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The quantum flux in quasi one-dimensional conductors

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Abstract A method is presented which quantizes electromagnetic fluxes directly in flux space. It is based on the commutation law $[\hat{\phi}_B, \hat{\phi}_E] = i$, where $\hat{\phi}_B$ is the magnetic flux, and $\hat{\phi}_E$ the longitudinal electric flux of a quasi one-dimensional conductor. The relevance of such a method for the description of the quantized Hall plateaus is discussed. In a second step, the polarization electric flux is introduced, together with a method for quantization of hybrid variables formed with pure electromagnetic fluxes plus electronic variables.

1. Introduction

After Laughlin[1] evoked gauge invariance to explain the quantization of the Hall conductance[1-7], it is becoming clearer that the effect should be caused by a macroscopic quantum mechanical phenomenon[5-7]. Here I propose a theory which formalizes this idea and might account for some basic features of the quantized Hall effect. The theory deals with a single (extended) degree of freedom: the magnetic flux and its canonical momentum, the electric flux.

Laughlin ideal Hall effect device is a ring where many Landau orbitals run in parallel. Each one of the Landau orbitals can be represented by the system shown in fig.1.

It is a ring of radius R and cross section S, with a magnetic field $\vec{B}$ parallel to its symmetry axis z. Electrons are confined in a thin slice of the ring, forming a quasi one-dimensional gas in the plane of closed curve I'. The systems' observables are[5] the Hall voltage V, and the longitudinal voltage V,.

If ϕ_E is the electric flux through S, and ϕ_B the magnetic flux through any one of the surfaces encircled by I', one can show that the quantum operators $\hat{\phi}_E$ and $\hat{\phi}_B$ associated with them obey the commutation relation

$$[\hat{\phi}_B, \hat{\phi}_E] = i \tag{1}$$

This commutator can be decluced directly from the equal time commutation relations of the electromagnetic field. Eq.(1) is true for any pair of surface S and closed curve Γ with interception. If they do not intercept, the commutator of fluxes vanishes.

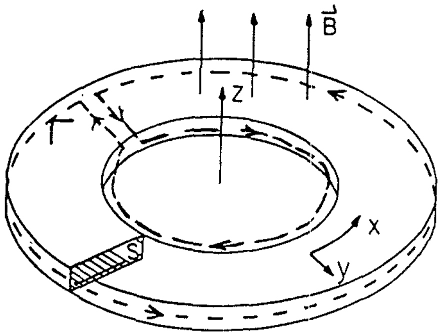

Fig.1 - Idealized orbital Γ is the dotted curve. The horizontal full line in cross-section S shows the plane of the quasi one-dimensional electron gas.

$\hat{\phi}_E$ and $\hat{\phi}_B$ are therefore a pair of extended operators, built up as linear combinations of the local fieid $\vec{A}(\vec{x})$ and $\vec{E}(\vec{x})$, and which form a canonical pair.

One observes that the observables, V, and V„ are respectively proportional to the expectation values of the fiuxes $< \hat{\phi}_B >$ and $< \hat{\phi}_E >$

$$V_x \cong \frac{2\pi R}{S} < \hat{\phi}_E > \tag{2}$$

That is, the two observables are related to a canonical pair. It would then seem reasonable to suppose, as I do here, that: the pair of fluxes should effectively decouple from other degrees of freedom, and develop a dynamics on its own.

In the ϕ_E representation, where states are represented by functions $\psi(\phi_E)$, the action of $\hat{\phi}_B$ in a given state must be given by $i\frac{\partial\psi}{\partial\phi_E}$. Notice that $\hat{\phi}_B$ is the total magnetic flw, i.e. it accounts for the flux of external magnetic fields, as well as for that generated by currents on the electron gas.

Before exploring the flux dynamics one has first to establish the domain of the wave functions $\psi(\phi_E)$ in ϕ_E space.

In a high magnetic field, a Hall effect device is, I suppose here, an ensemble of a large number M of well separated and almost identical narrow tubes of few electrons, like the idealized system of fig. 1. Each tube is in fact defined by the electronic wave function of a Landau orbital.

Let us consider one particular tube. To suppose that electric flux fluctuations through its cross-sections S should be produced ***only*** by the few electrons moving inside it, seems to be an educated guess, because the flux produced there, by the immense number of distant electrons moving in other tubes, would result in an average classical source field, perturbing the orbital dynamics.

The real Hall effect system allows direct control of the external magnetic field, of the gate voltage,and of the ***total*** current in the ensemble. But one has no externally direct control of the number of electrons in each tube. In this paper however I analyse a simpler system, where supposedly one can define the number of electrons at one's own will. This system is nevertheless good enough to illustrate the method and the consequences of flux quantization. Here, I treat the one electron case, leaving to the end of the paper a discussion about multi- carrier situations.

If the system contains a single electron, the electric flw on S must lie in the interval $-e/2 \leq \phi_E \leq +e/2$ and this interval must be the domain of the wave function $\psi(\phi_E)$.

Now one can easily obtain the eigenfunctions and eigenvalues of the magnetic flux operator

$$\hat{\phi}_B = i\frac{\partial}{\partial\phi_E}$$

in the domain above. If p is an integer, the p^{th} normalized eigenfunction reads

$$\psi_p(\phi_E) = \frac{1}{\sqrt{e}} e^{ip\frac{2\pi}{e}\phi_E} \tag{3}$$

whereas its associated **eigenvalue** is $\phi_{B\,p} = 2\pi p/e$ **resulting** therefore in magnetic flux quantization consistent with the observed anomalous quantized Hall **effect**[4,6] (although in the present case the magnetic flux can be an even as **well** as an odd multiple of $2\pi/e$).

From now on the p^{th} eigenstate of $\hat{\phi}_B$ shall be represented by $|p>$; and a generic state $\psi_\alpha(\phi_E)$ by $|\alpha>$. I also define $\phi_{E\,\alpha}$ and $\phi_{B\,\alpha}$ as the expectation values of electric and magnetic **fluxes** in states a.

In particular, notice that $\phi_{E\,p} = 0$, **which** means that ϕ_E vanishes if the system is a pure eigenstate of magnetic flux, and this explains why the longitudinal **voltage** V„ given in **eq.(2)**, must drop to zero whenever one has a very sharp Hall plateau.

With the purpose of illustrating why plateaus appear in dynamical models, I will now quantize a model Hamiltonian that is quadratic in the magnetic and electric flux operators belonging to a single Landau orbital. So I define the **Hamil**tonian

$$H = \frac{1}{L}(H_B + H_E) \tag{4}$$

where H_B and H_E are respectively its magnetic and electric **parts**

$$H_B = \frac{1}{2}\hat{\phi}_B^2 - b\ \hat{\phi}_B \tag{5}$$

and

$$H_E = \frac{1}{2}\nu\ \hat{\phi}_E^2 - \epsilon\ \hat{\phi}_E \tag{6}$$

H **has** therefore a self **induction** term $\hat{\phi}_B^2/2L$ a longitudinal electric ***flux*** capacity term $\hat{\phi}_E^2/2c$, with c $= L/2$ as the capacity for longitudinal ***flux only***; and two external source couplings. For brevity's sake, I call b the external magnetic field. Notice that the external electric source may be related to the action of a global source in the ensemble of orbitals.

This is one of the simplest non trivial Hamiltonians one could imagine for the system of **fig.1.** It is not however the real Hall effect Hamiltonian. The true Hamiltonian should result from the integration of electric degrees of freedom, and would of course couple **flux** operators belonging to different orbitals in the ensemble. Nonetheless the quadratic Hamiltonian already shows well the role that **flux** quantization may have in quasi one-dimensional conductors.

Let us first ignore the electric Hamiltonian, and solve H_B. Its eigenstates are the same as those of $\hat{\phi}_B$: $\psi_p(\phi_E)$ or $|p>$. The corresponding eigenvalues are then given by

$$E_p^0 = \frac{1}{2}\left(\frac{2\pi p}{e}\right)^2 - b\frac{2\pi p}{e} \tag{7}$$

The last equation means that, for a given magnetic source b, the ground state of H_B is the state $|P_c>$, with energy $E^0_{p_c}$, where p_c is the closest integer to $be/2\pi$.

I call this system with no electric energy Hamiltonian the ideal Hall effect system. Below I Iist some of its properties:

a) When $be/2\pi$ is half integer, the ground state has a double degeneracy.

b) Besides this discrete set of cases of degenerate vacua, the ground state magnetic flux ϕ_B is given by $2\pi p_c/e$. That is, for every integer p_c, in the interval $p_c - \frac{1}{2} \leq \frac{be}{2\pi} \leq p_c + \frac{1}{2}$, the magnetic **flux** is in the exact plateau $\phi_B = 2\pi p_c/e$.

These well defined quantized plateaus are typical of the ideal case. They occur only in situations that allow one to diicard the energy of the longitudinal electric field, as compared to the magnetic energy. This is a necessary condition for having simultaneous eigenstates of H and $\hat{\phi}_B$. In this case the semiclassical **flux** quantization made by Laughlin[1] would hold.

c) For $be/2\pi$ different from any half integer the system shows no longitudinal resistivity because, as discussed above, in magnetic flux eigenstates the electric **flux** expectation value must vanish, and so the longitudinal voltage V_z vanishes.

d) If $be/2\pi$ is **exactly** a half integer, p $+\frac{1}{2}$, then the ground state of H_B is degenerate, **and** the ideal system can be in any state of the form

$$|\theta, u>=\cos de^{iu}|p> + \text{sen}\ \theta e^{-iu}|p+1> \tag{8}$$

(for any pair 0, u, such **that** $0 \leq 0, u \leq \pi/2$) **with** equal probability. Then the electric flux expectation **value,** which in the **vacuum** $|\theta, u>$ is

$$\phi_E = \frac{e}{2\pi}\text{sen}\ 2\theta\text{sen}\ 2u \tag{9}$$

would be undertermined.

In what **follows** I show that the **full** Hamiltonian, which incorporates the electric energy, **leads** to **nosharp** plateaus. I compute the ground state of total Hamiltonian H,**by** means of' a variational **calculation,** suggested by the ideal case **analysis.**

If the externa1 **magnetic** field is between two given consecutive **multiples** of the magnetic **flux** quantum, p $2\pi/e$ and (p $+$ 1) $2\pi/e$, then the natural trial **is** a ground state which **mixes** states $|p>$ and $|p+1>$. Since state $|\theta, u>$ of **eq.(8)** is the most general normalized. state one can make up with those two ϕ_B eigenstates, I choose it to be the trial ground state. 0 and u are now variational parameters one must adjust to get the expectation value $<\theta u|H|\theta u>$ to its minimum, so as to **approach** the **exact ground** state.

Let θ_0 and u_0, be the values of θ and u which minimize the Hamiltonian expectation value in the trial state

$$\cos 2u_0 = \left[1+\left(\frac{2\pi}{e}\frac{\epsilon}{\nu}\right)^2\right]^{1/2}$$

and

$$\text{sen}\ 2\theta_0 = A\left[\left(p+\frac{1}{2}-\frac{be}{2}\right)^2 + A^2\right]^{-1/2} \tag{10}$$

The constant A is of second order in the fine structure constant

$$A = 2\left(\frac{e}{2\pi}\right)^4 \nu\left[1+\left(\frac{2\pi}{e}\frac{\epsilon}{\nu}\right)^2\right]^{1/2} \tag{11}$$

Then one can use formula (9) to get the longitudinal electric flux expectation value associated with external sources b and ϵ. The magnetic flux is in turn given by

$$\phi_B = \frac{2\pi}{e}\left(1 + \frac{1}{2} - \frac{1}{2}\cos 2\theta_0\right) \qquad (12)$$

Fig.(2) shows the electric and magnetic fluxes expectation values when the constant A equals 0.05.

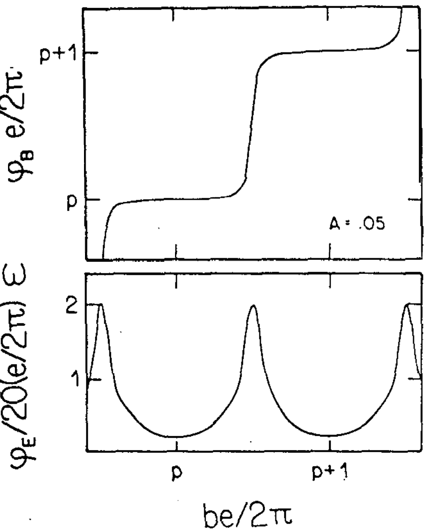

Fig.2 - The expectation values of electric and magnetic fluxes as functions of the external magnetic source for A = 0.05.

The bigger A is, the more different the system will be from the ideal quantized Hall effect. The variational approach used here indicates that the Hall plateus start being poorly defined if A $\gtrsim$ 0.5.

The trial function choice also introduces spurious discontinuities on ϕ_B and ϕ'_E, at integer values of $be/2\pi$, which are not visible in the scale fo fig.(2). The adoption of a more symmetric trial function, that mixes $|\theta s>$ with the state $|p-1>$, would remove those false discontinuities.

The theory might in principle be extended to multi-electron and multi-hole systems. In a gas of n carries, the maximum electric flux through cross-section S should be $ne/2$, so that perhaps the wave function domain could be enlarged to the interval $(-ne/2, +ne/2)$. That would lead to plateau states of the type $\phi_{B\,pn} = 2\pi p/en$. On the other hand there are factors which tend to inhibit configurations of longitudinal electric flw around $ne/2$, with n larger than one. For example, (1) those configurations should be rare in phase space, since they amount to having two or more electrons in the neighborhood of the same cross-section S-which is quite unlikely an event; and (2) because of the proximity of the electrons these configurations would also have an huge Coulomb interaction energy. One must then be cautious concerning the extension of the flux wave function domain to regions larger than the interval $(-e/2, +e/2)$. This is clearly a point which calls for a detailed analysis of the particular physical system where one intends to apply this method.

In order to pass from our results, which refer to a single system, to the whole ensemble of M systems (that supposedly cooperate and are all in the same quantum state) one must multiply fluxes by M, energies by M, and must also rescale the externa1 sources properly. The parameters L and C of eqs. (4) will depend on geometry and should be taken as the orbital impedance and capacity. A rough estimates gives $L \sim C \sim 2\pi R$, so that $\nu \sim 1$, $A \sim (e/2\pi)^4 \sim 10^{-5}$, for the one electron system. This figure gives one an idea of how far the system should be from the ideal case.

It then seems possible that this method of flw quantization in the very flux space might have a bearing to the description of the observed quantized Hall plateaus. In this regard I outline the following facts: (i) The longitudinal voltage V, is proportional to ϕ_E, and the Hall conductance is given by ne/ϕ_B; so that if n is allowed to change with b, our results might be use ful for interpreting the experimental data. (ii) The eiectrostatic interaction (related to the radial electric flux, not to the longitudinal one) might be described by adding to the Hamiltonian eq.(4) a term of the type

$$\frac{1}{2\tilde{C}}\hat{N}^2 - V_g\hat{N}$$

where $\hat{N}$ is the electron number operator. The term $\hat{N}^2$ accounts for the Coulomb energy, whereas V, is the externa1 gate **voltage** that acts against the system's tendency to discharge. Since the new Hamiltonian does not depend on $\hat{\theta}$, the momentum of $\hat{N}$, its ground state will always have a well defined number of **electrons** n. Then **if,** at **fixed** b, one keeps the system in a flux eigenstate $(\phi_{B p} = 2\pi p/e)$ **and increases** V, one will find that the Hall conductance as a function of V, shows a sequence of plateaus, analogous to the pattern observed in experiment. (iii) One effect of impurities is to **create** quantum fluctuations in the number of carriers. Thus their role could be introduced by means of another Hamiltonian part of the **type**

$$\alpha\left(\sqrt{\hat{N}}e^{i\hat{\theta}} + e^{-i\hat{\theta}}\sqrt{\hat{N}}\right)$$

which of course would lead to a ground state without a well-defined n. (iv) The results presented in this paper still hold if the **system's** geometry is not cylindrical.

This method is a cannonical extension of Dirac's quantum flux method.

Polarization[8]

This second section complements the quantum flux method by introducing the effect of electric polarization.

Let us take our system to be a box of length L and cross section S, with periodic boundary conditions. It could be either the ring represented in figure 1, or a rectilinear box, with is length parallel to the x direction.

The electric part of the electromagnetic cloud that **dresses** the electron, **and** is responsible for the low energy Coulomb interaction, is already taken into account by the electric flux $\hat{\phi}_E$ in commutator **eq.(1),** since this commutator can be deduced from a *local* field commutator. However, what that electric flux still does not account for is the effect of electronic polarization, because (in the traditional formulation of quantum electrodynamics) the electromagnetic and the electron fields are independent variables, in the sense that each one of them has its own completeness.

Polarization is an effect 'induced by external sources, which drives a charged particle in a system to occupy a new avarage position different from its former natural position.

Suppose that our box of volume LS has one electron inside. Thus the electronic position variable will be in the interval $-L/2 \leq x \leq L/2$. So, I define the polarization operator

$$\hat{P} = -\frac{ex}{LS} \tag{A.1}$$

that measures the deviation from the normal electron position $x = 0$; and the associated polarization electric flw

$$\phi_E^p = S\hat{P} = -\frac{ex}{L} \tag{A.2}$$

Then I interpret the difference between the pure electromagnetic electric flux and the polarization flw as the true electric field flux

$$\phi_E^{\text{total}} = \hat{\phi}_E - \phi_E^p = \hat{\phi}_E + \frac{ex}{L} \tag{A.3}$$

This construction bears .a (remote) analogy to the standard procedure, used in classical electromagnetism of material media, of introducing an auxiliary displacement vector $\vec{D}$, but making the complete electric field as the difference between $\vec{D}$ and the polarization vector.

On the other hand, the canonical momentum of the total electric flux is also another hybrid operator of magnetic type

$$\hat{\phi}_B^{\text{total}} = \hat{\phi}_B - \frac{iL}{e}\partial_x \tag{A.4}$$

where $-i\partial_x$ is the electron *generalized* momentum, and $\hat{\phi}_B$ the pure electromagnetic flux of commutator eq.(1).

Now, one sees that the total magnetic flux ϕ_B^{total} results to be just **proportional** to the **electron's** physical momentum, as defined through the **rule** of minimal coupling

$$\hat{p} = -i\partial_x + \frac{e}{L}\hat{\phi}_B = \frac{e}{L}\phi_B^{total} \qquad (A.5)$$

One notices., in this **regard,** that the operator $\frac{1}{L}\phi_B$ **is** in fact the ***space average*** of the vector potential operator $(1/L\int A, dz)$ in the orbital.

The **commutator** of the total fluxes **is**

$$\left[\phi_B^{total}, \frac{1}{2}\phi_E^{total}\right] = -\,i \qquad (A.6)$$

so that it **is** strictly the variable $\frac{1}{2}\phi_E^{total}$, and not #E$^{ta'}$, that makes a canonical pair with ϕ_B^{total}. Nevertheless, is spite of the **factor 1/2,** this alternative quantum flux scheme, which explicitly includes the polarization effects, also leads to ***the same set*** of physically allowed magnetic flux eigenvalues obtained in the former case: $2\pi p/e$. This is because an account of the additional polarization term, the domain of variation of the variable $\frac{1}{2}\phi_E^{total}$ is the same as that of $\hat{\phi}_E$: $\left(-\frac{e}{2}, +\frac{e}{2}\right)$.

This method can also be generalized to describe many-electron orbitals, where one may **take** x **and** $-\partial_x$ to be the position and generalized momentum of the electron cluster.

In applications of the method there are at **least** two distinct situations: (a) when the system's Hamiltonian **is** dominated by the pure magnetic energy $1/2\int \vec{B}^2 d\vec{x}$, the important operator shall be $\hat{\phi}_B$; and (b) if, however, the **dominant** Hamiltonian term is the electron kinetic energy, then the important operator shall be $\hat{\phi}_B^{total}$.

Following papers, by M. Simões e I. Ventura , present (a) the statistical Mechanics of the quantum flux for an ensemble of orbitals; and (b) the application of the canonical quantum flux method to the direction description, at finite temperatures, of the transistor where the quantized Hall effect is observed.

An analysis, by I. Ventura[1'], on the extension of the canonical flux quantization to systems which are essentially three-dimensional, is being concluded and shall be published elsewhere.

At a section of Academia Brasileira de Ciências, in October 8^{th}, 1985, in São Paulo, I. Ventura presented an application of the canonical quantum flux method to reinterpret the Bohn-Aharonov effect, and also to explain the flux quantization associated with the motion of a Cooper pair in a superconductor ring. I thank the referee for pointing out to me the paper by Webb et al[11] which reports the observation of Bohm-Aharonov oscillations in submicron normal-metal rings. It seems possible that the quantum flux method introduced here, might have some relevance to the description of that phenomenon.

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Resumo

O artigo apresenta um método que quantiza fluxos diretamente no espaço dos fluxos. O método se baseia na lei de comutação $[\hat{\phi}_E, \hat{\phi}_B] = i$, onde $\hat{\phi}_B$ é o fluxo magnético, e $\hat{\phi}_E$ o fluxo elétrico longitudinal de um condutor quasi unidimensional. Discute-se a relevância desse método para a descrição dos plateaus do efeito Hall quantizado. Numa segunda etapa introduz-se o fluxo elétrico de polarização, junto com o método de quantização de variáveis híbridas, formadas com fluxos eletromagnéticos puros e variáveis eletrônicas.