Revista Brasileira de Física, Vol. 19, nº 4, 1989

Riemannian approach for gravitational Yang-Mills *equations in the macroscopic case*

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Received May 24, 1989

Abstract A Yang-Mills approach for gravitation is considered here. It shows two dynamical equations, one for curvature and other for torsion. In the Riemannian limit Einstein's equation is obtained.

1. Introduction

An argument to consider a Yang-Mills (YM) classical approach for gravitation is its geometrical analogy with general relativity (GR). If we look for a space-time gauge model for gravitation, it is necessary to investigate the features of space-time gauge-like characteristics. On any differentiable manifold there is the bundle of affine frames, naturally defined, whose structural group is the affine linear group $AL(n, R) = GL(n, R) \times T_n$. For the space-time case, in particular, the requirement of Lorentz frames reduces $AL(n, R)$ to the Poincaré group $P = \mathcal{L} \times T_4$.

However, there is a lot of criticism concerning a YM gravitational model for the Poincaré group. The main point frequently made is that GR does not have the entire Poincaré local symmetry of space-time. Gauge theories for the Poincaré and de Sitter groups have been extensively studied as alternative theories for gravitation[1].

In a previous paper[2] we have shown a YM gravitational model for the Poincaré group, under a Inonu-Wigner contraction of the gauge fields. The absence of metric, because P is not a semi simple group, does not allow one to establish a lagrangian for the theory. Such a problem has been circumvented by means of Lie

* Work supported by CNPq, Brazilian Government Agency.

algebra invariants. This approach turns out to be a de Sitter theory, supplemented by weak constraints.

Stelle and West[3] analysed in detail the local geometrical structure of GR, as a gauge theory for the de Sitter group $SO(3,2)$. To reproduce the structure of Einstein-Cartan theory, the $SO(3,2)$ gauge symmetry was spontaneously broken down to the Lorentz group. In this approach the gravitational vierbein and spin connections were derived from the original $SO(3,2)$ gauge fields, by passing over to a set of non-linearly-transforming fields, through a redefinition involving a Goldstone field. The original $SO(3,2)$ gauge fields generated pseudo-translations and rotations in the so-called internal anti-de Sitter space, under a kind of parallel transport.

Norris et al.[4] proposed an underlying fibre-bundle structure for gauge theories of gravitation, and an extension to an affine structure group. They considered an extension of the linear frame bundle to the affine frame bundle, and pointed out that the torsion is just one part of the "total curvature" (curvature + torsion).

Mielke[5], within the framework of differential geometry, considered a Yang's parallel displacement gauge theory with respect to pure gravitational fields. He showed that, in a four-dimensional Riemannian manifold, double self-dual solutions obey Einstein's vacuum equation with a cosmological term, whereas the doubie anti-self dual configurations satisfy the Raynich conditions of geometrodynamics. Under duality conditions the Stephenson-Kilmister-Yang theory not only embraces $R_{\mu\nu} = 0$, but Nordstrom's vacuum theory as well.

The lagrangian structure of Poincaré gauge field equations for gravitation, and their Einsteinian content, under duality conditions for the sourceless case, is already known[6].

The disagreements to a Poincaré gauge model for gravitational may be justified by two main reasons: it is not a lagrangian theory under gauge-like conditions and if one tries to quantize it, vertices are not well defined[7]. The principal argument to discard GR, as a candidate to a gauge model, is because it is not renormalizable. An amended theory, Iike the de Sitter theory, which is renormalizable in the sourceless case, becomes divergent if one considers the stress-energy $T_{\mu\nu}$ as a source. Moreover, $T_{\mu\nu}$ cannot be a source for a microscopic case, because it is a macroscopic quantity (it describes a distribution of mass-energy).

In this paper our objecive is to deal with YM equations, for the Poincaré group, and show how Einstein's equation can be derived by considering a macroscopic case.

2. Field equations

Poincaré Lie algebra is a vector space, and it is the direct sum of the Lorentz and translation sectors. In a basis with generators $\{J_{ab}, I_c\}$, an affine connection $\bar{\Gamma}$ on the P bundle decomposes into

$$\bar{\Gamma} = \Gamma + S \tag{2.1}$$

where $\Gamma = J_a{}^b \Gamma^a{}_{b\mu} dx^\mu$ is a Lorentz connection form and $S = I_c h^c_\lambda dx^\lambda$ is the solder form[8].

Such a decomposition affects the curvature of $\bar{\Gamma}$

$$\bar{F} = F + T \tag{2.2}$$

where F and T are the curvature and torsion of Γ

$$F = d\Gamma + \Gamma \wedge \Gamma \tag{2.3}$$

$$T = dS + \Gamma \wedge S + S \wedge \Gamma \tag{2.4}$$

The above decomposition of the Lie algebra gives rise to Bianchi identities

$$dF + [\Gamma, F] = 0 \tag{2.5}$$

$$dT + [\Gamma, T] + [S, F] = 0 \tag{2.6}$$

Yang-Mills equations can be written, by duality symmetry, for any group, once its structure constants are known. For the P group they are

$$d * F + [\Gamma, *F] = 0 \tag{2.7}$$

$$d * T + [\Gamma, *T] + [S, *F] = 0 \tag{2.8}$$

stating field equations for the sourceless case. Sources may be inserted in the above equations, by considering a break of such a symmetry. These sources should be the Noether current densities, whose charges are the generators of the P group. Therefore, at a first sight, they could be the density of relativistic angular momentum M and the stress-energy θ :

$$d * F + [\Gamma, *F] = *M \tag{2.9}$$

$$d * T + [\Gamma, *T] + [S, *F] = *\theta \tag{2.10}$$

One notices that torsion is always present in the bundle of frames, and its vanishing must Iead to general relativity.

3. Riemannian limit

The above equations may be projected onto the base-manifold, Minkowski space-time, of the P-bundle, by means of the four-legs h^a_α. Localy this base-manifold is endowed with a Riemannian structure, if we consider a Levi-Civita connection:

$$d * F + [\Gamma, *F] = *M \tag{3.1}$$

$$[S, *F] = *\theta \tag{3.2}$$

which means in components

$$\partial^\lambda \tilde{F}^a_{b\mu\lambda} + \Gamma^{a\lambda}_{c} \tilde{F}^c{}_{b\mu\lambda} - \Gamma^{c\cdot\lambda}_{b} \tilde{F}^a{}_{c\mu\lambda} = M^a_{b\mu} \tag{3.3}$$

$$S^b_\lambda \tilde{F}^{a\cdot\lambda}_{b\cdot\mu} = \theta^a{}_\mu \tag{3.4}$$

where $\tilde{F}$ is the dual of F.

In the dual basis[8] these equations become, in Riemann space-time,

$$\nabla^{\lambda}\tilde{R}^{\alpha}{}_{\beta\mu\lambda} = M^{\alpha}{}_{\beta\mu} \tag{3.5}$$

$$\tilde{R}^{\alpha}{}_{\beta\cdot\mu}^{\cdot\beta} = \theta^{\alpha}_{\mu} \tag{3.6}$$

where $\tilde{R}$ is the dual Riemann tensor, whose components are

$$\tilde{R}^{\alpha\beta\lambda\sigma} = \frac{1}{4}\epsilon^{\alpha\beta\gamma\delta}R_{\gamma\delta\tau\rho}\epsilon^{\tau\rho\lambda\sigma} \tag{3.7}$$

By lowering and raising suffixes, we may write

$$\tilde{R}^{\alpha\beta}{}_{\lambda\sigma} = -\frac{1}{4}\epsilon^{\alpha\beta\gamma\delta}R^{\tau\rho}{}_{\gamma\delta}\epsilon_{\lambda\sigma\tau\rho} \tag{3.8}$$

and contracting with a $= \lambda$ we get

$$\tilde{R}^{\alpha\beta}{}_{\sigma\alpha} = \frac{1}{4}\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha\sigma\tau\rho}R^{\tau\rho}{}_{\gamma\delta} \tag{3.9}$$

Using the property

$$\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha\sigma\tau\rho} = 2(\delta^{\beta}_{\sigma}\delta^{\gamma}_{\tau}\delta^{\delta}_{\rho} + \delta^{\beta}_{\tau}\delta^{\gamma}_{\rho}\delta^{\delta}_{\sigma} + \delta^{\gamma}_{\rho}\delta^{\gamma}_{\sigma}\delta^{\delta}_{\tau}) \tag{3.10}$$

we are led to

$$\tilde{R}^{\alpha\beta}{}_{\alpha\sigma} = -\frac{1}{2}\delta^{\beta}_{\sigma}R + R^{\beta}_{\sigma} = G^{\beta}_{\sigma} \tag{3.11}$$

which are the components of Einstein tensor.

So, eq.(3.5), by contraction $(\alpha = \mu)$, Ieads to

$$\nabla^{\lambda}G_{\beta\lambda} = M_{\beta} \tag{3.12}$$

which'violates the conservation law $\nabla^{\lambda}G_{\beta\lambda} = 0$. We conclude that M cannot be inserted in eq.(2.9) as a sorirce. However, if we take $\theta^{\alpha}_{\mu} = kT^{\alpha}_{\mu}$ (k being a constant) eq. (3.6) becomes Einstein's equation

$$R^{\alpha}_{\mu} - \frac{1}{2}\delta^{\alpha}_{\mu}R = -kT^{\alpha}_{\mu} \tag{3.13}$$

4. Conclusion

The scenario developed here points out that Einstein's equations emerge from a break of dual symmetry of Bianchi's identity for torsion. Such a break cannot be taken for curvature, otherwise the conservation of Einstein's tensor will not be satisfied. Moreover, this approach yields a dynamical equation of the torsion field, for a general connection.

The author is indebted to Prof. R. Aldrovandi for discussions and suggestions given to this paper.

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8. In reality Γ can be defined in many different ways, with S being a horizontal form too, of a more general type. The solder form is particularly convenient

because, when written in a frame given by the four-leg field h^a_α, its components are those of the dual basis. See, for example, S. Kobayashi and K. Nomizu, Foundations of *Differential* Geometry, Insterscience, N.Y. vol. I (1963).

Resumo

Considera-se aqui um modelo de Yang-Mills para a gravitação. Este apresenta duas equações dinâmicas, uma para a curvatura e outra para a torção. No limite Riemaniano as equações de Einstein são obtidas.