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Feynman's propagator for an oscillator in a changing magnetic field

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Abstract We evaluate Feynman's propagator exactly for the time-dependent three-dimensional charged harmonic oscillator in a time-varying magnetic field, by solving the Schrõdinger equation through an adequate scale transformation on space and time.

Despite the vast range of operational versatility of Feynman's path-integration, the evaluation of the propagator for certain time-dependent systems, if carried out in a straightforward manner, can become much more difficult than to obtain the solution to the corresponding Schrõdinger equation.

As an example of the state of the art, we point out the recent, formidable calculation of the propagator for the time-dependent forced harmonic oscillator with damping by Cheng[1], via a generalized version of a method introduced by Montroll[2]. In contrast, the exact solution to the corresponding Schrõdinger equation can be found in a miich simpler way[3]. In another illustrative example, the exact evaluation of the propagator for a charged particle in a time-varying electromagnetic field was possible to be carried out only for the case of a constant cyclotron frequency[4].

It would be, therefore, somewhat discouraging to proceed further on applying the afore-mentioned path-integration techniques for other more elabdrated time-dependent problems. Rather, they appeal for alternative versatile methods for the evaluation for propagators without undergoing tedious and lengthy calculations, in such a way as to make Feynman's path-integration aesthetically more attractive[5-9].

In this work, we tackle the problem of a time-dependent three-dimensional charged harmonic oscillator in a time-varying magnetic field through a different approach. We solve directly the corresponding Schrõdinger equation through an adequate change of variable and time reparametrization. The essential ideal[0] in this paper is to make use of the nonlinear superposition law of Ray and Reid[11,12], which is a general procedure to find a global transformation of space and the time by introducing two arbitrary functions, say, $s(\tau)$ and $\mu(\tau)$, where τ is the new time parameter, which will permit us to reduce the original Schrõdinger equation for the standard harmonic oscillator with constant cyclotron frequency and mass. This will be done through a convenient choice of $s(\tau)$ and $\mu(\tau)$.[13]

We begin by writing the Hamiltonian for our system as[9]

$$H(\vec{p},\vec{x},t) = \frac{1}{2m(t)}\left[\vec{p}+\frac{q}{c}\vec{A}(t)\right]^2 + \frac{1}{2}m(t)\omega^2(t)[x^2+y^2+z^2], \tag{1}$$

where the time-varying magnetic field $B(t)$ is applied along the z-axis and the gauge is chosen such that the vector potential $\vec{A}$ is given by $(\frac{1}{2}B(t)y, -\frac{1}{2}B(t)x, 0)$. Then, the corresponding Schrõdinger equation reads

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m(t)}\nabla^2\psi + \frac{\hbar\omega_c(t)}{2i}\left(y\,\frac{\partial}{\partial x} - x\,\frac{\partial}{\partial y}\right)$$
$$+\frac{1}{2}m(t)[\Omega^2(t)(x^2+y^2)+\omega^2(t)z^2]\psi\ , \tag{2}$$

where $\Omega^2(t) \equiv \omega^2(t) + \frac{1}{4}\omega_c^2(t)$, with $\omega(t)$ and $\omega_c(t)$ $[= qB(t)/m(t)c]$ being the harmonic and cyclotron frequencies, respectively.

Let us make the following transformations,

$$x = s(\tau)\bar{x}\ ,\ y = s(\tau)\bar{y}\ ,\ z = s(\tau)\bar{z}\ , \tag{3a, b, c}$$

where r is a single-valued function related to t by

$$\tau(t) = \int^t \mu(\lambda)d\lambda,\ (d\tau(t)/dt = \mu(t)) \ . \tag{4}$$

In order to write the Schrôdinger equation in terrns of the new variables $\bar{x},\bar{y},\bar{z}$ and r, we also have to use the changes in the partial derivatives, that is,

$$\frac{\partial}{\partial t} = \mu\left(\frac{\partial}{\partial\tau} - \frac{s'}{s}\bar{x}\frac{\partial}{\partial\bar{x}} - \frac{s'}{s}\bar{y}\frac{\partial}{\partial\bar{y}} - \frac{s'}{s}\bar{z}\frac{\partial}{\partial\bar{z}}\right) \tag{5a}$$

$$\frac{\partial}{\partial x} = \frac{1}{s}\frac{\partial}{\partial\bar{x}}\ ,\ \frac{\partial}{\partial y} = \frac{1}{s}\frac{\partial}{\partial\bar{y}}\ ,\ \frac{\partial}{\partial z} = \frac{1}{s}\frac{\partial}{\partial\bar{z}}\ , \tag{5b,c,d}$$

where the prime denotes differentiations with respect to the parameter r. Using eqs.(3) and (5) in eq.(1) and making $\hbar = 1$, we obtain

$$\left\{i\mu\left(\frac{\partial}{\partial\tau} - \frac{s'}{s}\bar{x}\frac{\partial}{\partial\bar{x}} - \frac{s'}{s}\bar{y}\frac{\partial}{\partial\bar{y}} - \frac{s'}{s}\bar{z}\frac{\partial}{\partial\bar{z}} - \right) + \frac{1}{2ms^2}\left(\frac{\partial^2}{\partial\bar{x}^2} + \frac{\partial^2}{\partial\bar{y}^2} + \frac{\partial^2}{\partial\bar{z}^2}\right) + \right.$$

$$\left. - \frac{\omega_c}{2i}\left(\bar{y}\frac{\partial}{\partial\bar{x}} - \bar{x}\frac{\partial}{\partial\bar{y}}\right) - \frac{1}{2}m[\Omega^2s^2(\bar{x}^2+\bar{y}^2) + \omega^2s^2z^2]\right\}\phi(\bar{x},\bar{y},\bar{z},\tau) = 0 \tag{6}$$

where the function $\phi(\bar{x},\bar{y},\bar{z},$r) can be regarded as the wave function of the original problem written in terms of the new variables $(\bar{x},\bar{y},\bar{z},\tau)$.

Now, let us make the ansatz[10]

$$\phi(\bar{x},\bar{y},\bar{z},\tau) = \exp[if(\bar{x},\bar{y},\bar{z},\tau)]\chi(\bar{x},\bar{y},\bar{z},\tau) \ . \tag{7}$$

Substituting eq.(7) in eq.(6) we obtain

$$\left\{i\mu\frac{\partial}{\partial\tau} + \frac{1}{2ms^2}\left(\frac{\partial^2}{\partial\bar{x}^2} + \frac{\partial^2}{\partial\bar{y}^2} + \frac{\partial^2}{\partial\bar{z}^2}\right) - \frac{\omega_c}{2i}\left(\bar{y}\,\frac{\partial}{\partial\bar{x}} - \bar{x}\,\frac{\partial}{\partial\bar{y}}\right) + \right.$$

$$\left. - \frac{1}{2}\,ms^2[\Omega^2(\bar{x}^2+\bar{y}^2) + \omega^2\bar{z}^2]\right\}\chi(\bar{x},\bar{y},\bar{z},\tau) +$$

$$i\frac{\partial\chi(\bar{x},\bar{y},\bar{z},\tau)}{\partial\bar{x}}\left(\frac{1}{ms^2}\frac{\partial f}{\partial\bar{x}} - \mu\frac{s'}{s}\bar{x}\right) + i\frac{\partial\chi(\bar{x},\bar{y},\bar{z},\tau)}{\partial\bar{y}}\left(\frac{1}{ms^2}\frac{\partial f}{\partial\bar{y}} - \mu\frac{s'}{s}\bar{y}\right) +$$

$$+i\,\frac{\partial\chi(\bar{x},\bar{y},\bar{z},\tau)}{\partial\bar{z}}\left(\frac{1}{ms^2}\frac{\partial f}{\partial\bar{z}}-\mu\,\frac{s'}{s}\bar{z}\right)+\left\{\frac{1}{2ms^2}\left[i\left(\frac{\partial^2 f}{\partial\bar{x}^2}+\frac{\partial^2 f}{\partial\bar{y}^2}+\frac{\partial^2 f}{\partial\bar{z}^2}\right)+\right.\right.$$

$$-\left.\left[\left(\frac{\partial f}{\partial\bar{x}}\right)^2+\left(\frac{\partial f}{\partial\bar{y}}\right)^2+\left(\frac{\partial f}{\partial\bar{z}}\right)^2\right]\right]+\mu\,\frac{s'}{s}\left(\bar{x}\frac{\partial f}{\partial\bar{x}}+\bar{y}\frac{\partial f}{\partial\bar{y}}+\bar{z}\frac{\partial f}{\partial\bar{z}}\right)+$$

$$\left.-\mu\left(\frac{\partial f}{\partial\tau}\right)-\frac{\omega_c}{2}\left(\bar{y}\frac{\partial f}{\partial\bar{x}}-\bar{x}\,\frac{\partial f}{\partial\bar{y}}\right)\right\}\chi(\bar{x},\bar{y},\bar{z},\tau)=0\;. \tag{8}$$

Now, we will choose f $(\bar{x},\bar{y},\bar{z},\tau)$ in order to have:

$$\frac{1}{ms^2}\frac{\partial f}{\partial\bar{x}}-\mu\frac{\mathrm{SI}}{s}\bar{x}=0\rightarrow f(\bar{x},\bar{y},\Gamma,\mathrm{r})=\frac{1}{2}\mu mss'\bar{x}^2+f_1(\bar{y},\mathrm{r},\mathrm{r}), \tag{9a}$$

$$\frac{1}{ms^2}\frac{\partial f}{\partial\bar{y}}-\mu\frac{s'}{s}\bar{y}=0\rightarrow f(\bar{x},\bar{y},\bar{z},\tau)=\frac{1}{2}\mu mss'\bar{y}^2+f_2(\bar{x},\bar{z},\tau), \tag{9b}$$

$$\frac{1}{ms^2}\frac{\partial f}{\partial\bar{z}}-\mathrm{p}\frac{s'}{s}\mathrm{i}=0\rightarrow f(\bar{x},\bar{y},\bar{z},\tau)=\frac{1}{2}\mu mss'\bar{z}^2+f_3(\bar{x},\bar{y},\tau),\;. \tag{9c}$$

The equations (9a,b,c) lead to the solution

$$f(\bar{x},\bar{y},\mathrm{Z},\mathrm{r})=\frac{1}{2}\mu mss'(\bar{x}^2+\bar{y}^2+\bar{z}^2)+g(\tau)\;, \tag{10}$$

where $g(\tau)$ is an arbitrary function of r still to be determined.

Inserting eq.(9) and eq.(10) in eq.(8) and rearranging terms we obtain

$$\mu\left[i\frac{\partial}{\partial\tau}+\frac{1}{2ms^2\mu}\left(\frac{\partial}{\partial\bar{x}^2}+\frac{\partial}{\partial\bar{y}^2}+\frac{\partial}{\partial\bar{z}^2}\right)-\frac{\omega_c}{2i\mu}\left(\bar{y}\,\frac{\partial}{\partial\bar{x}}-\bar{x}\,\frac{\partial}{\partial\bar{y}}\right)+\right.$$

$$\left.-\frac{1}{8}\frac{ms^2\omega_c^2}{\mu}(\bar{x}^2+\bar{y}^2)\right]\chi(\bar{x},\bar{y},\bar{z},\tau)=$$

$$=\left[\mu\,\frac{dg}{d\tau}-i\,\frac{3}{2}\mu\,\frac{s'}{s}+(\bar{x}^2+\bar{y}^2+\bar{z}^2)\left(\frac{1}{2}\,ms^2\omega^2-\frac{1}{2}\,m\mu^2(s')^2+\right.\right.$$

$$\left.\left.+\frac{\mu}{2}\frac{d}{d\tau}(\mu mss')\right]\chi(\bar{x},\bar{y},\mathrm{i},\tau)\;. \tag{11}$$

Now, we will try to find $g(\tau)$ in such a way that the right-hand side of eq.(11) is zero. Then

$$\mu\frac{dg}{d\tau} = i\ \frac{3}{2}\mu\ \frac{s'}{s}\ , \tag{12a}$$

and

$$\frac{1}{2}\ ms^2\omega^2 - \frac{1}{2}\ m\mu^2(s')^2 + \frac{\mu}{2}\frac{d}{d\tau}\ (\mu mss') = 0\ . \tag{12b}$$

Integration of eq. (12a) leads to

$$g(\tau) = i\ln(s^{3/2})\ , \tag{13}$$

where we have appropriately chosen the constant of integration, and the integration of eq.(12b) leads to

$$\ddot{s} + \frac{\dot{m}}{m}\dot{s}^2 + \omega^2(t)s = 0\ . \tag{14}$$

Now, we are ready to choose the arbitrary function $\mu(\tau)$ in order to reduce the complicated differential equation given by eq. (1i)into a much simpler one, with no time-dependent terms. Then let us make[g]

$$ms^2\mu = M_0\ ,\ M_0 = \text{const}\ . \tag{15a}$$

$$\omega_c(t) = \omega_{0c}\ \mu(t)\ ,\ \omega_{0c} = \text{const}\ ; \tag{(15b)}$$

$$\omega_c(t) = M_0\ \omega_{0c}/s^2m\ . \tag{15c}$$

Substituting eqs. (12) and (15) into eq. (11) we obtain

$$\mu\Big[i\frac{\partial}{\partial\tau} + \frac{1}{2M_0}\Big(\frac{\partial^2}{\partial\bar{x}^2} + \frac{\partial^2}{\partial\bar{y}^2} + \frac{\partial^2}{\partial\bar{z}^2}\Big) - \frac{\omega_{0c}}{2i}\Big(\bar{y}\frac{\partial}{\partial\bar{x}} - \bar{x}\frac{\partial}{\partial\bar{y}}\Big) +$$

$$- \frac{M_0\omega_{0c}}{8}(\bar{x}^2 + \bar{y}^2)\Big]\chi(\bar{x},\bar{y},\bar{z},\tau) = 0\ . \tag{16}$$

This reduces the original problem to the well-known harmonic oscillator in a constant magnetic field with mass and frequency constants, given respectively by M_0 and ω_{0c}. Therefore, the desired solution is given by

$$\psi(\vec{x},t) = \{\exp[if(\bar{x},\bar{y},\bar{z},\tau)]\chi(\bar{x},\bar{y},\bar{z},\tau)\} \quad \begin{array}{l}\bar{x} = x/s(\tau)\\ \bar{y} = y/s(\tau)\\ \bar{z} = z/s(\tau),\ \tau = \tau(t)\end{array} \tag{17}$$

However, let us obtain explicitly the propagator $K(\vec{x},t;\vec{x}_0,t_0)$ of our problem, instead of writing the wave-function $\psi(\vec{x},t)$. The propagator is simply the special solution of the Schrödinger equation for $t > t_0$, subject to the condition

$$\lim_{t\to t_0} K(\vec{x},t;\vec{x}_0,t_0) = \delta(\vec{x}-\vec{x}_0) \ , \tag{18}$$

with

$$\vec{x} \equiv (x,y,z) \quad \text{and} \quad \vec{x}_0 \equiv (x_0,y_0,z_0)$$

This K (kernel or propagator) give us the solution for any arbitrary initial state $\psi(\vec{x}_0,t_0)$:

$$\psi(\vec{x},t) = \int K(\vec{x},t;\vec{x}_0,t_0)\psi(\vec{x}_0,t_0)dx_0dy_0dz_0 \ . \tag{19}$$

Analogously, for the harmonic oscillator in a constant magnetic field, the following also holds

$$\chi(\vec{x},t) = \int_{-\infty}^{\infty} K^B_{\text{h.o.}}(\vec{\bar{x}},\tau;\vec{\bar{x}}_0,\tau_0)\chi(\vec{\bar{x}},\tau_0)d\bar{x}_0d\bar{y}_0d\bar{z}_0 \ , \tag{20}$$

where $K^B_{\text{h.o.}}(\vec{\bar{x}},\tau;\vec{\bar{x}}_0,\tau_0)$ is the respective propagator and $\vec{\bar{x}} \equiv \bar{x},\bar{y},\bar{z}$, $\vec{\bar{x}}_0 \equiv \bar{x}_0,\bar{y}_0,\bar{z}_0$.

By using these results, we have

$$K(\vec{x},t;\vec{x}_0,t_0) = \{\exp[if(\vec{\bar{x}},\tau)]K^B_{\text{h.o.}}(\vec{\bar{x}},\tau;\vec{\bar{x}}_0,\tau_0)\exp[-if^*(\vec{\bar{x}}_0,\tau_0)]\} \ , \tag{21}$$

where $f^*(\vec{\bar{x}}_0,\tau_0)$ means complex conjugate. Substituting eqs. (10), **(13)** into eq. (21) and by using the well-known result of $K^B_{\text{h.o.}}(\vec{\bar{x}},\tau;\vec{\bar{x}}_0,\tau_0)$ [11] we finally obtain the sought propagator K

$$K(\vec{x},t;\vec{x}_0,t_0) = \left(\frac{M_0}{2\pi i\hbar s s_0(\tau-\tau_0)}\right)^{3/2}\left(\frac{\omega_{0c}(\tau-\tau_0)/2}{\sin[\omega_{0c}(\tau-\tau_0)/2]}\right)\times$$

$$\exp\left\{\frac{i}{\hbar}\left[\frac{m\dot{s}}{2s}(x^2+y^2+z^2) - \frac{m_0\dot{s}_0}{2s_0}(x_0^2+y_0^2+z_0^2)\right\}\times\right.$$

$$\exp\left\{\frac{iM_0}{2\hbar}\left[\frac{(\bar{z}-\bar{z}_0)^2}{(\tau-\tau_0)} + \frac{\omega_{0c}}{2}\text{cotg}\frac{\omega_{0c}(\tau-\tau_0)}{2}[(\bar{x}-\bar{x}_0)^2+(\bar{y}-\bar{y}_0)^2+\right.\right.$$

$$\left.\left.+\,\omega_{0c}(\bar{x}_0\bar{y}-\bar{x}\bar{y}_0)]\right\} \quad \begin{array}{c}\bar{x}=x/s(\tau)\\ \bar{y}=y/s(\tau)\\ \bar{z}=z/s(\tau),\ \tau=\tau(t)\end{array}, \qquad (22)$$

where we have brought back $\hbar$ and made the identification

$$s' = \left(\frac{ds}{dt}\right)\left(\frac{dt}{d\tau}\right) = \dot{s}\ ,$$

(the dot (.) here means differentiation with respect to t).

It is interesting to note that $g(\tau)$ is imaginary and thus, $\exp[ig(\tau)]$ is not a phase, but assumes a real value, contributing to the pre-exponential factor. In other words, this term is exactly the jacobian arising from the change of variable in the path integral rneasure when one works with Feynman's formalism.[10]

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Resumo

Neste trabalho, calculamos exatamente o propagador de Feynman para o oscilador harmônico tridimensional dependente do tempo em um campo magnético também dependente do tempo, resolvendo a equação de Schrödinger por intermédio de uma adequada transformação de escala no tempo e no espaço.