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# Grassmann's fields and generalized magnetic monopoles

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**Abstract** We present a theory of dual charges with the introduction of a generalized potential and a generalized field which are locally respectively elements of the odd and even parts of the Grassmann algebra of space-time, with values in the Lie algebra of a gauge group $G$. Defining a generalized Dirac operator and its dual, we get the field equations of the theory. When G $= U(1)$ we obtain a theory of electrodynamics with magnetic monopoles without string. We show that the generalized field is invariant under harmonic gange transformations and we obtain Dirac's quantization condition for the dual charges.

## 1. Introduction

It is well known that the generalized Maxwell equations including magnetic monopoles can be written in the language of differential form as:

$$\delta F = J_e \tag{1a}$$

$$dF = *J_m \tag{1b}$$

where $J_e = (p_e, \vec{j}_e)$ is the eletric-current 1-form and $J_m = (p, ,\vec{j}_m)$ is the magnetic current 1-form and F is the usual electromagnetic field 2-form $^*$, is the Hodge star operator and $\delta$ is the Hodge coderivative. For the definitions of these objects see Appendix B.

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If we wish to write the field F in terms of a potential as required by the lagrangian approach, we are brought up against a serious problem, since if $F = dA$, where **A** is a global electromagnetic potential, we have

$$dF = d(dA) = 0 \qquad (\text{since} \quad d^2 = 0)$$

which is incompatible with the second equation (1b).

Differently from Dirac's (introducing singular potentials) and the topological solutions, we propose in this paper an algebraic solution of this problem with a new definition of potential and field that generalizes the usual definitions for abelian gauge groups (electrodynamics).

The idea is simply to consider a potential not as an a 1-form but as a odd element of the Grassmann Algebra of space-time, $\Lambda(M)$. The natural differential operator $\mathbb{D}$ in $\Lambda(M)$ is an obvious generalization of the Dirac operator $d + \delta$ for a general gauge group G. Then we show that the field **F** is nothing but an even element of the same Grassmann Algebra $\Lambda(M)$.

Despite the fact that there are no conclusive evidences for magnetic monopoles, the physical motivation for the present approach is that although, in the principal fiber bundle formulation of electromagnetism and general gauge theories, monopoles appear as solutions of the dynamical equations of theory, there is a price to be paid. For example, for the existence of the Dirac monopole it is necessary to change in the topology of the base manifold of the $U(1)$ bundle from $\mathbb{R}^4$ to $\mathbb{R}^4 - \{\text{line}\}$. Obviously there are no empirical evidences for such a drastic change in the topology of space-time. A complete discusson of these points can be found in refs. [1,2,3].

The present approach is a preliminary step towards the presentation of the completely geometrical theory of the non topological monopole, formulated in **a** spliced-bundle[1].

## 2. General theory

Let $P(M,G,\pi)$ be a principal fiber–bundle over space–time M here considered as a Lorentzian manifold where the metric is taken with signature $(+1,-1,-1,-1)$. Let a and $\alpha'$ be two connections defined in $P(M,\mathrm{G},\pi)$ with values in the Lie–algebra $\hat{G}$ of G, and such that the pull–backs to $M$ are respectively the gauge potentials A and $B$.

**Definition 1.** The Generalized Potential is

$$\omega = A + *B \in \Lambda^1(M,\hat{G}) \oplus \Lambda^3(M,\hat{G}) \tag{2}$$

**Definition 2.** The Generalized Dirac operator associated to $w$ is

$$\mathbb{D}^\omega = D^A + \delta^B \tag{3}$$

where $D^A$ and $\delta^B$ are the usual covariant derivatives and coderivatives of the usual gauge theories with gauge group G [4]. Next, we need

**Definition 3.** The generalized field is given by

$$\begin{aligned}\Omega \stackrel{\circ}{=} \mathbb{D}^\omega \omega \stackrel{\circ}{=} &(D^A + \delta^B)(A + *B)\\ &+ \underbrace{\Omega^A + *R^B}_{2-form} + \underbrace{b^B A}_{0-form} \underbrace{+ D^A(*B)}_{4-form}\end{aligned} \tag{4}$$

where $R^A = D^A A$ and $\Omega^B = D^B B$ as usual.

Eqs. (2) and (3) show that in the general theory the potential is an element of the odd part of the Grassmann algebra of space–time $\Lambda(M,\hat{G})$ and the generalized field is an element of the even part of $\Lambda(M,\hat{G})$.

The first important remark is that contrary to the usual gauge theories[4] here we do not have the validity of Bianchi's identity. Instead we have

$$\mathbb{D}^\omega \Omega = (D^A\delta^B + \delta^B D^A)(A + *B) \tag{5}$$

which is in general different from zero. The tripotential $*B$ allows degrees of freedom to describe a generalized magnetic monopole. As we will see below it is then responsible for the non–integrability of the generalized field R.

In order to present the field equations of the general theory we need

Definition 4. The dual operator to $\mathbb{D}^{\omega}$ is

$$\Delta^{\omega} = *\,\mathbb{D}^{\omega}\,* = *(D^{A} + \delta^{B})* = D^{B} + \delta^{A} \tag{6}$$

We have,

$$\begin{aligned}\Delta^{\omega}\Omega = {} & D^{B}\,\Omega^{A} + D^{B}\,(*\,\Omega^{B}) + D^{B}\,\delta^{B}\,A + D^{B}\,D^{A}\,(*\,B) + \delta^{A}\,\Omega^{A} + \\ & + \delta^{A}\,(*\,\Omega^{B}) + \delta^{A}\,\delta^{B}\,A + \delta^{A}\,D^{A}\,(*\,B)\end{aligned} \tag{7}$$

Eq. (7) can be simplified since as dim $M = 4$ and $A, B \in \Lambda^{1}(M,\hat{G})$ we have identically

$$D^{B}\,D^{A}\,(*\,B) = 0, \qquad \delta^{A}\,\delta^{B}\,(A) = 0 \tag{8}$$

We now introduce the field equations through:

Postulate 1[2] The field equation of the general theory is

$$\delta^{\omega}\,\Omega = J + *\,G \tag{9}$$

where $J$ and $*\,G$ describe the sources of the generalized field.

In what follows we call dual *charges* the charges associated to the current $*G$.

Eq. (9) can be written using eq. (7) and eq. (8):

$$\begin{cases}\delta^{A}\,\Omega^{A} + *\,D^{A}\,\Omega^{B} + D^{B}\,\delta^{B}\,A = J \\ \delta^{B}\,\Omega^{B} + *\,D^{B}\,\Omega^{A} + D^{A}\,\delta^{A}\,B = G\end{cases} \tag{10}$$

[2] **In this paper we have used the opposite signs for the electric and magnetic currents of ref. 1.**

In the usual gauge theory based on a principal fiber bundle we have associated to the connections $\alpha$ and $a'$ the field equations

$$\delta^A \Omega^A = J \qquad \text{and} \qquad \delta^B \Omega^B = G \tag{11}$$

The additional terms on tbe left-hand side of eqs. (10) show that the general theory contains a non–trivial interaction between the potentials $A$ and B, which are represented by interaction currents.

At this point we make an interesting remark: We would like the potentials $A$ and B to be independent from each other, to a certain degree. This is provided by the Generalized Lorentz Gauge:

$$\delta^A B = \delta^B A = 0 \tag{12}$$

This condition implies that $A$ and $B$ are independent from each other (say, in the first derivative) but not the corresponding fields $\Omega^A$ and $\Omega^B$ (second derivative). If we use the condition eq. (12), the equations **(9)** can be written

$$\delta^A \Omega^A + * D^A \Omega^B = J \tag{13a}$$

$$\delta^B \Omega^B + * D^B \Omega^A = G \tag{13b}$$

These equations can be obtained from an spliced bundle formalism in a very elegant way[1].

**Definition 5.** The interaction currents are

$$\begin{cases} J_{\text{int}}^{(A,B)} = * D^A \Omega^B + D^B \delta^B A \\ J_{\text{int}}^{(B,A)} = * D^B \Omega^A + D^A \delta^A B \end{cases} \tag{14}$$

If we use the Generalized Lorentz Gauge eq.(12) we have:

$$J_{\text{int}}^{(A,B)} = * D^A \Omega^B$$

$$J^{(A,B)}_{\text{int}} = *D^B\Omega^A \tag{14'}$$

We can see that in usual electrodynamics:

$$J^{(A,B)}_{\text{int}} = *D^A\Omega^B = *d(dB) = 0$$

since $D^A = d$ for any potential A with values in an abelian group as $U(1)$. This shows that the photon field does not have self–interactions in the present theory as in the usual electrodynamics. The eqs.(14') may have a role in strong interaction theories in which magnetic monopoles are included since they establish a kind of "minimal coupling".

Using Cartan's structural equation $\Omega^a = da + \frac{1}{2}[a,a]$ for $a = A, B$ we can obtain an expression in components for the generalized field. We get

$$\begin{aligned}\Omega^i_{\mu\nu} &= \partial_\mu A^i_\nu - \partial_\nu A^i_\mu - \varepsilon_{\mu\nu\rho\sigma}\,\partial^\rho(B^i)^\sigma + \\ &+ \frac{1}{2}A^k_\mu A^j_\nu C^i_{kj} - \varepsilon_{\mu\nu\rho\sigma}\,\frac{1}{2}(B^k)^\rho(B^j)^\sigma C^i_{kj}\end{aligned} \tag{15}$$

where $[E_k, E_j] = C^i_{kj} E_i$, with $E_i \in \hat{G}$. Eq. (15) is a generalization of the Cabibbo–Ferrari relation[5] for a non–abelian group G.

## 3. Grassmann electrodynamics

When $G = U(1)$ we have $[A,A] = [B,B] = 0$ and $D^w = A^w = d + \delta$ and eq(3) gives

$$\Omega = (dA + *dB) + \delta A + d(*B) \tag{16}$$

The field equations (Postulate 1) then are

$$(d+\delta)\Omega = J + *G \Longrightarrow \Box A = J; \quad \Box B = G, \quad \Box = (d+\delta)^2.$$

The equations $\Box A = J$; $\Box B = G$ are always true in our formalism independently of the "gauge" since in the Grassmann electrodynamics $\Omega$ is the sum of scalar, pseudo-scalar and two-form terms. This is in contrast with the approach of ref [5] where eqs(17) are valid only in the Lorentz gauge. In order to have $\Omega$ a two-form its is necessary to fix the Lorentz gauge for the potentials. Indeed, when $\delta A = \delta B = 0$, eq(16) yields

$$\Omega = dA + * dB \qquad (18).$$

Here we observe that recently Teitelboim and Hennaux[6] presented an electrodynamics including magnetic monopoles where the potential is described by a pform, p $\neq 1$. In *[6]*electric and magnetic charges are extended objects. In our approach charges and monopoles (dual charges) continue to be point like-objects. This is done through the introduction of the generalized potential as element of the odd part of $\Lambda(M,\hat{G})$ and the generalized field as element of the even part of $\Lambda(M,\hat{G})$. Here the Dirac operator D = d + $\delta$ is the natural operator in the sense that

$$D: \underbrace{\Lambda^1(M) \oplus \Lambda^3(M)}_{potentials\ and\ currents} \longmapsto \Lambda^0\underbrace{(M) \oplus \Lambda^2(M) \oplus \Lambda^4(}_{fields}M).$$

This generalization of the concept of field and potential can suggest a new concept for matter fields if we take also into account that the Dirac spinor field can be represented as an element of $\Lambda^0(M) \oplus \Lambda^2(M) \oplus \Lambda^4(M)$ [7]. The theory in the case of electrodynamics can also be formulated in a Clifford bundle[1,9].

## 4. Harmonic gauge invariance of the generalized potential

Recall that in usual electrodynamics the field $F = dA$ is gauge invariant under the gauge transformation

$$A \mapsto A + dg, \quad g: I\!R^4 \to R \quad \text{a differentiable function.}$$

However, the generalized field given by eq(16) is not invariant under arbitrary transformations given by (19). The generalized field is invariant under a more restrict class of transformations, where g is harmonic. Indeed, let

$$A \longmapsto A + dg, \qquad B \longmapsto B + dh \tag{20}$$

We have that

$$\Omega \rightarrow \Omega + \delta dg + * \delta\, d\, h \tag{21}$$

and impossing invariance of the field we get

$$\delta dg = (\delta d + d\delta)g = \Box g = 0; \quad \delta dh = (\delta d + d\delta)h = \Box h = 0 \tag{22}$$

## 5. Quantization condition for the generalized electrodynamics

Here we show that our theory satisfies Dirac's quantization condition under a reasonable condition.

Let $\phi(x, \Gamma')$ be Mandelstam's path dependent wave function[10] for a charged particle in an usual electromagnetic field $F = dA$. If $\phi(x)$ is the usual wave function of the particle we have

$$\phi(x, \Gamma) = \varphi(x) \exp \int_\Gamma^x -i\, eA \tag{23}$$

where $\Gamma$ is an arbitrary path from $\infty$ to x. If we choose two paths $\Gamma$ and $\Gamma'$ differing only by for a finite part we get, using Stokes theorem

$$\phi(x, \Gamma'') = \phi(x, \Gamma) \exp \int_c -i\, edA \tag{24}$$

where $S$ is an arbitrary surface such that $\partial S = \Gamma'' - \Gamma$.

We now want to know how to generalize eq(24) for the case where the charge e interacts with the generalized potential given by eq(16). To start we observe that as $S$ is a bidimensional surface we must use the Lorentz gauge i.e., we put $\delta A = \delta B = 0$ and then $\Omega = dA + * dB$. We next have

Postulate 2. The interaction of an electric charge e with the generalized field is described by the path–dependent wave function $\phi(x,\Gamma)$ which satisfies

$$\phi(x,\Gamma') = \phi(x,\Gamma)\exp\int_S -i\,e\Omega \tag{25}$$

The independence of eq.(25) on the surfaces S implies

$$\exp\oint_{S_0} -i\,e(dA + * dB) = 1 \tag{26}$$

where $S_0$ is a closed surface. By Stokes' theorem we can write eq.(26) as

$$\exp\int_V -i\,e * \delta dB = 1; \quad S_0 = \partial V \tag{27}$$

Supposing now, without loss of generality that the origin is inside V and that we have a static monopole at the origin we have G = $(g\delta(\vec{r}), 0, 0, 0)$ and

$$\Box B = (d\delta + \delta d)B = \delta dB = G \tag{28}$$

Using eq.(28) in eq.(27) we have

$$\exp\int_V -i\,e * G = \exp(-i\,eg) = 1 \Longrightarrow \frac{eg}{4\pi} = n/2, \quad n \in Z \tag{29}$$

We observe here that the new point in order to obtain Dirac's quantization condition $eg/4\pi = n/2$, n $\in Z$ is the postulate 2. This point is not clear in ref [5]. We end this paper with the remark that postulate 2 is consitent with a quantization scheme of the monopole–charge system which give the right equations of motion and from which the Dirac quantization condition can be obtained in a very elegant manner .

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## Appendix A: Some algebras and their relations

Let be V an n–dimensional vector space over the real field IR. In the following we present the definitions and relations between the various algebraic structures used in the paper.

(i) The ***temor algebra*** over *IR* is the R–vector space of the direct sum of the powers $\oplus^p V$ with the usual tensor product $\otimes$ of its elements. We have

$$T(V) = \left(\oplus_{p=0}^{\infty} \otimes^p V, \otimes\right) \tag{A.1}$$

$T(V)$ is $Z$–graded: $(\otimes^p V) \otimes (\otimes^q V) \subset \otimes^{p+q} V$ and has infinite dimension if $n \geq 1$. As V is of finite dimension we can identify V with its image $\otimes^1 V$ in $T(V)$ and also define $\otimes^0 V = IR$.

In $T(V)$ there are two important involutive morphisms (both being linear automorphisms of $\oplus_{p=0}^{\infty} \otimes^p$ V) :

(a) The main automorphism a

$$\begin{cases} a(A \otimes B) = \alpha(A) \otimes \alpha(B) \\ \alpha(A) = A, \text{ if } A \in \otimes^0 V; \;\; \alpha(A) = -A, \text{ if } A \in \otimes' V \end{cases} \tag{A.2}$$

(b) The main antiautomorphism $\beta$ (or reversion), mapping $T(V)$ on the reversed algebra

$$\begin{cases} \beta(A \otimes B) = \beta(B) \otimes \beta(A) \\ \beta(A) = A \text{ if } A \in \otimes^0 V \oplus^1 V \end{cases} \tag{A.3}$$

(ii) The exterior algebra $\Lambda(V)$ is defined as the quotient algebra $T(V)/J$ where $J \subset T(V)$ is the bilateral ideal generated by element of the form $a \otimes a$, $a \in V$. *As* usual we denote the exterior product by A. As $J$ is homogeneous in the $Z$–gradation of $T(V)$ it follows that $\Lambda(V)$ is also $Z$–graded. $\Lambda(V) = \oplus \Lambda^p(V)$, with $\Lambda^p(V) \wedge \Lambda^q(V) \subset \Lambda^{p+q}(V)$, Here we identify $\Lambda^1(V) = V$ and $\Lambda^0(V) = R$. The subspaces $\Lambda^p(V)$ are of dimensions $\binom{n}{p}$ and $\Lambda(V)$ has dimension $2^n$. For $A \in \Lambda^p(V)$ and $B \in \Lambda^q(V)$ the exterior product $A \wedge B$ can be either commutative or anticommutative, i.e.,

$$A \wedge B = (-)^{pq} B \wedge A \tag{A.4}$$

The morphisms a and $\beta$ of $T(V)$ pass to the quotient $\Lambda(V)$.
Denoting these morphisms in $\Lambda(V)$ by the same letters we have

$$\begin{cases} a(A \text{ A } B) = \alpha(A) \text{ A } \alpha(B) \\ \beta(A \text{ A } B) = \beta(B) \wedge \beta(A) \end{cases} \tag{A.5}$$

If $A$ E $\Lambda^p$(V), then

$$\alpha(A) = (-)^p A \quad , \quad \beta(A) = (-)^{\binom{p}{2}} A \tag{A.6}$$

(iii) The Grassmann Algebra $\Lambda(V,$ Q) is the pair $(\Lambda(V),$ Q), formed by an exterior algebra $\Lambda(V)$ an interna1 product, (, ) : $\Lambda(V) * \Lambda(V) \to$ R induced in $\Lambda(V)$ by the fundamental quadratic form Q in V(Q : V $\to$ R) and defined as follows

(a) If $A \in \Lambda^p$(V) and B $\in \Lambda^q$(V), with p $\neq$ q then (A, B) = 0,

(b) If A = $a_1, \wedge a_2$ A ... $\wedge$ a, and $B = b_1$ A $b_2$ A $\cdots \wedge$ b,, $a_i, b_i \in \Lambda'(V)$, then $(A$,B) = $\det(\mathbb{B}(a_i, b_j))$ where $\mathbb{B}$ is the bilinear form associated to Q by

$$2\mathbb{B}(x,y) = Q(x+y) - Q(x) - Q(y) \tag{A.7}$$

(c) When A, B $\in \Lambda(V)$ , (a) and (b) extends by linearity.

## Appendix B: Some vector bundles associated with the cotangent bundle

The algebraic structures considered in A possess an R–linear structure inherited from the vector space V. Then, for its generalization to manifolds it is necessary to use the formalism of vector bundles.

In the text M is a space–time, i.e., a triple $(\mathcal{L}, g,$ V) where $\mathcal{L}$ is a Lorentzian manifold, oriented and time oriented, g is the Lorentz metric and V is the Levi–Civita connection of g in $\mathcal{L}$[11].

(i) The basic vector bundle for all the constructions used in this paper is the cotangent bundle denoted by $\tau_M^*$. Cross sections c $\in$ sec$(\tau_M^*)$, c : M $\to \tau_M^*$ are called one–form fields. The tangent bundle is denoted by $\tau_M$.

(ii) Given a cross section g $\in$ sec$(\tau_M * \tau_M)$, such that in each fibers $\pi^{-1}$(z), g, is a Lorentz metric, then the pair $(\tau_M^*$ ,g) will be called the Lorentzian vector bundle.

(iii) The Cartan bundle, denoted $\Lambda\tau_M^*$ is the vector bundle where the fibers are exterior algebras $\Lambda T_x^*$M over V $= T_x^*$M of the differential forms over M. As is well known, over the Cartan bundle the exterior derivative can be uniquely characterized by the following conditions

$$\text{(a) } d(A + B) = dA + dB,$$

$$\text{(b) } d(A \wedge B) = dA \wedge B + \alpha(A) \wedge dB,$$

$$\text{(c) } d^2 = 0,$$

$$\text{(d) } x \rfloor (df) \equiv x(f) \qquad (B.1)$$

for $\forall A$, B $\in$ sec $\Lambda\tau_M^*$, f $\in \Lambda^0\tau_M^*$ and X E sec $\tau_M$ and where $\rfloor$ is the contraction operator.

In particular d is homogeneous of grade +1 in the $Z$–gradation of the ring of cross sections of $\Lambda\tau_M^* = (\oplus\Lambda^p\tau_M^*$, A).

(iv) The pair $(\Lambda\tau_M^*$ ,g) where each fiber $(\Lambda(T_x^*M)$, g,) is a Grassmann Algebra is called the Hodge bundle over M with metric g. (This is the fundamental bundle for the calculations of the present paper).

(v) Associated to d we have a "divergence" 6, called the Hodge coderivative and that is the operator formally g–adjoint to d. In this paper (differently from ref. [1,7,8,9]) we define

$$\delta\omega_p = *d*\omega_p\,, \quad \omega_p \in \Lambda^p\tau_M^* \qquad (B.2)$$

where $*$ the Hodge star operator is defined by the linear isomorphism

$$* : \Lambda^p\tau_M^* \to \Lambda^{n-p}\tau_M^*$$
$$\psi \quad \to *\psi$$

given by

$$\varphi \wedge *\psi = (\varphi, \psi)\tau \qquad (B.3)$$

for all $p$–forms $\varphi \in \sec \Lambda^p\tau_M^*$ where $\tau$ is the volume n–form.

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## Resumo

Apresentamos uma teoria de cargas duais com a introdução de um potencial generalizado e um campo generalisado que são localmente respectivamente elementos das partes impar e par da álgebra de Grassmann do espaço–tempo com valores na álgebra de Lie do grupo de gauge G. Definindo um operador de Dirac generalizado e seu dual obtemos as equações de campo da teoria. Quando $G = U(1)$ obtemos uma teoria com monopolos magnkticos sem cordas. Mostramos que o campo generalizado k invariante sob transformações de gauge harmonicas e obtemos a condição de quantização de Dirac para as cargas duais.