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Gauge strueture and geometry in quantum adiabatic system

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Abstract The mathematical structure of a nontrivial geometric phase factor in adiabatic quantum processes is explored. Some geometric aspects of the compact Reimannian manifold are analyzed and the relation with the parameter space is discussed.

1. Introduction

In the past few years, there has been a widespread revival of interest in the adiabatic quantum system. It was started with the observation by Berry that a nontrivial phase factor would be developed for a quantum state in an adiabatic process even if the system was restored to its original condition[1]. This remarkable quantum adiabatic phenomenon is now known as Berry's phase, and various applications have been made in solid state physics[2], molecular physics[3] and even in optics[4]. The connections between the nontriviality of the adiabatic phase and the gauge anomalies in chiral fermion systems have also been established[5] recently.

On the other hand, a mathematical interpretation of this quantum adiabatic theorem was given in the context of a Hermitian line bundle over the parameter manifold. It was found that the geometric phase factor of Berry is precisely the holonomy in line bundle[7].

In this article, we shall first demonstrate how the gauge structure can be induced through an adiabatic process in a nongauge quantum system. We shall first

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review the general nature of tha adiabatic quantum phase in a nondegenerate system. The gauge connection will be calculated explicitly in the magnetic monopole system in the following section.

The relation between the quantum adiabatic process and the parallel transport of the complex line bundle or complex vector bundle over the parameter space will be analyzed in Section 3. Berry's holonomy in the line bundle is calculated and identified as the adiabatic quantum phase. The adiabatic quantum system with degenerate states is also discussed and its non-Abelian nature of gauge structure is explored. In Section 4, a quantum system of a spinless electron interacting with a slow moving nucleus confined on a S^N parameter manifold is taken as an example to illustrate the equivaiency between the non-Abelian gauge connection in the adiabatic quantum evolution of degenerate systems and the connection of a tangent bundle over S^N. Section 5 is devoted to a few propositions and theorems in Riemann geometry. Information concerning the eigenspectra of the quantum system allows one to determine the geometry of the parameters space. A brief remark is contained in the last section.

2. Quantum adiabatic phase

When a quanturn system is perturbed in a controllable fashion, the Hamiltonian of the system can he expressed in terms of a set of external parameters in addítion to the usual dynamical observables $\vec{x}$ and $\vec{p}$. Let the set of external controllable parameters be denoted by $\lambda = \{\lambda_1, ..., \lambda_m\}$, the the Hamiltonian takes the form $H = H(\vec{x}, \vec{p}; \lambda)$, where $\lambda_i (i = 1, ..., m)$ are functions of time t. The evolution of the system is by a continuous change of the set of parameters $\lambda = \lambda(t)$ from time $t = 0$ to $t = T$, and the dynamics of the system is governed by the time dependend Schrodinger equation,

$$i\frac{\partial}{\partial t}|n, \lambda(t) >= H(\lambda(t))|n, \lambda(t) >, \tag{2.1}$$

with the condition that the system is initially in the eigenstate n,

$$\psi_n(0) = |n, \lambda_0 > \tag{2.2}$$

If the system is under adiabatic perturbation, the wave function at time T, according to the Born-Oppenheimer approximation, can be expressed as

$$\psi_n(\lambda_T) = \exp(-i \int_0^T E_n(\lambda(t))dt_i\gamma_n)|n, \lambda_0> \quad , \tag{2.3}$$

where $E_n(\lambda(t))$ is the nondegenerate eigenenergy,

$$H(\lambda(t))|n, \lambda(t) >= E_n(\lambda(t))|n, \lambda(t) > \quad , \tag{2.4}$$

and

$$\gamma_n = \int_0^t dt < n, \lambda(t)|i \frac{d}{dt}|n, \lambda(t) > \tag{2.5}$$

The first term in the exponential of eq.*(2.3)* is the usual dynamic phase factor, while the second one is on nondynamical is origin. It was traditionally ignored due to the common belief that an extra phase factor in the state vector could be chosen to compensate the nondynamical one[8]. To be more precise, let us recast eq.(2.5) into the following form

$$\begin{aligned} \gamma_n &= \int_{\lambda_0}^{\lambda_T} d\lambda \cdot < n, \lambda|i\nabla_\lambda|n, \lambda > \\ &= \int_{\lambda_0}^{\lambda_t} A_n \cdot d\lambda \quad , \end{aligned} \tag{2.6}$$

where

$$A_n = i < n, \lambda|\nabla_\lambda|n, \lambda > \tag{2.7}$$

can be treated as the gauge potential in the parameter space. It exhibits $U(1)$ gauge structure when the following transformation is performed,

$$A_n \to A'_n = A_n - \nabla_\lambda \xi_n \quad , \tag{2.8}$$

then

$$\gamma_n \to \gamma'_n = \gamma n - (\xi_n(\lambda_T) - \xi_n(\lambda_0)) \quad . \tag{2.9}$$

Therefore γ'_n can be put to zero if $\xi_n(\lambda_T) \# \xi_n(\lambda_0)$ and $\gamma_n = \xi_n(\lambda_T) - \xi_n(\lambda_0)$.

However if we consider a quantum adiabatic process in which the system is brought back to its initial configuration, i.e. $\lambda_T = \lambda_0$, and the evolution of the quantum system a closed contour C in the parameter space, then a nonvanishing adiabatic quantum phase factor can be calculated by the loop integral

$$\gamma_n = \oint_C An \,.\, d\lambda \;\; mod \;\; 2\pi \tag{2.10}$$

where $mod\ 2\pi$ arises from the single-valuedness of the arbitrary function $\xi_n(\lambda)$, namely

$$\xi_n(\lambda_T) - \xi_n(\lambda_0) = mod \;\; 2\pi.$$

In analogy with the usual Abelian gauge theory in classical field, if one defines the gauge curvature tensor

$$\begin{aligned} F_n^{ij} &= \frac{\partial A_n^i}{\partial \lambda_j} - \frac{\partial a_n^j}{\partial \lambda_i} \\ &= -i\sum_m \{ < n|\nabla_i|m >< m|\nabla_j|n > -(i \leftrightarrow j)\} \\ &= \sum_{m\neq n} \frac{i}{(E_m - E_n)^2}\{ < n|(\nabla_i H)|m >< m|(\nabla_j H)|n > -(i \leftrightarrow j)\} \end{aligned} \tag{2.11}$$

where in the last equation, one makes use of the nondegenerate condition $E_m \neq E_n$ and

$$< n|\nabla_i|m >= \frac{1}{E_m - E_n} < n|(\nabla_i H)|m > \tag{2.12}$$

The contour integral can then be converted to surface integral by Stokes' theorem

$$\lambda_n = \int_s F_n^{ij} d\sigma_{ij} \tag{2.13}$$

where $d\sigma_{ij}$ is the surface element of integration in the parameter space.

As an example, let us consider the two level system of a spin $1/2$ particle in a magnetic field. The 2 x 2 Hamiltonian takes the form

$$H(\lambda) = \vec{\lambda} \cdot \vec{\sigma} = \begin{pmatrix} \lambda_3 & \lambda_1 - i\lambda_2 \\ \lambda_1 + i\lambda_2 & -\lambda_3 \end{pmatrix} \tag{2.14}$$

with the eigenvalues $\pm\lambda = \pm|\vec{\lambda}|$.

The renormalized eigenvector corresponding to eigenvalue λ is given by

$$|\lambda> = \frac{1}{\sqrt{2\lambda(\lambda+\lambda_3)}} \begin{pmatrix} \lambda + \lambda_3 \\ \lambda_1 + i\lambda_2 \end{pmatrix} \tag{2.15}$$

The degeneracy occurs at $\lambda = 0$, namely the origin of the parameter space. Applying eq.(2.7) to eq.(2.11), we obtain

$$\vec{A} = \Big(\frac{1}{2}\frac{\lambda_2}{\lambda(\lambda+\lambda_3)}, -\frac{\lambda_1}{2\lambda(\lambda+\lambda_3)}, 0\Big) \tag{2.16}$$

$$\vec{F} = -\frac{1}{2\lambda^3}(\lambda_1, \lambda_2, \lambda_3) \tag{2.17}$$

One recognizes that eq.(2.16) and eq.(2.17) are respectively the gauge potential and the magnetic field of a Dirac monopole of strength $-\frac{1}{2}$ located at the origin of λ-space. In fact, it also corresponds to a string lying along the negative λ_3-axis on which the gauge potential is not defined.

The adiabatic quantum phase evaluated from eq.(2.13) and eq.(2.17) is them the total magnetic flux through the surface subtended by the contour C. It is also precisely equal to the solid angle subtended by the contour multiplied by the strength of monopole charge.

Since the gauge potential eq.(2.16) is not globally defined, and a gauge transformation would bring $\vec{A}$ into $\vec{A}'$ corresponding to the singular string along the positive λ_3-axis, i.e.

$$\vec{A}' = \vec{A} + \nabla\varphi \tag{2.18a}$$

$$= \Big(-\frac{1}{2}\frac{\lambda_2}{\lambda(\lambda-\lambda_3)}, \frac{1}{2}\frac{\lambda_1}{\lambda(\lambda-\lambda_3)}, 0\Big) \tag{2.18b}$$

where

$$\varphi = \tan^{-1}\frac{\lambda_2}{\lambda_1} \tag{2.19}$$

is the azimuthal angle in λ-space. This gauge transformation corresponds to a change in the phase factor of the eigenvector $|\lambda>$

$$|\lambda> \rightarrow |\bar{\lambda}>= e^{-i\varphi}|\lambda>= \frac{1}{\sqrt{2\lambda(\lambda-\lambda_3)}}\begin{pmatrix}\lambda_1 - i\lambda_2 \\ \lambda-\lambda_3\end{pmatrix} \tag{2.20}$$

and one is able to verify that eq.(2.7) does hold for the transformed potential $\vec{A}'$, expressed in terms of the transformed vector $|\vec{\lambda}>$,

$$\vec{A}' = i < \bar{\lambda}|\nabla|\bar{\lambda}> \tag{2.21}$$

3. Quantum holonomy in line bundle

The eigenvector $|\lambda(t)>$ of $H(\lambda(t))$ can be regarded as a fiber at each point $\lambda = \{\lambda_1, ..., \lambda_m\}$ in the parameter space. For simplicity, we rescale the energy level so that the eigenvalues is zero, i.e.,

$$H(\lambda)|\lambda>= 0 \tag{3.1}$$

then $|\lambda>$ forms a line bundle over the parameter manifold A.

The adiabatic change of a quantum system by slowly varying $\lambda(t)$ in the parameter manifold is equivalent to parallel transport of a complex line bundle over h. In order to demonstrata this equivalency, let us restrict our discussion to the case of a nondegenerate quantum system. An infinitesimal Lie dragging of a complex line bundle from λ to $\lambda+\delta\lambda$ in the parameter manifold h is defined that the projection of the Lie dragged fiber $|\psi(\lambda+6X)>$ on the $|\psi(\lambda)>$ is invariant, namely

$$<\psi(\lambda+\delta\lambda)|\psi(\lambda)>= 1 + 0(\delta\lambda^2) \tag{3.2}$$

which reduces to the following parallel transport equation

$$<\frac{d\psi}{dt}|\psi>= 0 \tag{3.3}$$

To express the above equation in terms of local cross section,

$$|\psi(\lambda)>= e^{i\gamma}|\lambda> \quad , \tag{3.4}$$

one can rewrite the parallel transport equation as follows

$$\gamma - i < \lambda|\nabla_\lambda \lambda > \cdot\dot{\lambda} = 0 \; , \tag{3.5}$$

or equivalently

$$< \lambda|\psi > +, \lambda|\nabla_\lambda|\lambda > \cdot\dot{\lambda} < \lambda|\psi >= 0 \tag{3.6}$$

If one defines the connection in the complex line bundle

$$A(\lambda) = i < \lambda|\nabla_\lambda|\lambda > \; , \tag{3.7}$$

the parallel transport equation eq.(3.6) can be identified easily as the covariant derivative

$$(\text{Vx} + iA) < \lambda|\psi >= D_\lambda < \lambda|\psi >= 0 \tag{3.8}$$

where the first equality of the above equation defines the operator

$$D_\lambda = \nabla_\lambda + iA \tag{3.9}$$

For a parallel transport of a vector along the closed contour C on A, one obtains the holonomy of the connection in the complex line bundle by integrating eq.(3.5)

$$\gamma = i \oint_C d\lambda \cdot < \lambda|\nabla_\lambda|\lambda > \tag{3.10}$$

Let us consider the quantum system of p-fold degeneracy. If we denote the set of eigenvectors by $\{|S_a>, \; a = 1, ...p\}$, then a parallel transport of a vector ψ satisfies

$$< S_a(\lambda)|\psi(\lambda + \delta\lambda) > - < S_a(\lambda)|\psi(\lambda) >= 0(\delta\lambda^2) \tag{3.11}$$

for a $= 1, ...p$.

Expressing the vector ψ in terms of local section of the complex vector bundle, one reduces eq.(3.11) to

$$(\delta_{ab}\nabla - iA_{ab}) < S_b|\psi >= 0 \tag{3.12}$$

where

$$A_{ab} = \mathrm{i} < S_a(\lambda)|\nabla S_b(\lambda) > \tag{3.13}$$

Instead of a single holonomy angle in the degenerate case, one obtains a holonomy matrix of the connection in the complex vector bundle. The p $\times$ p matrix elements are given as

$$\gamma_{ab} = \oint A_{ab} d\lambda \quad , \tag{3.14}$$

Since A_{ab} od eq.(3.13) is a matrix valued connection of the complex vector bundle, the corresponding Werry's curvature exhibits the gauge structure of non-Abelian Yang-Mills type. If we express Berry's connection in the vector form $\vec{A}_{ab}$, the curvature can be calculated as follows

$$\vec{B}_{ab} = \vec{\nabla} \times \vec{A}_{ab} + (\vec{A} \times \vec{A})_{ab} \quad . \tag{3.14}$$

The non-Abelian gaugc: nature arisen from a nongauged quantum system in the adiabatic approximation was recently observed by Li[8] in a specific example of system with the fast and the slow variables. We shall explicitly calculate the connection of a tangent bundle over the parameter space of S^N in the following Section.

4. Complex vector bundle on S^N

Let us consider the motion of an electron which interacts with a nucleus at an instantaneous position R in (N + 1)-dimensional space. Because of the large mass ratio of the nucleus to the electron, the change in R. is so slow that the motion of the electron can be regarded as a quantum adiabatic process in which the parameter manifold is just a subspace of $\mathbf{R}^{N+1}$. The Hamiltonian describing the whole system is given by

$$H = \frac{\mathbf{P}^2}{2M} + \frac{\mathbf{p}^2}{2m} + V(\mathbf{R},\mathbf{r}) \tag{4.1}$$

According to the adiabatic approximation, the total wave function $\psi(\mathbf{R}, \mathbf{r})$ can be written as the direct product of the nucleus wave function and the electron wave function with the force center located at the instantaneous position of the nucleus R, i.e..

$$\psi\mathbf{R},\mathbf{r}) = \xi\mathbf{R}\varphi_n(\mathbf{R}, \tag{4.2}$$

where the electron wave function $\varphi_n(\mathbf{R},\mathbf{r})$ is related to the instantaneous eigenstate $|n, \mathrm{R}>$ by

$$< \mathbf{r}|n,\mathbf{R} >= \varphi_n(\mathbf{R},\mathbf{r}) \tag{4.3}$$

If we assume that the electron moves on an N-sphere, the wave function φ_n is of the type of hyperspherical harmonics. The Schrõdinger equation

$$\left\{\frac{\mathbf{P}^2}{2m} + V(\mathbf{R},\mathbf{r})\right\}|n,\mathbf{R} >= E_n|n,\mathbf{R} > \quad , \tag{4.4}$$

admits eigenvectors of $d(N,l)$-fold degeneracy with degenrate eigenvalue E_n = E(N,l), where

$$d(N,l) = \frac{(N_2l-2)(N+l-3)!}{(N-2)!l!} \quad , \tag{4.5}*$$

and

$$E(N,l) = \alpha l(N+l-1) \quad . \tag{4.6}*$$

The integer 1 is the degree of the harmonic polynomials and the constant a characterizes the square of curvature of S^N on which the motion of the electron is confined.

For the case l = 1, we have $d(N,1) =$ N, and the set of N-fold degenerate eigenvectors allows us to construct Berry's connection $A^{\mu}_{ab} = \mathrm{i} < \mathrm{a},\mathbf{R}|\nabla^{\mu}|b,\mathbf{R} >$. Since the Hamiltonian eq.(4.1) of $SO(N+1)$ symmetry will be reduced to

$SO(N)$ symmetry when R is taken instantaneous by fixed, then the coset space $SO(N+1)/SO(N)$ is just the S^N. Therefore if we take S^N as the parameter space, namely, if we further confine the motion of the nucleus to the subspace, $|\mathbf{R}|$ = constant in the R^{N+1} space, the evolution of the N-fold degenerate electron wave functions during the quantum adiabatic processes can then be treated as the parallel transport of an N-dimensional complex vectorbundle on S^N and $A^{\mu}_{ab} = i < a, \mathbf{R}|\nabla^{\mu}|b, \mathrm{R} >$, Berry's connection is then just the connection of the frame bundle on S^N.

Consider a stereographic projection of a point Q of S^N onto the equatorial plane at P from the north pole N (See fig. 1). The line passing N and P is given by

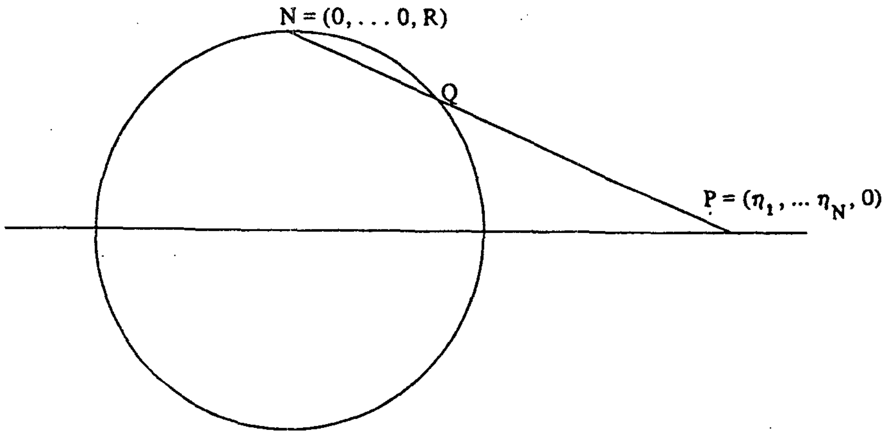

Fig.1 - Stereographic projection of S^N.

$$\vec{\alpha}(t) = t\vec{P} + (1-t)\mathrm{N} \tag{4.7}$$

and the coordinates of Q are determined by

$$|\vec{\alpha}(t)|^2 = R^2 \quad , \tag{4.8}$$

or

$$t = 2R^2/(P^2+R^2) \quad . \tag{4.9}$$

Since $\mathbf{P} = (\eta_1, ...\eta_N, 0)$, one obtains

$$Q = \frac{2R^2}{P^2+R^2}(\eta_1....\eta_N,0) + \frac{R(P^2-R^2)}{P^2+R^2}(0,...0,R) \quad . \tag{4.10}$$

Let

$$\psi \ : \ P \leftrightarrow Q \in S^N - N \tag{4.11}$$

the Riemannian metric of S^n can be calculated from the differential map

$$g(\tilde{V},\tilde{W}) = d\psi(\vec{V}) \cdot \psi(\vec{W}) \tag{4.12}$$

where

$$d\psi(\vec{U}) = \frac{d}{dt}\psi(\vec{P} + t\vec{U})|_{t=0} = \tag{4.13}$$

$$\frac{2R^2}{(P^2+R^2)^2}\{(P^2+R^2)\vec{U} - 2(\vec{P}\cdot\vec{U})\vec{P}, 2R^2(\vec{P}\cdot\vec{U})\} \quad . \tag{4.14}$$

for $\vec{V},\vec{W} \in R^N$ to be taken as basis vectors one finds that the metric g_{ab} becomes

$$\begin{aligned} g_{ab} &= d\psi(e_a)\cdot d\psi(e_b) \\ &= \frac{4R^2}{(P^2+R^2)^2}\ \delta_{ab} \quad , \end{aligned} \tag{4.15}$$

which enables us to derive the connection on the frame bundle by evaluating the connection coefficients

$$\begin{aligned} \Lambda^{\mu}_{ab} &= \frac{1}{2}\ g^{c\mu}(\partial_b g_{ac} + \partial_a g_{bc} - \partial_c g_{ab}) \\ &= \frac{-2}{P^2+R^2}\ (\eta_a\delta_{\mu b} + \eta_b\delta_{\mu a} - \eta_\mu\delta_{ab}) \end{aligned} \tag{4.16}$$

The off-diagonal gauge connection can be finally expressed in terms of $SO(N)$ generators

$$(X_{\mu\nu})_{ab} = \delta_{\mu a}\delta_{\nu b} - \delta_{\mu b}\delta_{\nu a}\ , \tag{4.17}$$

and

$$\Lambda^{\mu}_{ab} = i < S_a|\nabla^{\mu}|S_b >$$
$$= \frac{1}{R^2}\ \lambda_n u(X_{\nu\mu})_{ab} \quad . \qquad (4.18)$$

The non-abelian gauge struçture induced by parallel transport of the frame bundle over the parameter manifold is verified.

5. The geometry of the parameter manifold

As we have learned from the previous sections, a degenerate quantum system can provide a frame bundle over the parameter space. Therefore some general aspects of the geometry in the parameter space can be extracted from the knowledge of the connection in the bundle as well as from the curvature tensor. It is the purpose of this section to explore some geometric properties of certain parameter spaces. In order to achieve this goal, we shall establish the following propostions.

Let us consider an orientable compact Riemannian manifold, we have

Proposition 1: Any harmonic form must be a closed form as well as a co-closed one. Furthermore, a harmonic function is always a constant.

Proof: Let a and β be p-forms, and let

$$\Delta = d\delta + \delta d,$$

then

$$< \Delta\alpha, \beta > = < (d6 + \delta d)\alpha, \beta >$$
$$= < \delta\alpha, \delta\beta > + < da, d\beta >$$
$$= < \alpha, \Delta\beta > \quad .$$

For $\alpha = \beta =$harmonic form,

$$0 = < \Delta\alpha, \alpha > = < \delta\alpha, \delta\alpha > + < d\alpha, d\alpha >$$

or $d\alpha = 0,\ \delta\alpha = 0.$

Hence a is closed and co-closed. Furthermore, if α=harmonic zero form, i.e. a harmonic function, then it must be a constant because d a = 0. ○

Proposition 2: A compact Riemannian manifold is simply connected if the corresponding Ricci curvature is positive definite.

Proof: Consider the following integration

$$\int_M R_{ij}\alpha^i\alpha^j dv = \int_M \nabla_i(\alpha^j\ \nabla_j\ \alpha^i)dv - \int_M \nabla_i\alpha^j\nabla_j\ \alpha^i dv \tag{5.1}$$

where cr^i is the component of a harmonic 1-form, then according to Prop.1, d a = 0 implies that $\nabla_j\alpha^i = \nabla_i\alpha^j$. Since the first integration onthe right hand side of eq.(5.1) vanishes, i.e.

$$\int \nabla_i(\alpha^j\ \nabla_j\ \alpha^i)dv = -\int \delta(\alpha^j\ \nabla_j\ \alpha)dv = 0 \tag{5.3}$$

Therefore eq.(5.1) and eq.(5.2) dernand that $a = 0$ identically, and hence $b_1(M) = 0$. The absence of Betti number $b_1(M)$ implies that M is simply connected. ○

A Riemannian manifold M with vanishing covariant derivative of curvature is locally symmetric, and for a simply connected M, locally symmetric also implies globally symmetric[10]. It is known that there exist only four classes of simply connected and symmetric space with positive definite curvature: sphere, CP^N, QP^N and Cayley plane, namely the rank one symmetric spaces, with the sectional curvature taking the values between $\frac{1}{4}$ and 1 except the spheres[10,11]. Let us define

$$D(M) = Sup\{\rho(x,y)|x,y \in M\} \tag{5.4}$$

and quote the following two theorems without proofs;

T1: Minimal diameter theorem[12]: A complete simply connected manifold M, such that $D(M) > \pi$ and $1 \geq K_M \geq \frac{1}{4}$, is homeomorphic to S^N.

T2: Li-Yau theorem[13]: For a compact Riemannian manifold M such that $\partial M = 0$, R_{ij} positive definite, then the first eigenvalue of the Laplacian operator $\lambda_1 \geq \frac{1}{2}(\pi/D(M))^2$.

Now we are in a position to analyse the geometry of the parameter space for the degenerate quantum adiabatic system.

The curvature tensor and the Ricci tensor can be calculated from the gauge connection

$$A^{\mu}_{ab} = i < S_a|\nabla^{\mu}|S_b >$$

and

$$R^d_{abc} = A^d_{ab,c} - A^d_{ac,b} + A^e_{ab}A^d_{ec} - A^e_{ac}A^d_{eb} , \tag{5.5}$$

$$R_{ab} = A^c_{ca,b} - A^c_{ba,c} + A^c_{bd}A^d_{ca} - A^c_{cd}A^d_{ba} . \tag{5.6}$$

The determination of the eigenspectra of the quantum system, together with the Ricci tensor obtained from the set of degenerate wave vectors allows one to test the positivity of curvature. The general properties of the parameter space can then be understood by the previous propositions and theorems.

6. Conclusions and Remarks

The investigation of nonintegrable quantum phase in adiabatic systems by the method of fiber bundle has shown the richness of its mathematical structure in some details. The general aspects of the geometry in parameter manifold can be learned from Berry's connection in the line bundle for the case of a nondegenerate system and in the frame bundle for the case of a degenerate qnantum system. The criteria for classyfying the parameter space are formulated in propositions and theorems for which we provide the proofs to the propositions and leave the proofs of the theorems to the original papers because of the technical involvement and lengthy derivations. For the quanturn system of an electron interacting with a nucleus, the detailed knowledge of the degenerate eigenstates and the precise measurement of the energy spectra would provide some information on the geometry of the parameter space, arid hence the motion of the nucleus.

Appendix A: Calculation of eigenspectra of Laplace-Beltrami operator on S^N

Let $S^N(x) = \{x \in R^{N+1}; |x| = r)$, $S^N(1) = S^N$, and if the N-dimensional spherical coordinate is denoted as $\xi \in S^N$, then

$$x = P(r, \xi) = r\xi$$

where P is the map such that $[0,a] \times S^N \rightarrow R^{N+1}$.

Consider a chart (u, φ) on S^N and express x in terms of the local coordinate, since

$$\frac{\partial x}{r} = \xi \quad ,$$
$$\frac{\partial x}{\partial \varphi^i} = r\frac{\partial \xi}{\partial \varphi^i}$$

We can reduce the Riemannian metric h_{ij} on S^N with respect to the chart (u, φ) by

$$\left\langle \frac{\partial x}{\partial \varphi^{i'}} \frac{\partial x}{\partial \varphi^j} \right\rangle = r^2 h_{ij} \quad .$$

The corresponding Riemannian metric g_{lm} in R^{N+1} with respect to another chart (v, ψ) such that

$$\psi(x) = (r, \varphi(\xi)),$$

is given by:

$$g_{rr} = \left\langle \frac{\partial \psi}{\partial r}, \frac{\partial \psi}{\partial r} \right\rangle = 1$$

$$g_{ri} = 0 \quad \text{rmfor} \quad \mathbf{i} = 1, ...N$$

$$g_{ij} = r^2 h_{ij}$$

and det $g_{ij} = r^{2N}$ det h_{ij}, or $\sqrt{g} = r^N \sqrt{h}$. Consider the function $\phi(r, \xi)$; we calculate

$$\Delta_{R^{N+1}} \phi(r, \xi) = \frac{1}{\sqrt{g}} \sum_{km} \partial_k (g^{km} \sqrt{g} \partial_m \phi)$$
$$r^{-n} \partial_r (r^n \partial_r \phi) + \Delta_{S^N} \phi(r, \xi).$$

For a harmonic function, applying the method of separation of variables,

$$\Delta_{R^{N+1}}\phi = \Delta_{R^{N+1}} r^l W(\xi) = 0$$

can be reduced to

$$\Delta_{S^N} W = -\frac{1}{r^2} l(N + l - 1)W.$$

Therefore the eigenspectra of the Laplace-Beltrami operator is given by the expression

$$E(N,l) = \alpha l(N + l - 1)$$

with

$$\alpha = -\frac{1}{r^2}$$

Appendix B: Calculation of the order of degeneracy.

Let us denote W_l the set of homogeneous polynomials of degrees l in $x_1, ...x_N$, and let

$$H_l = \{f \in W_l | \Delta_{R^N} f = 0\}$$

be the element of the harmonic polynomial of degree l, then the order of degeneracy of the harmonic polynomial of degree 1 is equivalent to the dimension of H_l, namely, the number of linearly independent elements in H_l.

Since $x_1^{p_2} x_2^{p_2} ... x_N^{p_N}$ with $\sum P_i = l$ form the basis of W_l space which has dimension

$$dim\ \ W_l = H_l^N = C_l^{N+l-1} \ \ ,$$

then any W_l can be expressed as

$$F(x_1, ...x_N) = \sum_{k=0}^{l} \frac{x_1^k}{k!} F_k(x_2, ...x_N) \ ,$$

where F_k is the homogeneous polynomial of degree $1 - k$ in $x_2, ...x_N$.

Consider

$$\Delta_{R^N} F = \sum_{k=0}^{l-2} \frac{x_1^k}{k!} F_{k+2}(x_2, ...x_N) + \sum_{k}^{l} \frac{x_1^l}{k!} \Delta_{R^{N-1}} F_k(x_2, ...x_N)$$

the wnishing of $\Delta_{R^N} F = 0$ implies that

$$F_{k+2} = -\Delta_{R^{N-1}} F_k$$

for $0 \leq k \leq l - 2$; therefore $F \in H_l$ can be uniquely determined by F_0 and F_1, i.e.

$$F = F(F_0, F_1)$$

Let $F_0 \in U_l$ and $F_1 \in U_{l-1}$ be the polynomial of degree I and $l - 1$ respectively in $x_2, ...G_N$, and define the linear map ψ_0 and ψ_1 by

$$\psi_0 : U_l \to H_l \text{ or } F_0 \to F(F_0, 0)$$

$$\psi_0 : U_{l-1} \to H_l \text{ or } F_1 \to F(0, F_1)$$

then ψ_0 and ψ_1 are bijective. Therefore we conclude that

$$F(F_0, F_1) = F(F_0, 0) + F(0, F_1),$$

hence we have

$$H_l = \psi_0(U_l) + 41(U_{l-1}.$$

Since

$$dim\ U_l = H_l^N - H_{l-1}^N - C_l^{N+l-1} - C_{l-1}^{N+l-2}$$
$$\frac{(N + l - 2)!}{l!(N - 2)!},$$

one obtains that the order of degeneracy is

$$dim\ H_l = dim\ U_l + dim\ U_{l-1}$$
$$= (N \$21 - 2)(N \$1 - 3)!/l!(N - 2)!$$

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Resumo

A estrutura matemática de um fator de fase geométrico não-trivial em processos qiiânticos adiabáticos é explorada. Alguns aspectos geométricos da variedade

Riemaaniana compacta são analisados, e sua relação com o espaço de parâmetros é discutida.