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# Separable coordinates and particle creation III: Accelerating, Rindler and Milne vacua

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**Abstract** We compare the two vacua associated with accelerating observers to Rindler vacuum and to the Milne vacua, by means of Bogoliubov coefficients. This confirms previous results of the literature according to which two of these vacua are equivalent to Cartesian vacuum, and the three others behave like a thermal gas in the Cartesian vacuum.

## 1. Introduction

In this series of papers we study a massive scalar quantum field in coordinate systems that are not-static but also simple enough to allow the separation of the Klein-Gordon equation. We hope that the understanding of these non-trivial examples will throw some light on the concept of particles outside the frame of the Poincaré group. In the first paper[1] we described the separable orthogonal coordinate systems of the two-dimensional Minkowski space. In the second paper[2] we picked up one of these systems, where stationary observers are inertial in the past and become constant accelerated in the future. These observers are somehow more interesting than uniformly accelerated observers as they allow one to compare two particle definitions in the framework of only one coordinate system: We defined a set of positive frequency modes of the scalar field in the phase where the observers are inertial, and another one in the phase of constant acceleration. We found a thermal spectrum of inertial particles in the accelerated vacuum at a temperature proportional to the asymptotic acceleration of the observers, thus confirming the well known interpretation of the Fulling effect[3] in Rindler coordinates. We also compared these vacua to the cartesian one, that is plane waves, and saw that the accelerated vacuum also has a thermal spectrum at the same temperature but that the inertial vacuum is not thermal.

Here we investigate this problem further, first by trying to sample more information about the modes themselves, i.e., by calculating the Bogoliubov coefficients between them and the natural modes of Rindler and Milne coordinates. These coordinate systems are in a sense more adequate than the cartesian one because they are the right asymptotes of our coordinate system.

In a forthcoming paper we will finally conclude the investigation of this coordinate system by computing more physical magnitudes, like the Feynman propagator and the Hamiltonian. We will then proceed by studying a coordinate system where the observes are inertial in both time asymptotes, only suffering a boost on their velocities.

The paper is organized as follows. In the next section we define all the Fock spaces we will deal with. In section 3 we briefly compare them. The conclusions are obtained in section 4.

## 2. Characterization of the vacuum states

We will study the quantization of a massive scalar neutral field in curvilinear coordinate systems. For that we compare six different Fock spaces which are constructed in the usual manner through the complete function sets defined below.

### 2.1. Minkowski Cartesial modes

$$\psi_k^M(t,x) := \frac{1}{\sqrt{4\pi\epsilon}}\exp[-i\epsilon t - kx)] \tag{1}$$

These are plane waves with mass m, wave vector k and positive frequency $\epsilon = +\sqrt{k^2+m^2}$. The basis is completed with their complex conjugate (which we will omit hereafter). The concept of positive frequency appears in, at least, two ways.

First, we note that $\partial/\partial t$ is a Killing vector field and must have $\psi_k$ as eigenfunction. Its eigenvalue is the frequency

$$i\frac{\partial}{\partial t}\psi_k = \epsilon\ \psi_k \tag{2}$$

We use the additional fact that the Hamiltonian operator is defined by

$$\mathcal{H} = \frac{1}{2}\int_{-\infty}^{\infty}\left[\left(\frac{\partial\phi}{\partial t}\right)^2 + \left(\frac{\partial\phi}{\partial x}\right)^2 + m^2\phi^2\right]dx \tag{3}$$

in such a way that it generates time (t) translations

$$\left[\phi(t,x),\ \mathcal{H}(x)\right] = i\frac{a}{\partial t}\phi \tag{4}$$

Second, we may require that the positive frequency mode go to zero like $\exp(-\epsilon t)$ when $t$ goes to $-$i w. Of course the plane waves obey these two criteria.

## 2.2. Rindler modes

These modes are more easily defined in Rindler coordinates[4], that is

$$\begin{cases} t = X_R & \sinh aT_R \\ x = X_R & \cosh aT_R \end{cases}, \tag{5}$$

for $0 < X_R < \infty$ and $-\infty < T_R <$ w , where we write

$$\psi_\mu^R(T_R, X_R) = \frac{1}{\pi}\sqrt{\sinh\pi\mu}\exp(-i\mu aT_R)K_{i\mu}(mX_R) \tag{6}$$

where $K_{i\mu}$ is a modified Bessel function[5] and $\mu > 0$. These modes were first used by Fulling in his PhD thesis, and have since been well studied[6]. Rindler coordinates are adapted to observers with constant acceleration, in the sence that the coordinate lines describe world lines of constant acceleration. The field $\partial/\partial T_R$ is a Killing vector field, a fact that allowed Sanchez[7] to solve even the inverse problem of finding the coordinate transformation for a given vacuum spectrum. v is the frequency associated to the time $T_R$ in the two previous senses, as one may verify.

### 2.3. Sommerfield modes

These and the next are the modes associated to geodesic observers living in Milne universe[8]. They are indeed more easily written in Milne coordinates

$$\begin{cases} t = Y_M & \cosh aX_M \\ x = Y_M & \sinh aX_M \end{cases} \tag{7}$$

for $0 < Y_M < \infty$ and $-\infty < X_M < \infty$. It is useful to introduce another variable $T_M$, defined by

$$Y_M = \frac{1}{a}\exp(aT_M) \tag{8}$$

This mapping covers the future light cone. In this paper we will be also interested in the past light cone, where

$$\begin{cases} t = -Y_M & \cosh X_M \\ x = -Y_M & \sinh aX_M \end{cases} \tag{7'}$$

and

$$Y_M = \frac{1}{a}\exp(-aT_M) \tag{8'}$$

Here the two frequency definitions split: the time coordinate lines are not trajectories of a Killing vector field and the modes are not eigenfunctions of $\partial/\partial T_M$. Two proceduress can be followed: first we may, like Sommerfield[D], choose the first frequency concept by defining a dilatation operator $D$ as

$$D = \frac{1}{2}\int_{-\infty}^{\infty}\left[\left(\frac{\partial\phi}{\partial T_M}\right)^2 + \left(\frac{\partial\phi}{\partial X_M}\right)^2 + \exp(2aT_M)M^2\phi^2\right]dx \tag{9}$$

such that it generates time translations

$$[\phi(T_M, X_m), D(T_M)] = i\,\frac{\partial\phi}{\partial T_M} \tag{10}$$

If we expand 4 in the basis $(\psi_\nu^S, \psi_\nu^{S*})$, given by

$$\psi_\nu^S(Y_M, X_M) = \frac{-i}{2\sqrt{\sinh\,\pi\nu}}\exp(i\nu aX_M)J_{-i|\nu|}(mY_M) \tag{11}$$

and substitute it in the dilatation operator it becomes, in the light cone

$$D(T_M \to -\infty) \propto 1/2 \int_{\sim}^{\infty} \text{do } |\nu|[\text{a}^{\text{t}}(\nu)a(\nu) + a(\nu)a^{\dagger}(\text{o})] \tag{12}$$

One can check this using the expression

$$, \qquad \lim_{T_M \to -\infty} J_{i\lambda} \propto \frac{\exp(ia\lambda T_M)}{2^{i\lambda}\Gamma(1+i\lambda)} \tag{13}$$

These modes can be called positive dilatation frequency modes.

Second one may, with di Sessa[10], require that the positive frequency mode follows our second criterium. The problem is that these two criteria lead to two different modes $\psi^{s}$. The first is $\psi^{S}$ and the second, $\psi^{D}$, is defined below.

### 2.4. Di Sessa modes

$$\psi_{\rho}^{D}(Y_M, X_M) = \frac{-2}{2\sqrt{2}} \exp(\pi\rho/2) \exp(i\rho a X_M) H_{i\rho}^{(2)}(mY_M) \tag{14}$$

where $H_{i\rho}^{(2)}$ a Bessel function of the third kind. One can verify that the di Sessa condition is satisfied

$$\lim_{Y_M \to -i\infty} \psi_{\rho}^{D} \propto H_{i\rho}^{(2)}(-im\infty) = 0 \tag{15}$$

### 2.5. **Inertial** modes

The next two modes were defined in paper II of this series and are adapted to observes which become smoothly accelerated. The coordinates are defined as

$$\begin{cases} \text{t} + x = 2/a \sinh a(T_A + X_A) \\ \text{t} - x = -1/a \exp[-a(T_A - X_A)] \end{cases} \tag{16}$$

We can see that the proper acceleration

$$\alpha = \sqrt{g_{\nu\mu} \frac{D^2\xi^{\mu}}{Ds^2} \frac{D^2\xi^{\nu}}{Ds^2}} \tag{17}$$

of a stationary observer parameterized by $\xi^{\mu} = (T_A, X_A)$ is given by so that

$$\alpha_{A\infty} := \lim_{T_A \to \infty} \alpha_A = \exp(-aX_A).$$

$$\alpha_A = \frac{a\exp(2aX_A)}{\left[\exp(-2aT_A) + \exp(2aX_A)\right]^{3/2}} , \tag{18}$$

To write the modes it is better to use the variables $(Y_A, Z_A)$

$$\begin{cases} Y_A = 1/a\exp(-aT_A) \\ Z_A = 1/a\exp(+aX_A) \end{cases} \tag{19}$$

The modes are then $\psi^I_\sigma$ and $\psi^A_\tau$. We define $\psi^I_\sigma$ as

$$\psi^I_\sigma(Y_{A'}, Z_A) := \frac{1}{2}\frac{\sqrt{\sigma(1-\exp(-2\pi\sigma))}}{\pi} H^{(1)}_{i\sigma}(mY_A)K_{i\sigma}(mZ_A) \tag{20}$$

This first mode has a quasi-classical behavior in the asymptotic region. By quasi-classical, a concept that, in this context, was exactly defined in a previous paper[11], we mean

$$\lim_{Y_A \to m} \psi^I_\sigma \ \alpha \ \exp(-i\sigma mY_A) \tag{21}$$

These modes have positive frequency following di Sessa's criterium.

### 2.6. Accelerating modes

These are defined as

$$\psi^A_\tau(Y_A, Z_A) := \sqrt{\tau/\pi}J_{i\tau}(mY_A)K_{i\tau}(mZ_a) \tag{22}$$

and are quasi-classical in the accelerated region

$$\lim_{Y_r \to 0} \psi^A_\tau \propto Y_A^{i\tau} \tag{23}$$

They satisfy the Sommerfield definition of positive frequency.

## 3. Comparison of the vacua

Now that all modes we are interested on have been weel defined, we proceed to their comparison. It is well known that

$$| < \psi_k^M , \psi_\mu^R > |^2 = \frac{a^2}{2\pi\epsilon} \frac{1}{\exp(2\pi\mu) - 1} \tag{24}$$

Here it is possible to talk about a temperature because the equivalence principie allows one to compare the temperature measured in an inertial system with the temperature measured in the proper frame of the observer

$$\Theta = \sqrt{g}_{00} \Theta_0 \tag{25}$$

so that the temperature is proportional to the proper acceleration $(\alpha_R = 1/X_R)$

$$\Theta = \alpha_R / 2\pi \tag{26}$$

We also know the distributions of Milne particles in the Minkowski vacua

$$| < \psi_k^M , \psi_\nu^S > |^2 = \frac{1}{4\pi\epsilon} \frac{1}{\exp(2\pi\nu) - 1} \tag{27}$$

and

$$| < \psi_k^M , \psi_\nu^D > |^2 = 0 \tag{28}$$

For Sommerfield quantization, the Milne universe behaves as a big bang, where the temperature O is given by

$$\Theta = \frac{1}{2\pi} \exp(-aT_M) \tag{29}$$

so that as $T_M \to -\infty$, $\Theta \to \infty$, and as $T_M \to +\infty$, O $\to$ 0. The di Sessa modes, on the contrary, lead us to zero temperature.

In a previous paper we calculated the Bogoliubov coefficients between Minkowski and $\psi_\tau^A$-modes and between Minkowski and $\psi_\sigma^I$-modes

$$| < \psi_k^M , \psi_\tau^A > |^2 = \frac{1}{2\pi\epsilon a} \frac{1}{\exp(2\pi\tau) - 1} \tag{30}$$

(note that the associated proper temperature is proportional to ***a,)*** and

$$| < \psi_k^M , \psi_\sigma^I > |^2 = \sinh^{-3} \pi\sigma \frac{\sigma}{\epsilon a} Re^2 \left[ \frac{((\epsilon - k)/2a)^{-i\sigma}}{(-i\sigma)a} \right] \ . \tag{31}$$

This resonates in $(\epsilon - k)/2a$ and $a$ for $\epsilon >> k$ and goes to zero for $\epsilon \sim k$.

We also know the relationships

$$|<\psi^S_\nu, \psi^D_\sigma>|^2 = \frac{1}{\exp(2\pi\nu) - 1}\delta(\nu - \sigma) \tag{32}$$

and

$$|<\psi^A_\tau, \psi^I_\sigma>|^2 = \frac{1}{\exp(2\pi\sigma) - 1}\delta(\tau - \sigma) \tag{33}$$

The new results we are presenting in this paper come from the comparison of the accelerating vacua with Rindler and Milne ones. They are

$$|<\psi^R_\mu, \psi^A_\tau>|^2 = 0 \tag{34}$$

This is because the proper accelerations of gaussian observers coincide in the asymptotic region where both modes are quasi-classical. Secondly, we have

$$|<\psi^R_\mu, \psi^I_\sigma>|^2 = \frac{1}{\exp(2\pi\sigma) - 1}\delta(\tau - \sigma) \tag{35}$$

Finally we have

$$|<\psi^S_\nu, \psi^A_\tau>|^2 = 0 \tag{36}$$

and

$$|<\psi^D_\sigma, \psi^I_\sigma>|^2 = 0 \tag{37}$$

## 4. Conclusion

By drawing a diagram, one sees that the present concept of temperature may be seen as an equivalence relation.

$$\begin{array}{ccccc}
\boxed{\text{SOMMERFIELD}} & \Delta\theta = 0 & \boxed{\text{ACCELERATED}} & \Delta\theta = 0 & \boxed{\text{RINDLER}} \\
| & & | & & | \\
\Delta\Theta \neq 0 & & \Delta\Theta \neq 0 & & \Delta\Theta \neq 0 \\
| & & | & & | \\
\boxed{\text{DI SESSA}} & \Delta\theta = 0 & \boxed{\text{INERTIAL}} & \Delta\theta = ? & \boxed{\text{MINKOWSKI}}
\end{array}$$

The only exception is the Bogoliubov coefficient between the inertial and Minkowski modes. Castagnino[12] suggested that this result is due to the fact that the associated observers' fluid is not rigid. Notice that we have not computed the Bogoliubov coefficients between Sommerfield and inertial modes, not between Di Sessa and accelerated modes, because the positive frequency definitions are incompatible so the calculation would be senseless.

In a forthcoming paper we will check these results by calculating the Feynman propagator, Wightmann functions and the Hamiltonian which are needed to construct a detector.

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**Resumo**

Neste trabalho comparamos dois vácuos definidos por observadores assintoticamente acelerados com o vácuo de Rindler e os vácuos de Milne, através do cálculo dos coeficientes ou Bogoliubov. Estes cálculos confirmam a conjectura que afirma que dois destes vácuos são equivalentes ao vácuo associado a um observador inercial que utiliza coordenadas cartesianas enquanto os outros três se comportam como um estado térmico relativo a este vácuo "inercial".