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Fermionization of strings, and their conformal invariant solutions

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Abstract We discuss the ferrnionic description of bosonic string theory, which turns out to be a Thirring model. The relationship of continuous spin to compactification is discussed, and regular solutions with finitely many fields can be found if the spin is a rational number. The relation between Wess-Zumino-Witten theory and $SU(n)$ Thirring rnodel is also treated.

1. Introduction

The main ingrediente in the modern description of string theory implying its relevance to the unification of all interactions is reparametrization invariance[1]. In two-dimensional space time it is described by the conformal group, which in this case is infinite dimensional, the generators being the Laurent coefficients of the expansion of the energy momentum components - for further details see ref.2.

The description of strings in terms of fermions by means of bosonization techniques has been recently discussed[3]. As a result, different formulations of the Thirring models arose, some of them presenting twisted boundary conditions. We shall discuss these results in terms of conformal field theory. We also draw attention to the relation between the conformal invariant $SU(n)$ Thirring model solution[4a] and the chiral Wess-Zumino results of ref. 5.

In section 2 of the paper we set the problem in the abelian case, and write down the relevant bosonization formulae. The role of spin is discussed, as well as compactification constraints.

In section 3 the $SU(n)$ Thirring model and W.Z.W. theory are studied together with their solutions, which can be written pairwise. At the end of the section we treat the case of bound states, which are relevant to impose compactification constraints (equivalent to Gliozzi-Scherk- Olive projection)[15]. Section 4 closes the paper with conclusions, and an outlook of further ideas.

2. Abelian fermionization

We consider a bosonic string propagating on a torus T^d, often comparing with the simple case d = 1, namely the circle S^1. We do not take into account flat space components, which in this discution remain as passive bystanders. It is convenient to treat the tori as Riemann surfaces described by complex coordinates z and $\bar{z}$, which are related to the usual euclidean string variables a and r by

$$z = \exp(i(\sigma + i\tau))$$

$$\bar{z} = \exp(-i(\sigma - i\tau))$$

The action is the usual quadratic one, supplemented by a Wess-Zumino term which we rewrite as an antisymmetric tensor field (torsion) and reads

$$S = \frac{1}{2\pi}\int dz d\bar{z}\{\partial_z X^a \partial_{\bar{z}} X^a + B^{ab}\partial_z X^a \partial_{\bar{z}} X^b\} \tag{2.1}$$

where a = 1,...,d and B^{ab} is the antisymmetric tensor field.

Later on, we will discuss the existence or not of a conformally invariant theory in general manifolds.

In the following, conformal invariance in the quantum theory is taken for granted as a result of the antisymmetric field interaction[6,7]. Therefore left and right movers split. They are functions of z and $\bar{z}$ respectively.

We have the following mode expansion for the bosonic X^a_z and $X^a_{\bar{z}}$ fields[8].

$$X_z^a(z) = X_{0z}^a + \frac{i}{2}p_{0z}^a \ln z + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\alpha_{-n}^a z^{-n} \tag{2.2}$$

and

$$X_{\bar z}^a(\bar z) = X_{0\bar z}^a - \frac{i}{2}p_{0\bar z}^a \ln \bar z + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\tilde\alpha_{-n}^a \bar z^{-n} \tag{2.2a}$$

Let us restrict ourselves, for a moment, to the case d $= 1$. The field $X(z,\bar z)$ is the sum of right and left mover fields. We may also define the difference field

$$\tilde X(z,\bar z) = X(z) - \bar X(\bar z) \tag{2.3}$$

The following commutation relations are a consequence of the Dirac bracket quantization of the zero modes[0]

$$[P_{0z}, X_{0z}] = -\frac{i}{2} \tag{2.4a}$$

$$[P_{0\bar z}, X_{0\bar z}] = -\frac{i}{2} \tag{2.4b}$$

We have also the usual cornmutation relation for the creation and annihilation operators

$$[\alpha_n, \alpha_m] = n\delta_{n.-m} \tag{2.5}$$

Contraction of the string field is given by

$$< X(z)X(w) >= -\frac{1}{4}\ln(z-w) \tag{2.6a}$$

Whenever no confusion arises, $X^a(z)$ stands for $X_z^a(z)$.

This is also the first term in a Wilson expansion for the product of two X fields, which we write as

$$X(z)X(w) \sim -\frac{1}{4}\ln(z-w) \tag{2.6b}$$

In the above formulae we may assume a symmetry group $U(1)^d \otimes U(1)^d$. The non abelian generalization will generate a Kac-Moody algebra. For the time being we discuss further the generalized abelian case, namely a torus. The only difference with respect to the previous case is a kronecker delta on the right hand side of eq.(2.4).

We define the currents for the string model as

$$J^a(z) = \frac{i}{\sqrt{\pi}} \frac{\partial}{\partial z} X^a(z) \tag{2.7a}$$

$$\bar{J}^a(\bar{z}) = \frac{-i}{\sqrt{\pi}} \frac{\partial}{\partial \bar{z}} \bar{X}^a(\bar{z}) \tag{2.7b}$$

They have the following Wilson expansion

$$J^a(z) J^b(w) = \frac{1}{4\pi} \frac{1}{(z-w)^2} \delta^{ab} \tag{2.8}$$

This corresponds, in the usual Minkowski formulation to a current

$$J^a(x_+) = z\, J^a(z)\big|_{\tau \to i\tau} \tag{2.9}$$

where x_+ is the Minkowski left moving coordenate $(x_+ = \tau + a)$.

The commutator of the currents above defined reads

$$[J^a(x_+), J^b(y_+)] = \frac{i}{2} \delta^{ab} \delta'(x_+ - y_+) \tag{2.10}$$

Further Wilson expansions can be easily obtained and read

$$J^a(z) X^b(w) = -\frac{i}{4\sqrt{\pi}} \frac{1}{z-w} \delta^{ab}$$

$$\bar{J}^a(\bar{z}) \bar{J}^b(\bar{w}) = +\frac{1}{4\pi} \frac{1}{(\bar{z}-\bar{w})^2} \delta^{ab} \tag{2.11}$$

The energy momentum tensor is of the Sugawara form

$$T(z) = 2\pi : J(z)^2 :$$

$$\bar{T}(\bar{z}) = 2\pi : \bar{J}(\bar{z})^2 : \tag{2.12}$$

The following expansion hold for the products involving the energy-momentum tensor

$$T(z)J^a(w) = \frac{J^a(z)}{(z-w)^2} \tag{2.13a}$$

$$\bar{T}(\bar{z})\bar{J}^a(\bar{w}) = \frac{\bar{J}^a(2)}{(\bar{z}-\bar{w})^2} \tag{2.13b}$$

$$T(z)T(w) = \frac{d}{2(z-w)^4} + \frac{T(z)+T(w)}{(z-w)^2} \tag{2.13c}$$

$$T(\bar{z})T(w) = \frac{d}{2(\bar{z}-\bar{w})^4} + \frac{T(\bar{z})+T(\bar{w})}{(\bar{z}-\bar{w})^2} \tag{2.13d}$$

The above relations allow one to obtain the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{m,-n} \tag{2.14}$$

where

$$L_n = \oint dz z^{n+1} T(z)$$

Expanding in eq.(2.13a) $J^a(z)$ around w, one obtains

$$T(z)J^a(w) = \frac{J^a(w)}{(z-w)^2} + \frac{\partial_w J^a(w)}{z-w} \tag{2.15}$$

which tells us that J^a has canonical dimension, being conserved in the quantum theory.

It is possible to define creation and annihilation operators for the current mode expansion, obtaining the bosonic string spectrum on a torus.

Bosonization techniques are a very efective means to simplify field expression relating fermionic composite operators to bosonic fundamental fields and vice-versa. A possible starting point for the bosonization procedure is the observation that, given the Fourier representation for the massless fermionic field, which in the d = 1 case reads (for the abelian multi flavor case d # 1, the modífications are trivial)

$$\psi(x) = \int \frac{dk^1}{\sqrt{2\pi}}\{a^+(k^1)e^{ikx} + b(k^1)e^{-ikx}\}u(k) \tag{2.16}$$

where

$$u(k) = \begin{pmatrix} \theta(-k^1) \\ \theta(k^1) \end{pmatrix}$$

and

$$\{a(k^1), a^+(k'')) = \{b(k^1), b^+(k'')) = \delta(k^1 - k'')$$

one can compute the conserved current

$$j_\mu(x) =: \bar{\psi}(x)\gamma_\mu\psi(\bar{z}) : \tag{2.17}$$

obtaining its Fourier decomposition as[11]

$$j_\mu(x) = -\frac{i}{2\pi}\int dk \frac{k_\mu}{\sqrt{k_0}}\{C(k^1)e^{-ikz} - C^+(k^1)e^{ikz}\} \tag{2.18}$$

where

$$C(k) = \frac{1}{\sqrt{k_0}}\int dp^1 \{\theta(kp)\left[b^+(p)b(p+k) - a^+(p)a(p+k)\right] + \theta(p(k-p))a(k-p)b(p)\} \tag{2.19}$$

obeys canonical commutation relations, implying that j_μ(x) is the derivative of a free massless scalar field

$$j_\mu(z) = \frac{1}{\sqrt{\pi}}\partial_\mu J(x) \tag{2.20}$$

The above result has far-reaching consequences for the solution of the Thirring model, defined by the lagrangian density

$$\mathcal{L} = \bar{\psi}_i \; \partial\!\!\!/\psi + \frac{1}{2}g\bar{\psi}\gamma^\mu\psi\bar{\psi}\gamma_\mu\psi \tag{2.21}$$

whose formal equations of motion are

$$i\gamma^\mu\partial_\mu\psi = -gJ^\mu\gamma_\mu\psi \tag{2.22}$$

where $J_\mu = \bar{\psi}\gamma_\mu\psi$ is the current, which satisfies two conservation laws: namely its divergence and curl are zero.

We can write, in two dimensions

$$J_\mu(x) = \frac{1}{\sqrt{\pi}}\partial_\mu J(x) \tag{2.23}$$

Considering now the following field

$$\chi(x) = \exp\left\{ -\frac{ig}{\sqrt{\pi}}J(x)\right\}\psi(x) \tag{2.24}$$

one readily verifies that

$$\gamma^\mu \partial_\mu \chi(x) = 0 \tag{2.25}$$

and also

$$J_\mu(x) = \bar{\chi}(x)\gamma_\mu \chi(x) \tag{2.26}$$

therefore $J(x)$ is a two dimensional free massless field. The quantum operator $\psi(x)$ can be defined as

$$\psi(x) = \exp\left\{\frac{ig}{\sqrt{\pi}}J^{(+)}(x)\right\}\chi(x)\exp\left\{\frac{ig}{\sqrt{\pi}}J^{(-)}(x)\right\} \tag{2.27}$$

As a matter of fact the whole theory can be rewitten in terms of a massless bosonic field. In order to understand what happens, consider exponentials of free massless fields in two dimensions

$$\begin{aligned}\psi_{\alpha'\beta'}(x) &=: \exp i\big(\alpha'\phi(x) + \beta'\tilde{\phi}(x)\big) : \\ &= \exp i\big(\alpha'\phi^{(+)} + \beta'\tilde{\phi}^{(+)}\big)\exp i(\alpha'\phi^{(-)} + \beta'\phi^{(-)})\end{aligned} \tag{2.28}$$

where

$$\partial_\mu \phi = -\epsilon_{\mu\nu}\partial^\nu \tilde{\phi} \tag{2.29}$$

Notice that $\psi_{\alpha',\alpha'} = \psi_{\alpha',\alpha'}(x_+)$ and $\psi_{\alpha',-\alpha'} = \psi_{\alpha',-\alpha'}(x_-)$, and we rewrite the above fermion field as

$$\psi_{\alpha,\beta}(x) =: \exp(i(\alpha\chi(x_+) + \beta\chi(x_-))) : \tag{2.30}$$

where

$$[\chi^-(x), \chi^+(y)] = -\frac{1}{4\pi}\ln(x-y)$$

Products of the exponential fields generate local functions of $\chi(x_+)$ or $\chi(x_-)$. We have

$$\begin{aligned}\psi^+_{\alpha,0}(x+\epsilon)\psi_{\alpha,0}(x) &= \exp\left(-i\alpha\chi^{(+)}(x+\epsilon)\right)\ \exp\left(-i\alpha\chi^{(-)}(x+\epsilon)\right) \\ &\times \exp\left(i\alpha\chi^{(+)}(x)\right)\ \exp\left(i\alpha\chi^{(-)}(x)\right)\end{aligned} \tag{2.31}$$

joining the creation operators on the left (resp. annihilation to the right) we have

$$\begin{aligned}\psi^+_{\alpha,0}(x+\epsilon)\psi_{\alpha,0}(x) &= \exp\left(-i\alpha(\chi^{(+)}(x+\epsilon) - \chi^{(+)}(x))\right) \\ &\times \exp\left(-i\alpha(\chi^{(-)}(x+\epsilon) - \chi^{(-)}(x))\right) \\ &\times \exp\left(\alpha^2[\chi^{(-)}(x+\epsilon), \chi^{(+)}(x)]\right)\end{aligned} \tag{2.32}$$

Now we expand the first two exponentials to get

$$\psi^+_{\alpha,0}(x+\epsilon)\psi_{\alpha,0}(x) = (\epsilon_-)^{-\frac{\alpha^2}{4\pi}}\left(1 - i\alpha\epsilon_+\frac{\partial}{\partial\tilde{x}_+}\chi(x_+)\right) \tag{2.33}$$

Thus we define

$$J_+(x) = N(\psi^+(x)\psi(x)) = \frac{\alpha}{\pi}\partial^+\chi(x_+) \tag{2.34}$$

We have also

$$N(\psi^+(x_+)\partial^+\psi(x_+)) = \frac{\alpha^2}{\pi}(\partial^+\chi(x_+))^2 \tag{2.35}$$

In the Euclidean space we have the usual substitution $\exp(-ix_+) \to z$ and $\exp(-ix_-) \to \bar{z}$.

The solution of the Thirring model is written as follows

$$\psi(x) = \begin{pmatrix} \psi_{\alpha\beta}(x) \\ \psi_{\beta\alpha}(x) \end{pmatrix} \tag{2.36}$$

where $\beta = g/\sqrt{\pi}$. In terms of these fields, the Euclidean equations of motion are

$$\partial_{\bar{z}}\psi(z) = g\ J(z)\psi(z) \tag{2.37a}$$

$$\partial_{z}\psi(\bar{z}) = g\ J(\bar{z})\psi(\bar{z}) \tag{2.37b}$$

We obtain now the right.left moving components of the energy-momentum tensor

$$T(x_{\mp}) = i\psi^{\dagger}_{\frac{1}{2}}(x)(\partial_0 \pm \partial_1)\psi_{\frac{1}{2}}(x), \tag{2.38}$$

which due to eq.(2.34), and eq.(2.35) are of the Sugawara form, and in Euclidean formulation

$$T(z)[\bar{T}(\bar{z})] = 2\pi : J(z)^2 :\ [2\pi : \bar{J}(\bar{z})^2 :] \tag{2.39}$$

Comparing with the bosonic case we see a complete isomorphism between the currents and the energy - momentum tensor of the corresponding theories[12]. Therefore we get the Virasoro algebra. We identify the above bosonic field for the Thirring model, namely $\chi(x_+)$ and $\chi(x_-)$ with the corresponding left moving bosonic fields.

The short distance expansions in the Thirring model are

$$T(z)\psi_{\alpha,\beta}(w,\bar{w}) = \frac{\pi\alpha}{z-w} : J(z)\psi_{\alpha,\beta}(w,\bar{w}) : - \frac{i\alpha^2}{4(z-w)^2}\psi_{\alpha,\beta}(w,\bar{w}) \tag{2.40a}$$

$$\bar{T}(\bar{z})\psi_{\alpha,\beta}(w,\bar{w}) = \frac{\pi\beta}{z-w} : \bar{J}(\bar{z})\psi_{\alpha,\beta}(w,\bar{w}) : - \frac{i\beta^2}{4(z-w)^2}\psi_{\alpha,\beta}(w,\bar{w}) \tag{2.40b}$$

For a spin $-\lambda/2$ field in the Thirring model we have

$$\alpha^2 - \beta^2 = 4\lambda \tag{2.41}$$

and in order that 91, be a solution of the Thirring model we must have

$$\beta = g\sqrt{\pi} \tag{2.42}$$

We conclude that ψ above, in terms of the string field, is a two-parameter family of solutions to the Thirring model. We have to consider shifts in the string variables.

All results may be readily generalized to the $U(1)^d$ case, considering the fermionic action

$$\frac{1}{2\pi}\int d^2\xi\left[i\bar{\psi}\gamma^\mu\partial_\mu\psi - H_{ij}\psi_1^{i+}\psi_1^i\psi_2^{+j}\psi_2^j\right] \tag{2.43a}$$

with fields equations

$$\frac{\partial}{\partial x_+}\psi_1^i = -2\pi i F_{ai}J_+^a(x_+)\psi_1 \tag{2.43b}$$

$$\frac{\partial}{\partial x_-}\psi_2^i = -2\pi i K_{ai}J_-^a(x_-)\psi_2 \tag{2.43c}$$

where $H_{ij} = F_{ai}K_{aj}$.

Notice that there is no relation among the various $U(1)$ coupling constants.

The right and left movers and Euclidean holomorphic and antiholomorphic position operators are given by eq.(2.2) and eq.(2.2a).

The position operator is obtained by adding the two expressions above, while $\tilde{X}$ is the difference field eq.(2.3).

In a compactified space, we have symmetries associated to the period of non-trivial closed orbits; in the present case we shall, for the time being restrict ourselves to compactification and a d-dimensional torus, with a common radius R. Thus we have for each $X^a(z,\bar{z})$ a symmetry

$$X \to X + 2\pi R \tag{2.44}$$

This symmetry is rather intuitive; but it is not the only one. Consider the mode expansion on a torus

$$X = X_0 + \frac{m}{R}\tau + 2LR\sigma + \frac{i}{2}\sum_{n\neq 0}\frac{(\alpha_n z^{-n} + \tilde{\alpha}_n \bar{z}^{-n})}{n} \tag{2.45}$$

The momentum p, is quantixed in unities of $1/R$, corresponding to the above symmetry. On the other hand the momentum conjugate to X_0, p_0, has eigenvalue $2LR$, thus a multiple or 2R. This means that X_0 lies on a circle of radius $1/2R$; therefore the symmetry

$$X \to X + \frac{\pi}{R} \tag{2.46}$$

is obeyed.

The fermionic field constructed out of X and $\tilde{X}$ is given, in the case of a single $U(1)$ by

$$\psi_{\alpha,\beta}(z,\bar{z}) = \exp(i\alpha X_L(\bar{z}) + i\beta X_R(z)) = \exp\left(i\frac{\alpha+\beta}{2}X + i\frac{\alpha-\beta}{2}\tilde{X}\right) \tag{2.47}$$

and under eqs.(2.44), (2.46) transform as

$$\psi_{\alpha,\beta} \to \psi_{\alpha,\beta}\exp(i(\alpha+\beta)\pi R) \tag{2.48a}$$

$$\psi_{\alpha,\beta} \to \psi_{\alpha,\beta}\exp\left(i(\alpha+\beta)\frac{\pi}{2R}\right) \tag{2.48b}$$

One should note at this point that eq.(2.48a) and eq.(2.48b) correspond, in terrns of strings and moduli space, to modular transformations. Thus, modular invariant amplitudes require that physically relevant operators be invariant under the above transformationa. Thase are either bilinear of the type $\psi^+(x)\psi(x)$, or bound states

$$(\psi_{\alpha,\beta}(x))^F := \psi_{F\alpha,F\beta}(x),$$

such that

$$F(\alpha+\beta)R = 2n \tag{2.49a}$$

$$F(\alpha-\beta)\frac{1}{2R} = 2m \tag{2.49b}$$

where F, n, m, are integers. This implies, for the spin, the relation

$$S = \frac{\lambda}{2} = \frac{\alpha^2 - \beta^2}{8} = \frac{mn}{F^2} \tag{2.50}$$

which is a rational number.

We can work out in some detail the free field case, $\beta = 0, \lambda = \alpha^2/4$. For spin one, $\lambda = 2, \psi_{\alpha,0}$ is a physical field alone for $R = \sqrt{2}$, since ψ is invariant under eqs.(2.48a,b) in these circunstances. For half-integer spins, we need $R = 1$ and bound states of two fermions.

In general we have more complicated relations. The energy momentum tensor θ is of the Sugawara form.

The commutation relations between currents and elementary fields are

$$[J^a_-(x), \psi^i_1(y)] = -\frac{1}{2}A^{ai}\psi^i_1\delta(x_- - y_-) \tag{2.51a}$$

$$[J^a_+(x), \psi^i_1(y)] = -\frac{1}{2}B^{ai}\psi^i_1\delta(x_+ - y_+) \tag{2.51a}$$

$$[J^a_+(x), \psi^i_2(y)] = -\frac{1}{2}C^{ai}\psi^i_2\delta(x_+ - y_+) \tag{2.51c}$$

$$[J^a_-(x), \psi^i_2(x)] = -\frac{1}{2}D^{ai}\psi^i_2\delta(x_- - y_-) \tag{2.51d}$$

Thus, using the Sugawara form of the energy-momentum tensor we find

$$[\theta_-(x_-), \psi^i_1(x'_-, x'_+)] = -2\pi A^{ai} : J^a_-(x_-)\psi^i_1 : \delta(x_- - x'_-) + \frac{i}{4}A^{ai}A^{ai}\psi^i_1\delta'(x_- - x'_-)$$

$$[\theta_+(x_+), \psi^i_1(x'_-, x'_+)] = -2\pi B^{ai} : J^a_+(x_+)\psi^i_1 : \delta(x_+ - x'_+) + \frac{i}{4}B^{ai}B^{ai}\psi^i_1\delta'(x_+ - x'_+)$$

$$[\theta_+(x_+), \psi^i_2(x'_-, x'_+)] = -2\pi C^{ai} : J^a_+(x_+)\psi^i_2 : \delta(x_+ - x'_+) + \frac{i}{4}C^{ai}C^{ai}\psi^i_2\delta'(x_+ - x'_+)$$

$$[\varrho_-(x_-), \psi_2^i(x'_-, x'_+)] = -2\pi D^{ai} : J_-^a(x_-)\psi_2^i : \delta(x_- - x'_-) + \frac{i}{4} D^{ai} D^{ai} \psi_2^i \delta'(x_- - x'_-) \tag{2.52}$$

For the ψ's to satisfy the equations of motion we need $B^{ai} = 2F^{ai}$ and $D^{ai} = 2K^{ai}$. Demanding also that ψ have spin $s = \lambda/2$ we have

$$\sum_a A^{ai}A^{ai} - \sum_a B^{ai}B^{ai} = \sum_a C^{ai}C^{ai} - \sum_a D^{ai}D^{ai} = 4\lambda \tag{2.53}$$

There is a possible solution to these equations given by

$$A^{ai} = (Z^{ai} + \tilde{Z}^{ai} - Y^{ab}Z^{bi})\sqrt{\lambda} \tag{2.54a}$$

$$B^{ai} = (Z^{ai} - \tilde{Z}^{ai} + Y^{ab}Z^{bi})\sqrt{\lambda} \tag{2.54b}$$

$$C^{ai} = (Z^{ai} + \tilde{Z}^{ai} + Y^{ab}Z^{bi})\sqrt{\lambda} \tag{2.54c}$$

$$D^{ai} = (Z^{ai} - \tilde{Z}^{ai} - Y^{ab}Z^{bi})\sqrt{\lambda} \tag{2.54d}$$

where Z is symmetric, $\tilde{Z} = Z^{-1}$ and $Y^{ab} = -Y^{ba}$.

The fermionic operators are given by the usual Mandelstam formulae

$$\psi_1^i = \exp(-iA^{ai}X_-^a(x_-) + iB^{ai}X_+^a(x_+))$$
$$\psi_2^i = \exp(+iC^{ai}X_+^a(x_+) - iD^{ai}X_-^a(x_-)) \tag{2.55}$$

If we consider now that the torus is obtained dividing the space by a lattice Λ generated by the vectors E_μ^a we have the symmetry

$$X^a \to X^a + 2\pi E_\mu^a n^\mu \tag{2.56}$$

where n^μ are integers, while the dual lattice $\tilde{E}_\mu^a$ defined by the relation

$$\tilde{E}_\mu^a E^{\nu a} = \delta_\mu^\nu \tag{2.57}$$

generates the symmetry

$$\tilde{X}^a \to \tilde{X}^a + \pi \tilde{E}_\mu^a m^\mu \tag{2.58}$$

For the fermionic fields these transformations act as

$$\psi^i_{12} \rightarrow \psi_{12} \exp(\mp 2\pi i(\tilde{Z}^{ai} \mp Y^{ab} Z^{bi} \pm B^{ab} Z^{bi}) E^a_\mu n^\mu) \qquad (2.59a)$$

and

$$\psi^i_{12} + \psi^i_{12} \exp(-\pi i Z^{ai} \tilde{E}^a_\mu m^\mu) \qquad (2.596)$$

These twists are rather complicated; modular invariant operators must be defined in an invariant way. Some particular cases have been analised in eq.(16) but the general discussion is too complicated.

Therefore, in order to obtain modular invariant operators, we need to consider bound states of the above defined operators. The most important ones we consider in this work are the vertex operators. The first is the tachyon vertex

$$V(x) = \exp(ik\ X(x)) \qquad (2.60)$$

which in a compactified theory can be written as a product of the Minkowski space piece

$$V_{Mink}(x) = \exp(ik_\mu X^\mu(x)) \qquad \mathbf{(2.61)}$$

where $\mu = 0, ..., d_{Mink} - 1$ are unbounded coordinates, times the compactiffied piece

$$V_{comp}(x) = \exp(ik_i X^i(x)) \qquad (2.62)$$

where $i = d_{Mink}, ..., D-1$ are compactified coordinates. According to Gepner and Witten[18], we have to consider for the compactified part of the vertex

$$V_{comp} = g_{ij}(z, \bar{z}) \qquad (2.63)$$

which is equivalent to the above expression after using non abelian fermionization, and abelian bosonization prescriptions.

With these preliminaries out of the way we pass to the discussion of the non abelian case which is far more important, since non abelian symmetry groups appear naturally in compactification processes, such as that defined in heterotic strings compactification, namely **26** right moving bosonic coordinates turn into **10** open and **16** compactified, the latter having symmetry group $E(8) \times E(8)$.

3. The non abelian case

In general, the action of a σ-model with a Wess-Zumino-Witten term describing the compactification of the string variables in invariant under a non-abelian group, rather then $U(1)^d$, due to the solitons wrapping around the tori . It is by now well known that the non abelian fermionic determinant, computed in[13a], can be written in terms of a chiral model action plus a Wess-Zumino-Witten term[6,13b], which is the root of the non abelian bosonization procedure, generalizing previous formulae to

$$J_{+ij}(x;\epsilon) = ig^{+}_{ki}(+\epsilon)\partial_+ g_{kj}(x) \tag{3.1a}$$

$$J_{-ij}(x;\epsilon) = i\partial_- g_{ik}(x+\epsilon)(g_{ik}(x))^{+} \tag{3.1b}$$

and

$$g_{ij}(x) = \mu^{-1} : \psi^{+}_{-i}(x)\psi_{+j}(x) : \tag{3.1c}$$

These formulae may be obtained by various methods. **A** typical conformaliy invariant model was discussed some years ago by Dashen and Frishman[4]. It is the $SU(n)$ invariant Thirring model described by the lagrangean density

$$\mathcal{L} = \bar{\psi}_i \; \partial\!\!\!/ \psi - \frac{g}{2}(\bar{\psi}\gamma^\mu\tau^a\psi)(\bar{\psi}\gamma_\mu\tau^a\psi) - \frac{g'}{2}\bar{\psi}\gamma^\mu\psi\bar{\psi}\gamma_\mu\psi \tag{3.2}$$

where $[\tau^a, \tau^b] = i\ f^{abc}\ \tau^c$, and we define[4b]

$$j^a_\mu = \bar{\psi}\gamma_\mu\tau^a\psi \tag{3.3a}$$

$$j_\mu = \bar{\psi}\gamma_\mu\psi \tag{3.3b}$$

whose field equations are

$$i\,\partial\!\!\!/\psi = gj\psi + gj^a\tau^a\psi \tag{3.4}$$

The symmetries of the theory are given by the conservation of the current j_μ (charge conservation), j^5_μ (pseudo-charge conservation) and j^a_μ ($SU(n)$ transformations). The current $j^a_{\mu 5}$ is not conserved but obeys

$$\partial_\mu j^a_{\mu 5} = g f^{abc} j^{\lambda b} j^c_{\lambda 5} \tag{3.5}$$

The most important result we can use from refs. 4, 14 is the fact that for a particular value of the coupling constant g, given by

$$g = \frac{2\pi}{Q_\psi + k} = \frac{2\pi}{n+k}$$

where Q_ψ is the Casimir of the group (n for $SU(n)$ which is the only case to be discussed) and k is the coefficient of the central term of the Kac-Moody algebra, conformal invariant solutions for the quantum equations of motion may be found. The fundamental point lies in the fact that the theory can be written in terms of currents.

The relevant commutators are[4ab]

$$[j_\pm(x_\pm), j_\pm(x_\pm)] = 2iC_0\delta'(x_\pm - y_\pm) \tag{3.7a}$$

$$[j_\pm(x_\pm), \psi(y_\pm)] = -(a \pm \bar{a}\gamma_5)\psi(y_\pm)\delta(x_\pm - y_\pm) \tag{3.7b}$$

and

$$[j^a_\pm(x_\pm), j^b_\pm(y_\pm)] = if^{abc} j^c_\pm(x_\pm)\delta(x_\pm - y_\pm) + iC_1\delta^{ab}\delta(x_\pm - y_\pm) \tag{3.7c}$$

which implies a Kac-Moody type algebra for the currents. This is a Kac-Moody algebra of the level $k = 2\pi C_1$; a, $\bar{a}$ and C_0 are constants to be determined in terms of the spin and anomalous dimension of the theory.

The energy momentum tensor is of the Sugawara type

$$T(x_\pm) = \frac{1}{2C_0} : j_\pm(x_\pm)^2 : + \frac{1}{2\left(C_1 + \frac{n}{2\pi}\right)} : j^a_\pm(x_\pm)^2 : \tag{3.8}$$

implying a Virasoro-Kac-Moody algebra.

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n,-m} \tag{3.9a}$$

$$[L_n, J^a_m] = -mJ^a_{m+n} \tag{3.9b}$$

$$[J^a_m, J^b_n] = f^{abc}J^c_{m+n} + nk\delta^{ab}\delta_{m,-n} \tag{3.9c}$$

where the coefficient c is given hy

$$c = \frac{kD}{Q_\psi + k}$$

and D is the dimensionality of the group.

The full solutions for the 2 and 4 point functions are known. We list them for later comparison with other results.

$$< 0|\psi_1(x_+x_-)\psi_1^+(y_+y_-)|0> = f_0[i(x_+ - y_+) + \epsilon]^{2s}[-(x_+ - y_+)(x_- - y_-)$$

$$+ i\epsilon(x_+ + x_- - y_+ - y_-)]^{-(d-s)} \tag{3.10}$$

and

$$< 0|\psi_{1a}(x_{+1}x_{-1})\psi^+_{1a'}(y_{+1}y_{-1})\psi_{1b}(x_{+2}x_{-2}\psi_{1b'}(y_{+2}y_{-2})|0 >= u^{(d+s)}v^{(d-s)}$$

$$\times f_0^2\left\{\left[i(x_{+1} - y_{+1}) + \epsilon\right]\left[i(x_{+2} - y_{+2}) + \epsilon\right]\right\}^{-(d+s)}$$

$$\times \left\{\left[i(x_{-1} - y_{-1}) + \epsilon\right]\left[i(x_{-2} - y_{-2}) + \epsilon\right]\right\}^{-(d-s)} G_{aa'bb'}(\xi,\eta) \tag{3.11}$$

where

$$u = \frac{[i(x_{+1} - x_{+2}) + \epsilon][i(y_{+1} - y_{+2}) + \epsilon]}{[i(x_{+1} - y_{+2} + \epsilon][i(y_{+1} - x_{+2}) + \epsilon]} \tag{3.12a}$$

$$v = \frac{[i(x_{-1} - x_{-2} + \epsilon][i(y_{-1} - y_{-2}) + \epsilon]}{[i(x_{-1} - y_{-2} + \epsilon][i(y_{-1} - x_{-2}) + \epsilon]} \tag{3.12b}$$

and s is the spin and d the anomalous dirnension.

We can go to Euclidean space, where we use the following notation (to compare with ref.5 this is not the same as previously, therefore we use now greek letters ξ and $\bar{\xi}$ for complex variable)

$$\xi = x_1 + ix_2$$

$$\bar{\xi} = x_1 - ix_2$$

The solution is then

$$G_{aa'bb'}(\xi, \bar{\xi}) = \delta_{aa'}\delta_{bb'}H_1(\xi, \bar{\xi}) + \delta_{ab'}\delta_{ba'}H_2(\xi, \bar{\xi}) \tag{3.13}$$

In Dasher and Frishman[4] only the case k $= 1$ was discussed. The general.case (k $\neq 1$) may present non regular solutions and has been discussed recently[14]. For 6 $= +1$, H's are anti-holomorphic, while $\delta = -1$ implies that H's are holomorphic functions. It has been shown that they satisfy the hypergeometric equation. We write $H_{1,2} = H_{1,2}(\xi)$, which obeys

$$\xi(\xi - 1)\frac{\partial^2}{\partial \xi^2}H = \frac{1}{2\pi(C_1 + \frac{n}{2\pi})}\left[(2n + 2\pi C_1)\xi - 2n\right]\frac{\partial}{\partial \xi}H$$

$$- \frac{n-1}{\xi\left[2n(C_1 + \frac{n}{2\pi})\right]^2}(1 - 2\pi C_1)H = 0 \tag{3.14}$$

In order to have regular solutions, Dasher and Frishman[4] took C, $= 1/2\pi$ or k $= 1$ in the Kac-Moody algebra.

In the general case however, the solution is the hypergeometric function. In complex variables we have

$$\langle\psi^a(\xi_1)\psi^{b^+}(\xi_2)\psi^{c^+}(\xi_3)\psi^d(\xi_4)\rangle = [(\xi_1-\xi_4)(\xi_2-\xi_3)]^{-2\Delta}$$

$$\times\left\{\delta^{ab}\delta^{cd}\left(\Im_1^{(0)}(x)+h\Im_1^{(1)}(x)\right)\right.$$

$$\left.+\,\delta^{ac}\delta^{bd}\left(\Im_2^{(0)}(x)+h\Im_2^{(1)}(x)\right)\right\} \qquad (3.15)$$

where

$$x=\frac{(\xi_1-\xi_2)(\xi_3-\xi_4)}{(\xi_1-\xi_4)(\xi_3-\xi_2)} \qquad (3.16)$$

and

$$\Im_1^{(0)}(x)=x^{-2\Delta}(1-x)^{\Delta_1-2\Delta}F\Big(-\frac{1}{2\lambda},\frac{1}{2\lambda};1+\frac{N}{2\lambda};x\Big) \qquad (3.17a)$$

$$\Im_2^{(0)}(x)=-(2\lambda+N)^{-1}x^{1-2\Delta}(1-x)^{\Delta_1-2\Delta}F\Big(1-\frac{1}{2\lambda};1+\frac{1}{2\lambda};2+\frac{N}{2\lambda};x\Big) \qquad (3.17b)$$

$$\Im_1^{(0)}(x)=x^{\Delta_1-2\Delta}(1-x)^{\Delta_1-2\Delta}F\Big(-\frac{N-1}{2\lambda};\frac{N+1}{2\lambda};1-\frac{N}{2\lambda};x\Big) \qquad (3.17c)$$

$$\Im_2^{(0)}(x)=-Nx^{\Delta_1-2\Delta}(1-x)^{\Delta_1-2\Delta}F\Big(-\frac{N-1}{2\lambda};\frac{-N+1}{2\lambda};-\frac{N}{2\lambda};x\Big) \qquad (3.17d)$$

where $A=(N^2-1)/2N(N+\lambda),\Delta_1=N/(N+k)$ $1=-\frac{1}{2}(N+k)$ and F is the hypergeometric function

$$F(a,b;c;x)=1+\frac{ab}{1!c}x+\frac{a(a+1)b(b+1)}{2!c(c+1)}x^2+\ldots \qquad (3.17e)$$

We use also

$$h=\frac{1}{N^2}\,\frac{\Gamma\big((N-1)/(N+k)\big)\Gamma\big((N+1)/(N+k)\big)}{\Gamma\big((k+1)/(N+k)\big)\Gamma\big((k-1)/(N+k)\big)}\,\frac{\Gamma^2\big(k/(N+k)\big)}{\Gamma^2\big(N/(N+k)\big)} \qquad (3.18)$$

We may compare the results with those of ref.5 and verify their equality for[5,6,13] $g_{ij}(x) = \mu^{-1} : \psi^{+}_{-i}\psi_{+j} :$. We turn back to the discussion of compactification, requiring the theory to be invariant under[3]

$$\psi_{\substack{L\\R}} \longrightarrow \psi_{\substack{L\\R}} \exp(\pm\pi i R(\alpha+\beta)) \tag{3.19a}$$

$$\psi_{L,R} \longrightarrow \psi_{L,R} \exp(\pi i(\alpha-\beta)/R) \tag{3.19b}$$

which is a superselection rule about the correlation functions to be taken for the theory, implying that in general we have to deal with correlators involving products of operators at equal points. For the spin 1/2 case, $\lambda = 1$. In the free field case, $\beta = 0$, $a = 2\sqrt{\lambda}$, and the abelian discussion holds.

In the general case, to define modular invariant products we have to deal with correlators involving

$$[\psi^a(y+\epsilon)\psi^b(\xi) - \text{constant}]\epsilon^{\gamma}$$

where 7 is chosen in order to provide a finite result. As an example we take eq.(3.15).

We define the limit E and E' approaching zero, where

$$\xi_1 = \xi_2 + \epsilon = \xi + \epsilon \tag{3.20a}$$

$$\xi_4 = \xi_3 + \epsilon' = \xi' + \epsilon' \tag{3.20b}$$

with x in eq.(3.16) being now

$$x = \frac{\epsilon\epsilon'}{(E - E')''}$$

In that case we obtain

$$\langle\psi^a(\xi+\epsilon)\psi^{b^+}(\xi)\psi^{c^+}(\xi')\psi^d(\xi'+\epsilon')\rangle =$$

$$= (\xi-\xi')^{-4\Delta}\mu^{-8\Delta}\Big\{\delta^{ab}\delta^{cd}\Big[\frac{(\epsilon\epsilon')^{-2\Delta}}{(\xi-\xi')^{-4\Delta}}(1-x)^{\Delta_1-2\Delta}$$

$$F\Big(-\frac{1}{2k};\frac{1}{2k};1+\frac{N}{2k},x\Big)+h\Big(\frac{\epsilon\epsilon'}{(\xi-\xi')^2}\Big)^{\Delta_1-2\Delta}$$
$$-\delta^{ac}\delta^{bd}\Big[(2k+N)\frac{(\epsilon\epsilon')^{1-2\Delta}}{(\xi-\xi')^{2(1-2\Delta)}}$$

$$+hN\Big(\frac{\epsilon\epsilon'}{(\xi-\xi')^2}\Big)^{\Delta_1-2\Delta}\Big]\Big\} \tag{3.21}$$

Substituting

$$F\Big(-\frac{1}{2\lambda},\frac{1}{2\lambda};1+\frac{N}{2\lambda};x\Big)$$

by

$$1-\frac{2}{2\lambda(N+2\lambda)}$$

(see eq.(3.17e)) we are left with

$$\langle\psi^a(\xi+\epsilon)\psi^{b+}(\xi)\psi^{c+}(\xi')\psi^d(\xi'+\epsilon')\rangle=$$
$$=\delta^{ab}\delta^{cd}(\epsilon\epsilon')^{-2\Delta}+h\Big(\frac{\epsilon\epsilon'}{(\xi-\xi')^2}\Big)^{\Delta_1-2\Delta}(\delta^{ab}\delta^{cd}-N\delta^{ac}\Delta^{bd})$$
$$-\Big\{\delta^{ab}\delta^{cd}\frac{k-N}{Nk(N+k)}-k\delta^{ac}\delta^{bd}\Big\}\frac{(\epsilon\epsilon')^{1-2\Delta}}{(\xi-\xi')^2}+\ldots \tag{3.22}$$

Notice that the first contribution is trivial and we define the two-point function for the normal product of two ψ's

$$\Big\langle N(\psi^a\psi^{b+})(\xi)N(\psi^{c+}\psi^d)(\xi')\Big\rangle=h\frac{\mu^{-8\Delta}}{(\xi-\xi')^{2/N(N+k)}}(\delta^{ab}\delta^{cd}-N\delta^{ac}\delta^{bd}) \tag{3.23}$$

This is zero for k $=1$. In such a case we have to define

$$\langle N(\psi^a\psi^{b+})(\xi)N(\psi^{c+}\psi^d)(\xi')\rangle=-\frac{\mu^{-8\Delta}}{(\xi-\xi')^2}\Big[\delta^{ab}\delta^{cd}\frac{1-N}{N(N+1)}-\delta^{ac}\delta^{bd}\Big] \tag{3.24}$$

Notice the analogy of the short distance behavior between this last formula and the expectation value of two free current operators. Further normal products may be defined analogously.

4. Conclusions

We reviewed several facts about abelian bosonization, rewriting them in a language familiar to conformal field theory, and fermionized the bosonic string, obtaining a Thirring model. This model is soluble, and describes a fermion with an arbitrary spin $\lambda/2$. The case $\lambda = 1$ was discussed thoroughly in ref.3, and we showed that the extension to general(rational) λ is straightforward. The only difference being the type of G.S.O. projection[15] used to obtain a modular invariant result. The extension of the results to the non abelian case can also be done, once we have the solution for the G-invariant Thirring model. This is well known for G $= SU(n)$. The resulting correlation functions have been computed (in case of 2- and 4-point functions) and the result agrees with those of ref.5, which deals with the chiral model plus a W.Z.W. term, as far as the usual identification $g_{ij} \sim \mu^{-1} :\psi^{+}_{-i}\psi_{+j}:$ is made.

An important point is the discussion of the compactification and the type of boundary condition obeyed by ψ. In[3] it has been shown that left and right movers have vacua described by charges Q_L and Q_R, induced from twisted boundary conditions. Boundary conditions are important issues for the discussions of orbifolds, but it is clear that bosonization/fermionization formulae should be useful in that case as well[16]. Twisted boundary conditions for the bosonic case were discussed in''. In that case, constraints defining a sigma model on a symmetric space were modified in such a way that twisted boundary conditions showed up quite naturally. The relationship between these different types of boundary conditions and the construction of the equivalent fermionic model is an open and interesting question.

As a consequence of the bosonization prescription described in the paper, we may study vertex operators of compactified bosonic string theories, which turns out to be the elementary field operator in the fermionic language. Thus, a fermion operator $\psi \sim e^{i\alpha x}$ of spin $\alpha^2/4$ corresponds to a vertex operator of momentum α. Bound states of ψ obeying modular invariance can be computed, and in the case k $\neq$ 1, anomalous dimensions arise naturally, as discussed in the last section.

At last, in the non abelian theory, the number of free parameters is very much reduced, contrary to the abelian case, where compactification radii are completely uncorralated. Being conected the non abelian symmetry group, correlates all radii, and the only freedom left is in the abelian piece. This property may have some non trivial role in further developments.

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Resumo

Discutimos a descrição fermiônica da teoria de cordas bosônicas, a qual resulta em um modelo de Thirring. A relação entre spin contínuo e compactificação é discutida, e soluções regulares com um número finito de campos podem ser encontradas se o spin é um número racional. As relação entre a teoria W.Z.We o modelo de Thirring em SU(n) é também tratada.