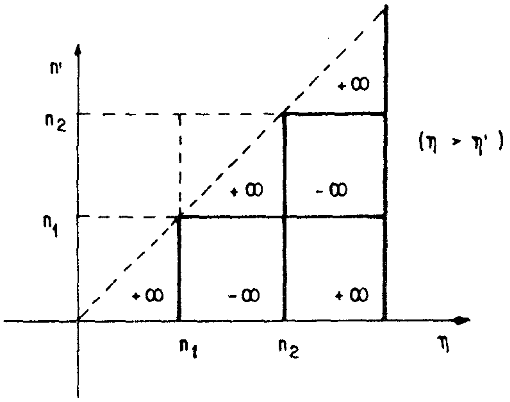

Fig. 2 - (η_i is the i'th root of $J_0(m\eta) = 0$).

In the evaluation of the Pauli-Jordan function, the divergence can be eliminated when we calculate $G^+ - G^-$ as the principal value of an integral.

$$G^+(\eta,\xi;\eta',\xi') - G^-(\eta,\xi;\eta'\xi') =$$

$$\frac{1}{4}\int_{-\infty}^{\infty}\frac{d\lambda}{\sinh\pi\lambda}e^{i\lambda(\xi-\xi')}J_{-i\lambda}(m\eta)J_{i\lambda}(m\eta')+$$

$$-\frac{1}{4}\int_{-\infty}^{\infty}\frac{d\lambda}{\sinh\pi\lambda}e^{i\lambda(\xi-\xi')}J_{i\lambda}(m\eta)J_{-i\lambda}(m\eta')\ . \tag{4.7}$$

Defining

$$f_\lambda(z_1,z_2,z_3) = \frac{1}{\sinh\pi\lambda}e^{i\lambda z_1}J_{-i\lambda}(mz_2)J_{i\lambda}(mz_3) \tag{4.8}$$

The expression (4.7) can be written as

$$G^+ - G^- = I_1 - I_2 \tag{4.9}$$

where

$$I_1 = \frac{1}{4}\int_{-\infty}^{\infty}d\lambda f_\lambda(\xi-\xi',\eta,\eta') \tag{4.10a}$$

$$I_2 = \frac{1}{4}\int_{-\infty}^{\infty} d\lambda f_\lambda(\xi - \xi', \eta', \eta) \tag{4.10b}$$

When $m\eta$ and $m\eta'$ are not roots of $J_0, |f_\lambda|$ tends to infinity as $|\lambda^{-1}|$, when $\lambda \to 0$. However, the integrals are finite if we adopt the principal value at the origin (A $= 0$). The function f_λ is analytic with respect to λ in the whole complex plane except at the points $\lambda = $ ni (n E Z). We have an infinite number of first order poles, and the residue of f_λ at these points is

$$\mathrm{Res}(f_\lambda; ni) = \frac{1}{A} e^{-nz_1} J_n(mz_2) J_n(mz_3) \tag{4.11}$$

Two distinct contours C and C' will be used to calculate I_1 and I_2 (see fig. 3).

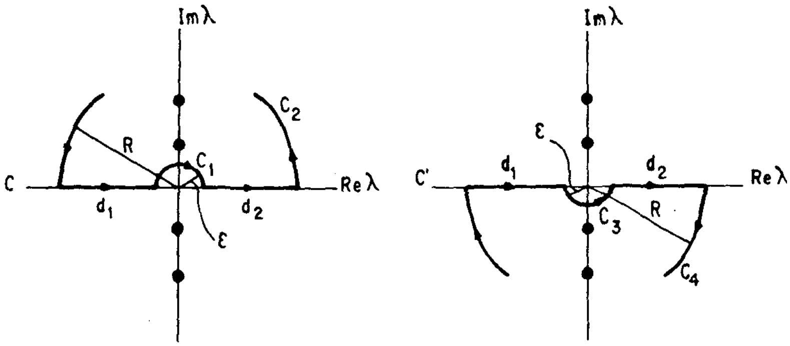

Fig. 3

C_2 and C_4 cross the imaginary axis at the middle point of the adjacent poles i.e.

$$R = q + \frac{1}{2} \quad q \in N$$

then

$$\lim \int_{C_2} d\lambda f_\lambda(z_1, z_2, z_3) = 0 \quad \text{if} \quad e^{z_1}\frac{z_3}{z_2} > 1 \tag{4.12a}$$

$$R = q + \frac{1}{2} \rightarrow \infty$$

and

$$\lim \int_{C_4} d\lambda f_\lambda(z_1, z_2, z_3) = 0 \quad \text{if} \quad e^{z_1}\frac{z_3}{z_2} < 1 \tag{4.12b}$$
$$R = q + \frac{1}{2} \rightarrow \infty$$

In Appendix A we demonstrate eqs. (4.12a) and (4.12b).

By the Cauchy theorern

$$\int_C f_\lambda(z_1, z_2, z_3) d\lambda = 2\pi i \sum_{n=1}^{q} \text{Res}(f_\lambda; ni) \tag{4.13}$$

If $e^{z_1}\frac{z_3}{z_2} > 1$, and we take the limit of the eq. (4.13) when $\epsilon \rightarrow 0$, $q \rightarrow \infty$ we get

$$\int_{-\infty}^{\infty} f_\lambda(z_1, z_2, z_3) d\lambda - \pi i \text{Res}(f_\lambda; 0) =$$
$$2\pi i \sum_{n=1}^{\infty} \text{Res}(f_\lambda; ni) \tag{4.14}$$

Therefore using eq. (4.11)

$$\int_{-\infty}^{\infty} d\lambda f_\lambda(z_1, z_2, z_3) = i J_0(mz_2) J_0(mz_3) +$$
$$+ 2i \sum_{n=1}^{\infty} e^{-nz_1} J_n(mz_2) J_n(mz_3) \quad \text{if} \quad e^{z_1}\frac{z_3}{z_2} > 1 \tag{4.15}$$

Similarly, when $e^{z_1}\frac{z_3}{z_2} < 1$ starting from

$$\int_{C'} f_\lambda(z_1, z_2, z_3) d\lambda = -2\pi i \sum_{n=-1}^{-q} \text{Res}(f_\lambda; ni) \tag{4.16}$$

we get, taking the limit

$$\int_{-\infty}^{\infty} f_\lambda(z_1, z_2, z_3)d\lambda + \pi i \ \mathrm{Res}(f_\lambda; 0) =$$

$$-2\pi i \sum_{n=-1}^{-\infty} \mathrm{Res}(f_\lambda; ni) \ . \tag{4.17}$$

Therefore using **eq.** (4.11)

$$\int_{-\infty}^{\infty} f_\lambda(z_1, z_2, z_3)d\lambda = -iJ_0(mz_2)J_0(mz_3) +$$

$$-2i \sum_{n=-1}^{-\infty} e^{-nz_1} J_n(mz_2)J_n(mz_3) \quad \text{if} \quad e^{z_1}\frac{z_3}{z_2} < 1 \tag{4.18}$$

Using eqs. (4.10a) and (4.10b) we have

$$I_1 = \frac{i}{4}\left[J_0(m\eta)J_0(m\eta') + 2\sum_{n=1}^{\infty} e^{-n(\xi-\xi')} J_n(m\eta)J_n(m\eta')\right]$$

$$\text{if } e^{(\xi-\xi')}\frac{\eta'}{\eta} > 1 \tag{4.19a}$$

$$I_1 = -\frac{i}{4}\left[J_0(m\eta)J_0(m\eta') + 2\sum_{n=-1}^{-\infty} e^{-n(\xi-\xi')} J_n(m\eta)J_n(m\eta')\right]$$

$$\text{if } e^{(\xi-\xi')}\frac{\eta'}{\eta} < 1 \tag{4.19b}$$

$$I_2 = \frac{i}{4}\left[J_0(m\eta)J_0(m\eta') + 2\sum_{n=1}^{\infty} e^{-n(\xi-\xi')} J_n(m\eta)J_n(m\eta')\right]$$

$$\text{if } e^{(\xi-\xi')}\frac{\eta}{\eta'} > 1 \tag{4.20a}$$

$$I_2 = -\frac{i}{4}\left[J_0(m\eta)J_0(m\eta') + 2\sum_{n=-1}^{-\infty} e^{-n(\xi-\xi')} J_n(m\eta)J_n(m\eta')\right]$$

$$\text{if } e^{(\xi-\xi')}\frac{\eta}{\eta'} < 1 \tag{4.20b}$$

The space-time interval o $= (y^0 - y^{0'})^2 - (y^1 - y^{1'})^2$ in the coordinates (r], ξ) can be written as

$$\sigma = \eta^2 + \eta'^2 - 2\eta\eta' \cosh(\xi - \xi') =$$
$$- \eta\eta' e^{-(\xi-\xi')} \left(e^{(\xi-\xi')} \frac{\eta'}{\eta} - 1 \right) \left(e^{(\xi-\xi')} \frac{\eta}{\eta'} - 1 \right) \tag{4.21}$$

If o $<$ 0 (which corresponds to space-like separated events) there are two possibilities

$$e^{(\xi-\xi')} \frac{\eta'}{\eta} > 1 \quad \text{and} \quad e^{(\xi-\xi')} \frac{\eta}{\eta'} > 1 \tag{4.22a}$$

or

$$e^{(\xi-\xi')} \frac{\eta'}{\eta} < 1 \quad \text{and} \quad e^{(\xi-\xi')} \frac{\eta}{\eta'} < 1 \tag{4.22b}$$

In the cases (4.22a) and (4.22b) $I_1 = I_2$ so $G(x, x') = 0$.

If a $>$ 0 (which corresponds to time-like separated events) there are again two possibilities

$$e^{(\xi-\xi')} \frac{\eta'}{\eta} > 1 \quad \text{and} \quad e^{(\xi-\xi')} \frac{\eta}{\eta'} < 1 \tag{4.23a}$$

or

$$e^{(\xi-\xi')} \frac{\eta'}{\eta} < 1 \quad \text{and} \quad e^{(\xi-\xi')} \frac{\eta}{\eta'} > 1 \tag{4.23b}$$

It should be noted that in (4.23a) rJ' $> \eta$ and in (4.23b) $\eta > \eta'$.

In the case (4.23a) (a > 0, $\eta' > \eta$)

$$I_1 - I_2 = \frac{i}{2} \sum_{n=-\infty}^{\infty} e^{n(\xi-\xi')} J_n(m\eta) J_n(m\eta') \, . \tag{4.24a}$$

In the case (4.23b) (o > 0, r] $> \eta'$)

$$I_1 - I_2 = -\frac{i}{2} \sum_{n=-\infty}^{\infty} e^{n(\xi-\xi')} J_n(m\eta) J_n(m\eta') \tag{4.24b}$$

Thus using eqs. (4.9), (4.24a) and (4.24b)

$$G^+ - G^- = -\frac{i}{2}\left[\Theta(\sigma)\epsilon(\eta-\eta')\sum_{n=-\infty}^{\infty} e^{n(\xi-\xi')}J_n(m\eta)J_n(m\eta')\right] \qquad (4.25)$$

where

$$\epsilon(\eta) = \begin{cases} 1 & \eta > 0 \\ -1 & q < o \end{cases}$$

The addition theorem of the Bessel functions states that

$$J_0(ms) = \sum_{n=-\infty}^{\infty} J_n(m\eta)J_n(m\eta')e^{in\Theta} \qquad (4.26)$$

where

$$s = (\eta^2 + \eta'^2 - 2\eta\eta'\cos\Theta)^{1/2}$$

Taking the analytical extension of $O = i(\xi - \xi')$ we get, using eq. (4.21)

$$s = \sigma^{1/2}$$

$$J_0(m\sigma^{1/2}) = \sum_{n=-\infty}^{\infty} J_n(m\eta)J_n(m\eta')e^{n(\xi-\xi')} \qquad (4.27)$$

(see Appendix B for a more detailed demonstration).

Finally substituting eqs. (4.27) in (4.25) and using eq. (4.1) we get

$$iG = G^+ - G^- = -\frac{i}{2}[\Theta(\sigma)\epsilon(\eta-\eta')J_0(m^2\sigma)^{1/2}] \qquad (4.28)$$

This coincides with the result obtained by di Sessa[6].

5. Summary and Discussion

In this paper we have studied two sets of functions commonly employed in literature to expand the field operator $\phi(x)$ that describes a neutral massive scalar field in a portion of Minkovski space-time that is covered by the Milne coordinate systems. These sets are usually regarded as complete and equivalent. We showed that the space generated by the set (2.12) does not contain the space generated by the set (2.11).

In a second step we proved that although we do not have a Bogoliubov transformation or an equivalence between the two sets, the Pauli-Jordan propagator is the same for both sets.

We proved that the Feynman propagator associated with the set (2.12) diverges for almost all the values of (g,ξ), (η',ξ'). In fact it will not diverge only in a set a zero measure. It is not clear if this divergence is connected with the infrared divergence presented by Wightman in a theory that describes a massless boson in a two-dimensional space-time. This point deserves further investigations.

The following question naturally arises from this work:

If

- $\{u_\lambda, u_\lambda^*\}, \{v_\lambda, v_\lambda^*\}$ are two sets of orthonormal mode solutions employed to expand a bosonic field operator in an infinite volume;
- only a discrete set (or for a set that in some sense has zero measure) of index, modes of $\{u_\lambda, u_\lambda^*\}$ does not admit expansion using the set $\{v_\lambda, v_\lambda^*\}$.

Will the Pauli-Jordan propagator be the same when calculated using $\{u_\lambda, u_\lambda^*\}$ and $\{v_\lambda, v_\lambda^*\}$?

This work intends to stress the importance of the equivalence and completeness relation between sets of orthonormal mode solutions.

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Appendix A

It is known that

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{(z/2)^{2k+\nu}}{\Gamma(\nu+k+1)} .$$

Defining

$$(a)_0 = 1$$
$$(a)_k = a(a+1)...(a+k-1) \qquad (A.1)$$

we get

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{(-z^2/4)^k}{k!(\nu+1)_k} \qquad (A.2)$$

The same formula can also be expressed using the hypergeometric functions[11]

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)} \, {}_0F_1\left(\nu+1; -\frac{1}{4}z^2\right) .$$

Supposing (and this is most important) $\exists n_0 > 0$ such that

$$\forall n \in Z \qquad |\nu + n| \geq n_0$$

Using the definition **(A.2)**

$$|(\nu+1)_k| \geq (n_0)^k \qquad \forall k \in \mathbf{N} \qquad (A.3)$$

Now using eqs. **(A.2)** and **(A.3)**

$$|J_\nu(z)| \leq \frac{|(z/2)^\nu|}{|\Gamma(\nu+1)|} \sum_{k=0}^{\infty} \frac{|(-\frac{z^2}{4})^k|}{k!|(\nu+1)_k|} \leq \frac{|(z/2)^\nu|}{|\Gamma(\nu+1)|} \sum_{k=0}^{\infty} \frac{|\frac{z^2}{4}|^k}{k!\; n_0^k}$$

This inequality can be simplified

$$|J_\nu(z)| \leq \frac{|(z/2)^\nu|}{|\Gamma(\nu+1)|} \exp\left(\frac{z^2}{4n_0}\right) \qquad (A.4)$$

if $|\nu + n| \geq n_0 \;\; \forall n \in Z$.

Let us suppose

$$|\nu| = q + \frac{1}{2} \qquad q \in \mathbf{N}$$

then

$$|\nu + k| \geq \frac{1}{2} \qquad k \in Z$$

and using eq. **(A.4)** we get

$$|J_\nu(z)| \leq \frac{|(z/2)^\nu|}{|\Gamma(\nu + 1)|} \exp\left(\frac{z^2}{2}\right) \tag{A.5}$$

if

$$|\nu| = q + \frac{1}{2} \quad \text{for some} \quad q \in \mathbf{N}$$

In the main text eq. **(4.8)** is

$$f_\lambda(z_1, z_2, z_3) = \frac{1}{\sinh \pi\lambda} e^{i\lambda z_1} J_{-i\lambda}(mz_2) J_{i\lambda}(mz_3) \ .$$

In the contour C_2 and C_4

$$|\pm i\lambda| = q + \frac{1}{2} \qquad q \in \mathbf{N} \tag{A.6}$$

Using eqs.(A.5) and (A.6), we get

$$|f_\lambda(z_1, z_2, z_3)| \leq \left|\frac{1}{\sinh \pi\lambda} e^{i\lambda z_1}\right| \left|\frac{(z_3/z_2)^{i\lambda}}{\Gamma(i\lambda + 1)\Gamma(-i\lambda + 1)}\right| \exp\left(\frac{m^2}{2}(z_2^2 + z_3^2)\right)$$

in C_2 and C_4.

Because

$$\Gamma(i\lambda + 1)\Gamma(-i\lambda + 1) = \frac{\pi\lambda}{\sinh \pi\lambda}$$

the inequality becomes

$$|f_\lambda(z_1, z_2, z_3)| \leq (^{\mathrm{I}} \quad e \quad z_2)^{i\lambda} \left|\frac{1}{\pi} \exp\left(\frac{m^2}{2}(z_2^2 + z_3^2)\right)\right.$$

in C_2 and C_4.

The expression

$$\frac{1}{\pi} \exp\left(\frac{m^2}{2}(z_2^2 + z_3^2)\right)$$

does not change in the contour C_2 and C_4 so we call it M, and we get

$$|f_\lambda(z_1,z_2,z_3)| \leq \left|\left(e^{z_1}\frac{z_3}{z_2}\right)^{i\lambda}\right|\frac{M}{|\lambda|} \qquad (A.7)$$

in C_2 and C_4.

In our problem $z_1 = \xi - \xi'$ is a real number and z_2, z_3 $(q$ or $\eta')$ are positive. Then there exists $K \in \mathrm{R}$ such that

$$e^{z_1}\frac{z_3}{z_2} = e^K \ . \qquad (A.8)$$

Let us study the case

$$e^{z_1}\frac{z_3}{z_2} > 1 \quad (K > 0).$$

Using eqs. (A.7) and (A.8)

$$\left|\int_{C_2} f_\lambda(z_1,z_2,z_3)d\lambda\right| \leq \int_{C_2}\left|f_\lambda(z_1,z_2,z_3)\right||d\lambda| \leq \int_{C_2} M|e^{i\lambda k}|\frac{|d\lambda|}{|\lambda|}$$

We can choose the parametrization

$$C_2 \ : \ \lambda(\Theta) = \left(q+\frac{1}{2}\right)e^{i\Theta} \qquad 0 \leq \Theta \leq \pi$$

The inequality above becomes

$$\left|\int_{C_2} f_\lambda d\lambda\right| \leq M\int_0^\pi |e^{i\lambda K}|d\Theta = M\int_0^\pi \exp\left(-k\left(q+\frac{1}{2}\right)\sin\Theta\right)d\Theta =$$

$$2M\int_0^{\pi/2} \exp\left(-\left(q+\frac{1}{2}\right)\sin\Theta\right)d\Theta \qquad (A.9)$$

If $0 \leq \Theta \leq \pi/2$ then $2\Theta/\pi \leq \sin\Theta$.

We are studying the case $K > 0$. So

$$\exp\left(-K\left(q+\frac{1}{2}\right)\mathrm{sen}\Theta\right) \leq \exp\left(-K\left(q+\frac{1}{2}\right)\frac{2\Theta}{\pi}\right) \qquad 0 \leq \Theta \leq \pi/2$$

and

$$\left|\int_{C_2} f_\lambda d\lambda\right| \leq 2M \int_0^{\pi/2} \exp\Big(-\frac{2K}{\pi}\Big(q+\frac{1}{2}\Big)\Theta\Big)d\Theta =$$
$$= -\frac{M\pi}{K\big(q+\frac{1}{2}\big)}\Big(\exp\Big(-K\Big(q+\frac{1}{2}\Big)\Big)-1\Big) \leq \frac{M\pi}{K\big(q+\frac{1}{2}\big)} \qquad (A.10)$$

Thus using eq.(A.10)

$$\lim_{\substack{q\to\infty \\ q\in N}} \left|\int_{C_2} f_\lambda d\lambda\right| = 0 \quad \text{if} \quad e^{z_1}\frac{z_3}{z_2} > 1 \ .$$

For the case $e^{z_1}\frac{z_3}{z_2} < 1$ $(K < 0)$ the contour C_4 is the adequate one. Similar calculations give us

$$\lim_{\substack{q\to\infty \\ q\in N}} \left|\int_{C_4} f_\lambda d\lambda\right| = 0 \quad \text{if} \quad e^{z_1}\frac{z_3}{z_2} < 1 \ .$$

Appendix B

We will define

$$g(z) = J_0\Big[m\Big(\eta^2+\eta'^2-\eta\eta'\Big(z+\frac{1}{z}\Big)\Big)^{1/2}\Big]$$
$$= J_0\Big[m\Big(\Big(\eta+\eta' z\Big)\Big(\eta+\frac{\eta'}{z}\Big)\Big)^{1/2}\Big] \ . \qquad (B.1)$$

J_0 is analytic in the whole complex plane and its expansion in power series centered at zero contains only even powers. Then the square root above can be naturally eliminated and $g(z)$ is analytic in the whole complex plane except the origin. We will define "another" function

$$h(z) = \sum_{n=-\infty}^{\infty} J_n(m\eta)J_n(m\eta')z^n \qquad (B.2)$$

The series **(B.2)** is convergent if $z \neq 0$. Then $h(z)$ is analytic in the whole complex plane except at $z=0$.

If we take $|z|=1$,

$$z = e^{i\Theta} \qquad \Theta \in R$$

$$h(z) = \sum_{n=-w}^{\infty} J_n(m\eta)J_n(m\eta')e^{in\Theta} =$$

$$J_0(m\eta)J_0(m\eta') + 2\sum_{n=l}^{\infty} J_n(m\eta)J_n(m\eta')\cos\ n\Theta\ .$$

Using the addition theorem and cylinder functions we get

$$h(z) = J_0[m(\eta^2 + \eta'^2 - 2\eta\eta'\cos\Theta)^{1/2}]$$

Now

$$g(z) = g(e^{i\Theta}) = J_0[m(\eta^2 + \eta'^2 - 2\eta\eta'\cos\Theta)^{1/2}]$$

we obtain $g(z) = h(z)$ if $|z| = 1$.
$g(z) - h(z)$ is analytic in $C - \{0\}$ and vanishes in $|z| = 1$, then it must be equal to zero in $C - \{0\}$.

This occurs because the zeros of any analytic function are isolated inside its domain (open and connected) or else the function vanishes in all the domain. What we obtained is that

$$J_0\left[m\left(\eta^2 + \eta'^2 - \eta\eta'\left(z + \frac{1}{z}\right)\right)^{1/2}\right] = \sum_{n=-\infty}^{\infty} J_n(m\eta)J_n(m\eta')z^n \qquad z \neq 0\ . \qquad (B.3)$$

If

$$z = e^{i\Theta} \quad \Theta \in C$$

then

$$z(\Theta) \neq 0 \qquad \forall\ \Theta \in C$$

and using eq. **(B.3)** we get

$$J_0\left[n(\eta^2+\eta'^2-2\eta\eta'\cos\Theta)^{1/2}\right]=\sum_{n=-\infty}^{\infty}J_n(m\eta)J_n(m\eta')e^{in\Theta}\quad\Theta\in C$$

If $\Theta = i([-\xi')$ we find

$$J_0[m(\sigma)^{1/2}]=\sum_{n=-\infty}^{\infty}J_n(n\eta)J_n(m\eta')e^{n(\xi-\xi')}$$

where

$$a=(\eta^2+\eta'^2-2\eta\eta'\cosh(\xi-E'))$$

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Resumo

Neste trabalho estudamos um campo escalar neutro e massivo em um espaço-tempo de Milne bi-dimensional. A quantização é feita em hipérboles que são superfícies Lorentz invariantes. A expansão do operador do campo é realizado usando-se um conjunto completo de modos ortonormais que têm frequência de dilatação positiva e negativa definida, Nós calculamos as funções de Green retardada e avançada e provamos que o propagador de Feynman diverge no sentido usual.