Revista Brasileira de Física, Vol. 19, nº 1, 1989

On a class of generalized elliptic-type integrals

Rafat N. Siddiqi

Department of Mathematics, Kuwait University, P.O.Box 5969, 19060 Safat, Kuwait

Received on April 13, 1988

Abstract In the present paper we study the family of integrals A, (a,k) = $\int_0^\pi e^{\alpha \sin^2 \phi/2}(1 - k^2 \cos\phi)^{-\nu-1/2} d\phi$ $0 \leq k < 1$ and $\nu > -1/2$ are real numbers. First we establish a relation of A, (a,k) with confluent hypergeometric function of two variables which leads to a series expansion of A, (a, k) for small k. We also obtain recursion formula of A, (a,k). We can easily verify that $\Lambda_0(0, k) = 2(1+k^2)^{1/2}K(\lambda)$ and $\Lambda_1(0, k) = 2(1+k^2)^{-1/2}(1-k^2)^{-1}B(\lambda)$ where $K(\lambda)$ and $E(\lambda)$ are complete elliptic integrals of first kind and of the second kind.

1. Introduction

Epstein and Hubbell[1] have treated a family of integrals

$$\Omega_j(k) = \int_0^\pi (1 - k^2 \cos\phi)^{-j-1/2} d\phi \tag{1}$$

where $0 \leq k < 1$ and j is a positive integer. These integrals are encountered in the application of a Legendre polynomial expansion method[2] to certain problems involving computation of the radiation field off-axis from a uniform circular disc radiating according to an arbitrary angular distribution law[S]. In a follow-up note on Epstein-Hubbell work[4], Weiss[5] obtained an expansion of eq.(1) in the neighbourhood of $k^2 = 1$ and established its relationship with Legendre and hypergeometric functions. The function defined in eq.(1) has a singularity at $k = 1$ and plays in important role in analysis.

For special values of j, the integral eq.(1) reduces to the complete elliptic integrals of the first and the second kind. It can be verified that

$$\Omega_0(k) = (\sqrt{2}\lambda/k)K(\lambda) \tag{2}$$

where

$$\lambda^2 = 2K^2/(1+k^2) \tag{3}$$

and $K(\lambda)$ is the complete elliptic integral of the first kind defined by

$$K(\lambda) = \int_0^{\pi/2} \left(1-\lambda^2 \sin^2 \phi\right)^{-1/2} d\phi \tag{4}$$

Similarly

$$\Omega_1(k) = (\sqrt{2}\lambda/k)(1/(1-k^2))E(\lambda) \tag{5}$$

where

$$E(\lambda) = \int_0^{\pi/2} (1-\lambda^2 \sin^2 \phi)^{1/2} d\phi \tag{6}$$

is the complete integral of the second kind.

Several mathematicians including Epstein[4], Hubbell[3], Weiss[5], Kalla[6], and Al-Saqabi[1] have studied generalized elliptic-type integrals. Recently Kalla, Conde and Hubbe11[7] have defined and studied certain generalized elliptic-type integrals.

In the present work, we study the family of integrals

$$\Lambda_\nu(\alpha,k) = \int_0^{\pi} exp\left(\alpha \sin^2 \phi/2\right)(1-k^2 \cos\phi)^{-\nu-1/2} d\phi$$

$0 \leq k < 1$, a and $v > -1/2$ are real numbers. The importance of $\Lambda_\nu(\alpha, k)$ stems from the fact that several elliptic-type integrals are simply special cases of them, and thus each recurrence formula, identity or asymptotic formula, developed here becomes a master formula from which a large number of relations for other functions can be deduced.

2. Some elliptic-type integrals

Recently Kalla[6] has studied the function

$$S_\mu(k,\nu) = \int_0^\pi \frac{\sin^{2\nu}\theta d\theta}{(1-k^2\cos\theta)^{\mu+1/2}} \tag{7}$$

where $0 \leq k < 1$, $Re(\nu) > -1/2$ and $Re(\mu) > -1/2$. He obtained series expansion and some other formulae. Further, Kalla, Conde and Hubbell[7] have studied a more generalized elliptic-type integral in the following form

$$R_\mu(k,\alpha,\nu) = \int_0^\pi \frac{\cos^{2\alpha-1}(\theta/2)\sin^{2\nu-2\alpha-1}(\theta/2)d\theta}{\left(1-k^2\cos\theta\right)^{\mu+1/2}} \tag{8}$$

$0 \leq k < 1$, $Re(\nu) > Re(\alpha) > 0$ and $Re(\mu) > -1/2$. We observe that for $\nu = 2\alpha$

$$R(k,\alpha,2\alpha) = 2^{1-2\alpha}S(k,\alpha-1/2) \tag{9}$$

and also

$$R_j(k,1/2,1) = S_j(k,0) = \Omega_j(k) \tag{10}$$

where j is a non-negative integer. Further

$$R_0(k,1/2,1) = \Omega_0(k) = (\sqrt{2}\lambda/k)K(\lambda)$$

and also

$$R_1(k,1/2,1) = \Omega_1(k) = (\sqrt{2}\lambda/[k(1-k^2)])E(\lambda)$$

where λ^2, $K(\lambda)$ and $E(\lambda)$ have been defined in eqs.(3), (4) and (6) respectively. Kalla, Conde and Hubbell[7] obtained a series expansion of $R_\mu(k,\alpha,\nu)$ and established its relationship with Gauss hypergeometric functions. They obtained asymptotic expansions of $R_\mu(k,\alpha,\nu)$ in the neighbourhood of $k^2 = 1$ and also obtained some recurrence formulas. Then they computed some numerical values of $R_\mu(k,\alpha,\nu)$ for selected values of μ, k, α and ν.

Kalla and Al-Saqabi[8] have studied the family of integrals of the form

$$K_\mu(k,m)=\int_0^\pi \frac{\cos^{2m}\theta d\theta}{(1-k^2\cos\theta)^{\mu+1/2}} \tag{11}$$

where $0 \leq k < 1, Re(\mu) > -1/2$ and **m** is a non-negative integer. They used the differencing techniques to express $K_\mu(k, m)$ and $S_\mu(k,\nu)$ in terms of the confluent hypergeometric functions. The methods of steepest descent was applied to obtain some relations with other functions. Some results were computed numerically and particular cases were mentioned. Recently, Al-Saqabi[1] further studied the family of integrals of the form

$$B_\mu(k,n,\beta)=\int_0^\pi \frac{\cos^{2n}\theta\sin^{2\beta}\theta d\theta}{(1-k^2\cos\theta)^{\mu+1/2}} \tag{12}$$

where $0 < k < 1$, $Re(\beta) > -1/2$ and n is a non-negative integer. It can be observed that for $\beta = 0$

$$B_\mu(k,n,0)=K_\mu(k,n) \tag{13}$$

and for $n=0$

$$B_\nu(k,0,\beta)=S_\nu(k,\beta) \tag{14}$$

Further, for $\beta = n = 0$ and $\mu = j$, we have

$$B_j(k,0,0)=K_j(k,0)=S_j(k,0)=\Omega_j(k) \tag{15}$$

and for $j=0$, we get

$$B_0(k,0,0)=K_0(k,0)=S_0(k,0)=\Omega_0(k)=(\sqrt{2}\xi/k)K(\xi) \tag{16}$$

and for $j=1$

$$B_1(k,0,0)=K_1(k,0)=S_1(k,0)=\Omega_1(k)=(\sqrt{2}\xi/k(1-k^2))E(\xi) \tag{17}$$

where

$$\xi^2=\frac{2k^2}{1+k^2};\quad K(\xi)\ \ and\ \ E(\xi)$$

are the complete elliptic integrals of the first kind and the second kind defined in eqs. (4) and (6) respectively. First Al.Saqabi[1] obtained a series expansion of $B_\mu(k,n,\beta)$ for small values of k then he established a relation with Gauss hypergeometric function. He also established an asymptotic formula for $B_\mu(k,n,\beta)$ by using a simple method.

3. Elliptic-type integrals $\Lambda_\nu(\alpha,k)$

In the present paper, we study the family of integrals

$$\Lambda_\nu(\alpha,k) = \int_0^\pi exp\left(\alpha\sin^2\phi/2\right)\left(1-k^2\cos\phi\right)^{-\nu-1/2}d\phi \tag{18}$$

$0 \leq k < 1$ and $\nu > -1/2$ are real numbers. First we establish a relation of $\Lambda_\nu(\alpha,k)$ with confluent hypergeometric function of two variables which leads to a series expansion of $\Lambda_\nu(\alpha,k)$ for small k. We also obtain a recursion formula of $\Lambda_\nu(\alpha,k)$. We can easily verify that

$$\Lambda_0(0,k) = 2(1+k^2)^{-1/2}K(\lambda) \tag{19}$$

and

$$\Lambda_1(0,k) = \frac{2}{\sqrt{1+k^2}}(1-k^2)^{-1}E(\lambda) \tag{20}$$

where $K(\lambda)$ and $E(\lambda)$ are complete elliptic integrals of the first kind and of the second kind defined in eqs.(4) and (6) respectively. If ν is a positive integer (say j) and $\alpha = 0$, then

$$\Lambda_j(0,k) = \int_0^\pi \left(1-k^2\cos\phi\right)^{-j-1/2}d\phi = \Omega_j(k) \tag{21}$$

We further observe that

$$\Lambda_\nu(\alpha,k) = 2\left(1+k^2\right)^{-\nu-1/2}\int_0^1 e^{\alpha t^2}(1-t^2)^{-1/2}\left(1-\frac{2k^2}{1+k^2}\right)^{-\nu-1/2}dt$$

$$= \left(1+k^2\right)^{-\nu-1/2}\int_0^1 e^{\alpha t}u^{-1/2}\left(1-u\right)^{-1/2}\left(1-\lambda^2 u\right)^{-\nu-1/2}du$$

$$= \pi\left(1+k^2\right)^{-\nu-1/2}\Phi_1\left(\frac{1}{2},\nu+\frac{1}{2},1;\lambda^2,\alpha\right)$$

where

$$\lambda^2 = \frac{2k^2}{1+k^2} \quad \text{and} \quad \Phi_1$$

is a confluent hypergeometric function of two variables defined as

$$\Phi_1(a,b;c;\omega,z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 u^{a-1}\left(1-u\right)^{c-a-1}\left(1-u\omega\right)^{-b}e^{uz}du$$

$Re(a)$, $Re(c-a) > 0$. Hence the series expansion for $\Lambda_\nu(\alpha,k)$ can be written as

$$\Lambda_\nu(\alpha,k) = \frac{\pi}{\left(1+k^2\right)^{\nu+1/2}}\sum_{k,\ell=0}^{\infty}\frac{\left(\frac{1}{2}\right)_{k+\ell}\left(\nu+\frac{1}{2}\right)_k\lambda^{2k}\alpha^\ell}{(1)_{k+\ell}k!\,\ell!}$$

For $\alpha = 0$, we get

$$\Lambda_\nu(0,k) = \int_0^\pi\left(1-k^2\cos\phi\right)^{-\nu-1/2}d\phi \tag{22}$$

Hence by the series expansion of Φ_1, we obtain

$$\Lambda_\nu(0,k) = \pi\left(\frac{2}{2-\lambda^2}\right)^{-\nu-1/2}{}_2F_1\left(\nu+\frac{1}{2},\frac{1}{2};1;\lambda^2\right)$$

The results eqs.(19) and (20) follows directly from the formula by giving the value of $\nu = 0$ and $\nu = +1$ respectively. Hence for special value of ν and $\alpha = 0$, the integral eq.(18) reduces to the complete elliptic integrals of the second kind. For the particular case $\nu = 0$, we can write

$$\Lambda_0(0,k) = \pi\left(\frac{2}{2-\lambda^2}\right)^{-1/2} {}_2F_1\left(\frac{1}{2};\frac{1}{2};1:\lambda^2\right)$$

and

$$\Lambda_1(0,k) = \left(\frac{2}{2-\lambda^2}\right)^{-3/2} {}_2F_1\left(\frac{3}{2};\frac{1}{2};1:\lambda^2\right)$$

4. Asymptotic expansion

Since

$$\int_0^\infty t^{z-1}e^{-t}dt = \Gamma(z)$$

$$\int_0^\infty e^{-pt}t^{\lambda-1}dt = \frac{p^\lambda}{\Gamma(\lambda)}, \quad Re(p) > 0$$

By pattingt $pt = u$, the integral on the left, hand side becomes

$$\frac{1}{p^\lambda}\int_0^\infty e^{-u}u^{\lambda-1}du = \frac{\Gamma(\lambda)}{p^\lambda}$$

for $Re(\lambda) > 0$. Hence

$$\frac{1}{\Gamma(\lambda)}\int_0^\infty e^{-pt}t^{\lambda-1}dt = \frac{1}{p^\lambda} = p^{-\lambda} \tag{24}$$

By cornparing eqs. (22) and (24) we obain $-\lambda = \nu + \frac{1}{2}$. Hence

$$\frac{1}{\Gamma\left(-\nu-\frac{1}{2}\right)}\int_0^\infty e^{-pt}t^{-\lambda-3/2}dt = p^{\nu+1/2}$$

where $Re(-\nu - 1/2) > 0$ or we can say $R(\nu) < -1/2$. By putting $p = (1 - k^2\cos\theta)$, we obtain

$$\left(1-k^2\cos\theta\right)^{-\nu-1/2}=\frac{1}{\Gamma\left(+\nu+\frac{1}{2}\right)}\int_0^\infty e^{-t(1-k^2\cos\theta)}t^{+\nu-1/2}\,dt$$

But **we have**

$$\Lambda_\nu\,(a,k)=\int_0 exp\left(\alpha\sin^2\frac{\phi}{2}\right)\left(1-k^2\cos\phi\right)^{-\nu-1/2}d\phi$$

$$=\frac{1}{\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^\pi exp\left(\alpha\sin^2\frac{\phi}{2}\right)\left[\int_0^\infty e^{-t(1-k^2\cos\phi)}t^{\nu-1/2}dt\right]d\phi$$

$$=\frac{1}{\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^\pi exp(osin^2\frac{\phi}{2})d\phi\left[\int_0^\infty t^{\nu-1/2}exp\left(-t(1-k^2\cos\phi)\right)dt\right]$$

$$=\frac{1}{\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^\infty t^{\nu-1/2}e^{-t}dt\int_0^\pi exp\left(k^2t\cos\phi+\frac{\alpha(1-\cos\phi)}{2}\right)d\phi$$

$$=\frac{e^{\alpha/2}}{\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^\infty t^{\nu-1/2}e^{-t}dt\int_0^\pi exp\left(k^2t\cos\phi-\frac{\alpha}{2}\cos\phi\right)d\phi$$

$$=\frac{\pi e^{\alpha/2}}{\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^\infty t^{\nu-1/2}e^{-t}I_0\left(k^2t-\frac{\alpha}{2}\right)$$

$$=\frac{\pi e^{\alpha/2}}{\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^\infty\left(\frac{x}{k^2}\right)^{\nu-1/2}e^{-x/k^2}I_0\left(x-\frac{\alpha}{2}\right)\frac{dx}{k^2}$$

$$=\frac{\pi e^{\alpha/2}}{\Gamma(\nu+\frac{1}{2})}\left(\frac{1}{k^2}\right)^{\nu-1/2}\int_0^\infty x^{\nu-1/2}e^{-x/k^2}I_0\left(x-\frac{\alpha}{2}\right)dx$$

Hence we obtain

$$\Lambda_\nu(\alpha,k) = \frac{\pi e^{\alpha/2}}{\Gamma\left(\nu+\frac{1}{2}\right)} k^{-2\nu-1} \int_0^\infty exp\left(-x\frac{1-k^2}{k^2}\right) x^{\nu-1/2} e^{-x} I_o\left(x-\frac{\alpha}{2}\right) dx \quad (25)$$

We have therefore exposed A, *(a,k) as* a Laplace transform in which the coefficient in the first exponent $((1-k^2)/k^2)$ approaches zero *as* k^2 approaches *1.* Hence we can apply an Abelian theorem for Laplace transforms[5] to determine the behaviour of $\Lambda_\nu(\alpha,k)$ in the neighbourhood of $k^2 = 1$. To do this we note an asymptotic expansion

$$e^{-x} I_0(x) \sim \frac{1}{\sqrt{2\pi x}} \sum_{n=0}^\infty \frac{(-1)^n}{(2x)^n} \frac{\Gamma\left(\frac{1}{2}+n\right)}{\Gamma\left(\frac{1}{2}-n\right)} \cdot \frac{1}{n!} \quad (26)$$

as x tends to infinity. Substituting eq.(26) into eq.(25), we find for the asymptotic expansion of $\Lambda_n u$(a,*k)* :

$$\Lambda_\nu(\alpha,k) \approx \frac{\pi e^{\alpha/2}}{\Gamma\left(\nu+\frac{1}{2}\right)} k^{-2\nu-1} \frac{1}{\sqrt{2\pi}} \sum_{n=0}^\infty \frac{(-1)^n}{2^n} \frac{\Gamma\left(\frac{1}{2}+n\right)}{\Gamma\left(\frac{1}{2}-n\right) n!} \frac{\Gamma(\nu-n)}{\left(\frac{1-k^2}{k^2}\right)^{\nu-n}}$$

where v is not a negative integer.

5. Recurrence relations

For finding the recursion formula of A, (a,*k),* we proceed in the following way

$$\begin{aligned}\Lambda_\nu(0,k) &= \pi\left(\frac{2}{2-\gamma^2}\right)^{-\nu-1/2} {}_2F_1\left(\nu+\frac{1}{2};\frac{1}{2};1:\gamma^2\right)\\ &= \pi\left(\frac{2}{2-\gamma^2}\right)^{-\nu-1/2}\left(1-\gamma^2\right)^{-\nu}\left(\frac{2}{2-\gamma^2}\right)^{1/2-\nu}\\ &= {}_2F_1\left(\frac{1}{2}-\nu;\frac{1}{2};1:\gamma^2\right)\cdot\left(\frac{2}{2-\gamma^2}\right)^{-\nu-1/2}\end{aligned}$$

Hence we obtain

$$\Lambda_\nu(0,k) = \left(\frac{2}{2-\gamma^2}\right)^{-2\nu}\left(1-\lambda^2\right)^{-\nu}\Lambda_{-\nu}(0,k)$$

It is known[0] that

$$\Phi_1\left(\alpha,\beta;\gamma;x;y\right) = \left(1-x\right)^{-\beta}e^y\phi_1\left(\gamma-\alpha,\beta;\gamma;\frac{x}{x-1},-y\right)$$

Hence

$$\begin{aligned}\Lambda_\nu(\alpha,k) &= \frac{\pi}{\left(1+k\right)^{\nu+1/2}}\Phi_1\left(\frac{1}{2},v+\frac{1}{2};1;\lambda^2,\alpha\right)\\ &= \frac{\pi}{\left(1+k^2\right)^{\nu+1/2}}\left(1-\lambda^2\right)^{-\nu-1/2}e^\alpha\ \Phi_1\left(\frac{1}{2},\nu^{\frac{1}{2}};1;\frac{\lambda^2}{\lambda^2-1},-\alpha\right)\\ &= \left(1-k^2\right)^{-\nu-\frac{1}{2}}e^\alpha\ \Phi_1\left(\frac{1}{2},\nu+\frac{1}{2};1;\frac{2k^2}{k^2+1},-\alpha\right)\\ &= e^{m}\mathrm{A},\ (-\alpha,\mathrm{ik})\end{aligned}$$

Hence we obtain

$$\mathrm{A},\ (\mathrm{a},\mathrm{k}) = e^{m}\mathrm{A},\ (-\alpha,ik)$$

Finally I would like to express my sincere thanks to Professor S.L. Kalla for his suggestions and comments for a better presentation of this paper. Thanks are also due to Research Council of Kuwait University for sanctioning my research project SM054 under which this paper was completed.

References

1. B.N. Al-Saqabi, Hadranic Jour. Vol. 10, (1987).
2. M.J. Berger and J.C. Lamkin, J.Res.NBS, 60, 109 (1958).
3. J.H. Hubbell, R.L. Bach and R.J. Herbold, J. Res. NBS. C. 65, 249 (1961).
4. L.F. Epstein and J.H. Hubbell, J.Res. NBS B 67, 1 (1963).
5. G.H. Weiss, J.Res. NBS, B. 68, 1 (1964).
6. S.L. Kalla, Bulgarian Acad. Science, (Special volume, 216 (1984).
7. S.L. Kalla, S. Conde and J.H. Hubbell, Applicable Analysis, 22, 273 (1986).

8. S.L.Kalla and B.N. Al-Saqabi, Rev.Bras.Fis. 16, *145 (1986)*.

9. V. Kim Tuan, S.L. Kalla, Rev. Tec. Ing. Univ. Zulia, **10(i)**, 81 *(1987)*.

Resumo

No presente trabalho estudamos a família de integrais $A_\nu(\alpha,k) = \int_0^\pi e^{\alpha \sin^2 \phi/2}(1 - k^2 \cos\phi)^{-\nu-1/2} d\phi$ onde $0 \leq k < 1$ e $\nu > -1/2$ são números reais. Estabelecemos a relação entre $A_\nu(\alpha,k)$ e as funções hipergeométricas confluentes de duas variáveis que conduz a uma expansão em série dos $A_\nu(\alpha,k)$ para k pequenos. Também obtemos fórmulas de recorrência para $A_\nu(\alpha,k)$. Verificamos que $A_0(0,k) = 2(1+k^2)^{1/2}K(\lambda)$ e $A_1(0,k) = 2(1+k^2)^{-1/2}(1-k^2)^{-1}E(\lambda)$ onde $K(\lambda)$ $E(\lambda)$ são integrais elípticas de primeira e segunda espécie.