Revista Brasileira de Física, Vol. 19, nº 4, 1989

The effective potential of QED, with N ferrnions

L.D. Almeida and A.A. Natale
Instituto de Física Teórica, Universidade Estadual Paulista, Rua Pamplona 145, São Paulo, 01405, SP, Brasil

Received October 4, 1989

Abstract We compute the effective potential for composite operators of 2+1 dimensional quantum electrodynamics with N fermions. Making use of a simple ansatz for the fermionic self-energy, we obtain values for the dynamically generated mass that are in fair agreement with the ones obtained solving the non-linear Schwinger-Dyson equation.

Some years ago Cornwall, Jackiw and Tomboulis[1] derived an effective potential for composite operators (at two-loop level) depending on the complete fermionic propagator $S(p)$. This potential is stationary with respect to variations of $S(p)$, and this condition leads exactly to the Schwinger-Dyson (SD) equation of the fermionic self-energy. Therefore the calculation of this potential at its minimum is equivalent to solving a non-linear gap equation.

The authors of ref. (1) initiated a program of studying chiral symmetry breaking, in such a way that one could recover the non-linearities of the theory if one introduced into the potential a linear solution of the gap equation. Examples of this procedure can be found in ref. (2) and references therein.

Although the extreme condition of the effective potential for composite operators gives the non-linear SD equation, it is far from obvious that we can obtain the exact answer of the full non-linear equation, only by using a linear solution of the gap equation as an ansatz for the computation of the effective potential. As far as we know a comparison of these two different calculations has never been done, and it is the purpose of this work to make this comparison in the case of 2+1 dimensional quantum electrodynamics (QED_3) with N fermions.

QED_3 is a super-renormalizable theory and, in spite of the fact that it is not realistic, it is quite similar to quantum chromodynamics in many respects[3]. Working with N fermions we can make use of the $1/N$ expansion to elaborate a systematic study of chiral symmetry breaking (χSB). The higher-order corrections of the vertex function in the SD equation can be neglected at large N, and the gauge boson acquires a dynamical mass that is calculable to leading order in the $1/N$ expansion. These and other aspects make of this model an arena for reliable investigation of dynamical symmetry breaking. The SD gap equation of QED_3 was solved numerically for small N by Appelquist et al.[4], and a value for $\Sigma(0)/\alpha$ was obtained, where $\Sigma(0)$ is the dynamical mass at the origin and $\alpha = e^2 N/8$. As α has mass dimensionality, $\Sigma(0)/\alpha$ is a dimensionless number which can also be obtained when the potential is minimized, and this will allow us to compare our results with the ones of ref. (4).

Let us briefly review a few aspects of QED_3 with N fermions. The massless lagrangian density is

$$L = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \sum_{j=1}^{N} \bar{\psi}_j \gamma^\nu (i\partial_\nu - eA_\nu)\psi_j \ , \tag{1}$$

where the coupling constant e has dimensionality $\sqrt{M}$. The global chiral symmetry is $U(2N)$ and a mass term $m\bar{\psi}\psi$ would break it to $U(N) \times U(N)$. The gauge-boson propagator in the Landau gauge is

$$D_{\nu\mu}(k) = \frac{g_{\nu\mu} - k_\mu k_\nu/k^2}{k^2[1 + \Pi(k)]} \ , \tag{2}$$

where to leading order in $1/N$ expansion, $\Pi(k)$ is given by

$$\Pi(k) = \alpha/k \ . \tag{3}$$

The inverse euclidean fermionic propagator is

$$S^{-1}(p) = -\gamma_\mu p^\mu [1 + A(p)] + \bar{\Sigma}(p) \ , \tag{4}$$

where $A(p)$ is the wave function renormalization, which is perturbatively generated. Neglecting $A(p)$ to leading order of the $1/N$ expansion and using the lowest order vertex $\Gamma^\mu \approx \gamma^\mu$, the SD gap equation for the fermionic propagator is, after angular integration and defining the new variables $x = p/\alpha$, $y = k/\alpha$ and $\Sigma(x) = \bar{\Sigma}(x)/\alpha$ [4],

$$\Sigma(x) = \frac{4}{\pi^2 N x} \int_0^\infty dy \frac{y\Sigma(y)}{y^2 + \Sigma(y)^2} Ln\left[\frac{x+y+1}{|x-y|+1}\right] . \tag{5}$$

It can be shown that in the infrared region eq. (5) behaves as a constant which, in the sequence, we shall designate as $\Sigma(0)$. If we apply the prescription of Maris, Herscovitz and Jacob[6] and Mandelstam[6] which consists in replacing $x^2 + \Sigma(x)^2$ in the denominator of eq.(5) by $x^2 + \Sigma(0)^2$, expanding the logarithmic function in eq.(5) and differentiating this equation twice, we obtain the equation (for $x < 1$)

$$\frac{d^2\Sigma(x)}{dx^2} + \frac{2}{x}\frac{d\Sigma(x)}{dx} + \frac{8}{\pi^2 N}\frac{\Sigma(x)}{x^2 + \Sigma(0)^2} = 0 , \tag{6}$$

whose unique solution consistent with a finite mass at the origin is

$$\Sigma(x) = \Sigma(0)\,{}_2F_1\left(\frac{1}{4} + \frac{1}{4}\gamma, \frac{1}{4} - \frac{1}{4}\gamma; \frac{3}{2}; -\frac{x^2}{\Sigma(0)^2}\right) , \tag{7}$$

where ${}_2F_1$ is the hypergeometric function and 7 is given by

$$\gamma = \sqrt{1 - \frac{32}{\pi^2 N}} . \tag{8}$$

For $x^2 >> \Sigma(0)^2$ and N $< 32/\pi^2$ the solution of eq.(6) can be reduced to

$$\Sigma(x) \approx \frac{C}{\sqrt{x}} \cos\left[\frac{\omega}{2} Ln(x) + \beta\right] ,$$

where

$$\omega = \sqrt{\frac{32}{\pi^2 N} - 1} . \tag{9}$$

For practical purposes we will use eq.(9) which is much easier to handle than eq. (7). For N $> 32/\pi^2$ only the trivial solution ($\Sigma(x) = 0$) exists[5].

For QED_3 the expression of the effectivepotential for composite operators as defined by Cornwall, Jackiw and Tomboulis[1,4] is given by

$$V = V_0 + V_1 \ , \tag{10}$$

where

$$V_0[\Sigma] = \frac{N}{\pi^2} \int_0^\infty x^2 dx \Big\{ \frac{2\Sigma(x)^2}{x^2 + \Sigma(x)^2} - Ln\Big[1 + \frac{\Sigma(x)^2}{x^2}\Big] \Big\} \ , \tag{11}$$

and

$$V_1[\Sigma] = -\frac{4}{\pi^4} \int_0^\infty dx \frac{x\Sigma(x)}{x^2 + \Sigma(x)^2} \int_o^\infty dy \frac{y\Sigma(y)}{y^2 + \Sigma(y)^2} Ln\Big[\frac{x + y + 1}{|x - y| + 1}\Big] \ , \tag{12}$$

where the angular integration has already been performed. Notice that V_0 and V_1 were divided by α^3, and V_0 (V,) corresponds to the one (two) - loop contribution. It can be seen that

$$\frac{\delta V[\Sigma]}{\delta \Sigma} = 0 \ , \tag{13}$$

gives exactly the SD eq.(5), i.e., its solutions are stationary solutions of the effective potential. In the case when there is more than one non-trivial solution, the one realized in "nature" will be the one leading to the deepest minimum of energy'. This choice can be done by computing the extreme values of eq.(12) which we indicate by $V_e[\Sigma]$. Its expression can be deduced introducing the SD eq.(5) into eq.(10) which entails

$$V_e[\Sigma] = \frac{N}{\pi^2} \int_0^\infty x^2 dx \Big\{ \frac{\Sigma(x)^2}{x^2 + \Sigma(x)^2} - Ln\Big[1 + \frac{\Sigma(1x)^2}{x^2}\Big] \Big\} \ , \tag{14}$$

An important feature of $V_e[\Sigma]$ is that, analysing the sign of the integrand, one finds $V_e[\Sigma] \leq 0$. *It* means that every non-trivial solution of the SD equation is energetically preferred to $\Sigma(x) = 0$ and induces χSB. It is clear that eq.(14) has been deduced assuming that $\Sigma(x)$ is an exact solution of eq.(5). However, in actual calculations we shall have only rough approximations of $\Sigma(x)$, and the better the approximation the smaller will be the difference between eq.(14) and the extremum value of eq. (10).

To compute the effective potential we will make use of the following ansatz for $\Sigma(x)$ in the full momentum range:

$$\Sigma(x) = a\Big\{\theta\Big(1 - \frac{x}{a}\Big) + f\Big(\frac{x}{a}\Big)\theta\Big(\frac{x}{a} - 1\Big)\Big\} , \tag{15}$$

where

$$f(z) = \frac{C}{\sqrt{z}} \cos\left[\frac{\omega}{2} Ln(z) + \beta\right] , \tag{16}$$

where $\theta(x)$ is the step function, $z = x/a$ and $a \equiv \Sigma(0)$. The parameter a will be determined when we compute the minimum of the effective potential for each value of N. Notice that it is the same parameter that is computed in ref. (4), and is a small number which decreases at least as fast as e^{-N}. C and β in eq.(16) are constants that will be determined later, but is is clear that for the continuity of eq.(15) we must impose the condition

$$C\cos(\beta) = 1 . \tag{17}$$

Introducing the ansatz eq.(15) into eq.(10) the one-loop contribution to the effective potential becomes

$$V_0[\Sigma] = \frac{N}{\pi^2}\Bigg\{\int_0^1 z^2 dz\Big\{\frac{2}{z^2+1} - Ln\Big[1 + \frac{1}{z^2}\Big]\Big\} + \\ + \int_1^{1/a} z^2 dz\Big\{\frac{2f(z)^2}{z^2 + f(z)^2} - Ln\Big[1 + \frac{f(z)^2}{z^2}\Big]\Big\}\Bigg\} , \tag{18}$$

and the two-loop one is

$$V_1[\Sigma] = \frac{-8a^2}{\pi^4}\Bigg\{\int_0^1 z^2 dz \int^z dt\frac{t}{t^2+1} + \int_1^{1/a} dz\frac{zf(z)^2}{z^2+f(z)^2}\int^1 dt\frac{t}{t^2+1} + \\ + \int_1^{1/a} dz\frac{zf(z)^2}{z^2+f(z)^2}\int_1^z dt\frac{tf(t)^2}{t^2+f(t)^2}\Bigg\} - Ln\Big[\frac{x+y+1}{|x-y|+1}\Big] . \tag{19}$$

We have restricted tbe calculation of the effective potential to the region $x < 1$ (or $z < 1/a$). The introduction of this cutoff is consistent with the results of Appelquist et al.[4], where it is shown that $\Sigma(x)$ is highly damped for x > 1.

The constants C and β in eq. (16) can be determined if we impose that eq. (9) and eq. (7) must have the same asymptotic behavior. With some simple algebra we see that C and β must satisfy the relation

$$C\sin(\beta) = \mathrm{Im}\left\{\frac{\sqrt{\pi}\,\Gamma\left(\frac{i}{2}\omega\right)}{\Gamma\left(\frac{1}{4}+\frac{i}{4}\omega\right)\Gamma\left(\frac{5}{4}+\frac{i}{4}\omega\right)}\right\} \tag{20}$$

besides the continuity condition given by eq.(17). With these two equations we find the values of C and β contained in table 1.

Table 1 - Values of the constants C ând β appearing in eq.(16).

N	C	β
1.0	1.216	-0.605
1.2	1.289	-0.683
1.4	1.369	-0.752
1.6	1.462	-0.818
1.8	1.571	-0.881
2.0	1.702	-0.943
2.2	1.854	-1.001
2.4	2.087	-1.071
2.6	2.400	-1.141
2.8	2.902	-1.219
3.0	3.937	-1.314

We are now in a condition to compute eqs. (18) and (19) but the presence of the trigonometric function in eq. (16) does not allow us to perform an analytical calculation, even approxirnated, and, in this case, we must do a numerical calculation. In table 2 we present the values of a obtained from the minimization of $V[\Sigma]$ together with the ones of ref.4. Notice that the agreement with the values obtained solving the non-linear equation is excellent, even more if we remember that we started from a rough linear ansatz for $\Sigma(x)$. Table 3 contains the value of the potential $V(a)$ at the minimum, where they are compared to the ones determined from $V_e[\Sigma]$ for different values of N. In table 3 we omit the results N = 2.8 and 3.0 because it was not possible to obtain reliable values without expending a very large computational time (the same happened for the value of a for N = 3.0 in table 2). The critical behavior at $N = 32/\pi^{2/5}$ is the responsible for the slow convergence of this calculation.

Table 2 - Values of $a = \Sigma(0)$ from the minimization of the effective potential and from ref. 4.

N	$-Ln(a)$	$-Ln(a)$ [4]
1.0	2.55	2.3
1.2	3.14	2.9
1.4	3.78	3.6
1.6	4.48	4.3
1.8	5.27	5.1
2.0	6.19	6.1
2.2	7.32	7.2
2.4	8.79	8.6
2.6	11.53	10.7
2.8	13.90	13.8
3.0	–	19.5

Table 3 - Comparison of the values of the effective potential at the minimum.

N	$V[\Sigma]$	$V_e[\Sigma]$
1.0	-1.242×10^{-5}	-1.510×10^{-5}
1.2	-2.626×10^{-6}	-3.243×10^{-6}
1.4	-4.714×10^{-7}	-5.770×10^{-7}
1.6	-6.815×10^{-8}	-8.351×10^{-8}
1.8	-7.298×10^{-9}	-9.038×10^{-9}
2.0	-5.104×10^{-10}	-3.587×10^{-10}
2.2	-1.931×10^{-11}	-2.461×10^{-11}
2.4	-3.408×10^{-13}	-3.364×10^{-13}
2.6	-1.356×10^{-16}	-1.314×10^{-17}

In conclusion, we have computed the effective potential for composite operators in the case of QED_3 with N fermions. Our intention was to make a comparison between the calculation of the dynamically generated mass by solving directly the non-linear Schwinger-Dyson equation for the fermionic propagator, with the determination of the effectivepotential using a linear ansatz for the dynamical mass. We found an excellent agreement between the different approaches to compute the dynamical mass as shown in table 2. It is impressive how powerful the technique of the effective potential is. Even considering that we started from a very rough approximation of $\Sigma(x)$, we arrived to values of a $= \bar{\Sigma}(0)/\alpha$ which do not differ from the ones of Appelquist et al.[4] by more than 5% to 10%. We also recall that the value of $V[\Sigma]$ at the minimum shows a reasonable agreement with the one of $V_e[\Sigma]$, as shown in table 3, providing another check for our calculation. We believe that in QED, as well as in more complex theories the construction of expansions in a linear (and non-perturbative) solution, as in the potential of Cornwall, Jackiw and Tomboulis', may recover the non-linearities of the theory, and the agreement observed here gives support to a large amount of work which adopted this procedure[2].

We are indebted to. Conselho Nacional de Pesquisas (CNPq) (AAN) and Coordenadoria de Aperfeiçoamento de Pessoal do Ensino Superior (CAPES) (LDA) for their financial support.

References

1. J.M. Corwall, R. Jackiw and E. Tomboulis, Phys. Rev. **D10**, 2428 (1974).
2. M. Peskin, in ***Recent Advances in Field Theories and Statistical Mechanics,*** 1982 Les Houches Lectures, edited by J.B. Zuber and R. Stora (North-Holland, Amsterdam, 1984). P. Castorina and S.-Y.Pi, Phys. Rev. **D31,** 411 (1985). A. Barducci et al., Phys. Rev. **D38**, 238 (1988).
3. J.M. Cornwall, Phys. Rev. D22, 1452 (1980).
4. T. Appelquist, M.J. Rowick, E. Cohler and L.C.R. Wijewardhana, Phys. Rev. Lett. 55, 1715 (1985). T. Appelquist, M.J. Bowick, D. Karabali, and L.C.R. Wijewardhana, Phys. Rev. D33, 3704 (1986).

5. T.W. Appelquist, D. Nash and L.C.R. Wijewardhana, Phys. Rev. Lett. **60**, 2575 (1988).L.D. Almeida and A.A. Natale, Europhys. Lett. 9, **321** (1989).
6. Th. A.J. Maris, V.E. Herscovitz and G. Jacob, Phys. Rev. Lett. 12, **13** (1964). S. Mandelstam, Phys. Rev. D20, **3223** (1979).
7. J.C. Montero, A.A. Natale, V. Pleitez and S.F. Novaes, Phys. Lett. **161B**, **151** (1985).

Resumo

Calculamos o potencial efetivo para operadores compostos da eletrodinâmica quântica com N fermions em 2+1 dimensões. Utilizando-se um ansatz bastante simples para a forma da auto-energia fermionica nós obtivemos valores para a massa gerada dinamicamente, os quais estão em bom acordo com aqueles obtidos através da solução da equação não-linear de Schwinger-Dyson.